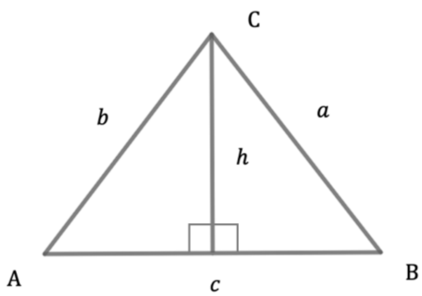
Geometry and Trigonometry Module 8 Trigonometric Identities and Conic Sections

Section 8.1 Law of Sines

Looking Back 8.1

Not every triangle is a right triangle; therefore, the Pythagorean Theorem cannot always be used to find missing sides when angles are known. Some triangles are acute, and others are obtuse. Non-right triangles are called oblique. You learned about oblique triangles in Section 6.13 of this text by investigating acute triangles.

Law of Sines:



$$\sin \mathrm{A} = \frac{h}{b} \qquad \sin \mathrm{B} = \frac{h}{a}$$

$$b \sin \mathrm{A} = h \qquad a \sin \mathrm{B} = h$$

$$b \sin \mathrm{A} = a \sin \mathrm{B}$$

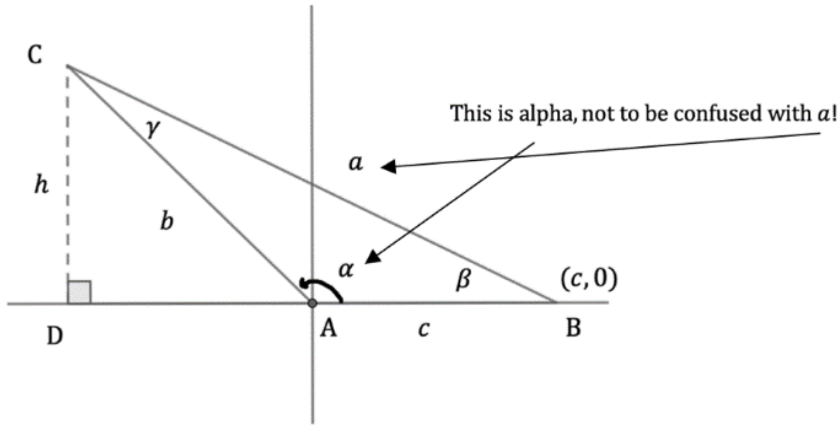
$$\frac{\sin \mathrm{A}}{a} = \frac{\sin \mathrm{B}}{b}$$

By the transitive property, two things equal to the same thing are equal to each other. The Law of Sines can be used to find a missing side or angle when the other sides or angles are known. In summary, the Law of Sines states the following:

$$\frac{\sin \mathrm{A}}{a} = \frac{\sin \mathrm{B}}{b} = \frac{\sin \mathrm{C}}{c}$$

In Pre-Calculus and Calculus, we will use α (alpha) for angle A, β (beta) for angle B, and γ (gamma) for angle C.

Let triangle ABC be an obtuse triangle. Point A is at the origin and point B lies on the positive x-axis at point $(c, 0)$; side a is across from $\angle$A, side b is across from $\angle$B, and side c is across from $\angle$C.



We know that $\sin \alpha = \frac{h}{b}$ because α is in standard position, and it is the same as the reference angle. The height, h, of ΔACD is the altitude because it is perpendicular to the extended side of $\overline{\text{BA}}$. From ΔBCD, $\sin \beta = \frac{h}{a}$. Therefore…

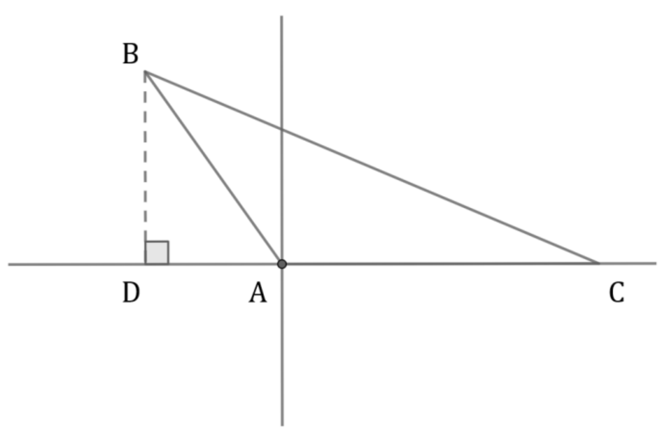
$$\sin \alpha = \frac{h}{b} \qquad \sin \beta = \frac{h}{a}$$

$$b \sin \alpha = h \qquad a \sin \beta = h$$

$$b \sin \alpha = a \sin \beta$$

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b}$$

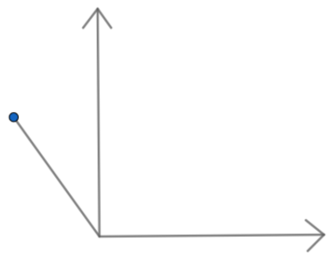
If we change the position of the triangle so c is on the x-axis at point $(b, 0)$ and $\overline{\text{BD}}$ is perpendicular to the extended side of $\overline{\text{CA}}$, then $\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$.



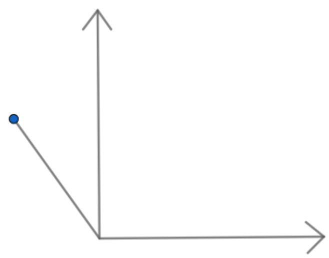
We learned that the Law of Sines may be used to find missing sides and angles when two angles and one side of a triangle are known such as angle-angle-side or angle-side-angle. Law of Sines does not work when three sides are known, or when two sides and the angle between them are known. We investigated four possibilities when two sides and an angle that is not in between the two sides are given for an acute angle in standard position. The acute angle was in standard position with angle β given, and side a extends to the positive x-axis.

Let us suppose obtuse angle β is given and is in standard position, and two sides a and b are given but angle β is not the included angle between them.

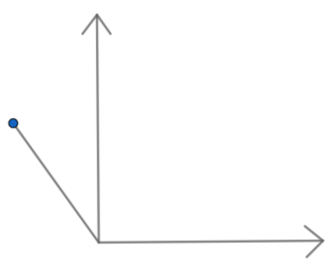
a) If side b is less than h, then a triangle cannot be made.



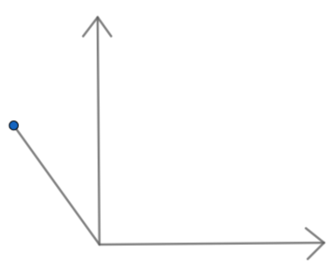
b) If b is equal to h, no obtuse triangle exists because vertex A is not on the positive x-axis.



c) If side b is longer than h, but shorter than or equal to side a, then no obtuse triangle exists.



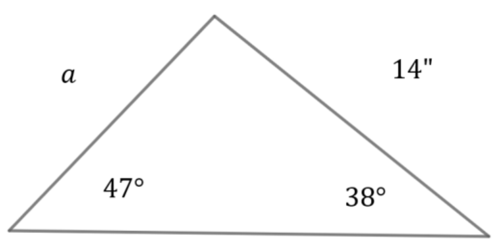
d) If side b is longer than h and longer than side a, only one unique triangle can be made.



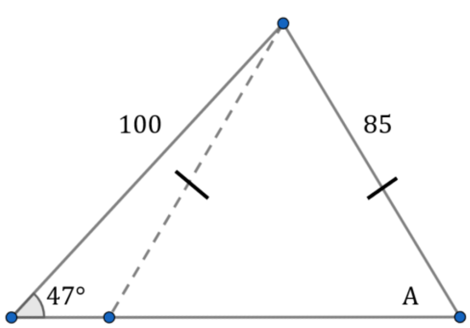
Looking Ahead 8.1

Example 1: Two planes flew over Cincinnati at the same time. One traveled west at 380 mph at a bearing of 250° and the other one traveled east at a bearing of 140°. What was the speed of the airplane traveling east? How far apart were they after 2 hours of traveling?

Example 2: Find the length of side a.



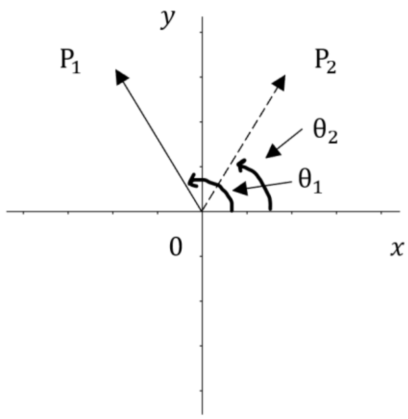
Example 3: Sometimes, two different angles (one acute and one obtuse) share the same value of sine. This is called an ambiguous case. Demonstrate an ambiguous case using the triangle below.



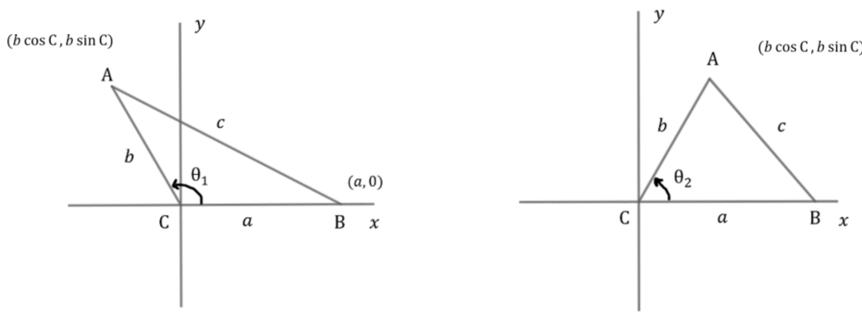
Section 8.2 Law of Cosines

Looking Back 8.2

Like the Law of Sines, the Law of Cosines is used to find sides and angles in oblique (non-right) triangles. Let θ be an angle in standard position and P be a point on the terminal side. Because $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$, then $r\cos\theta = x$ and $r\sin\theta = y$. The coordinates of P are $(r\cos\theta, r\sin\theta)$ and $r = \text{OP}$.

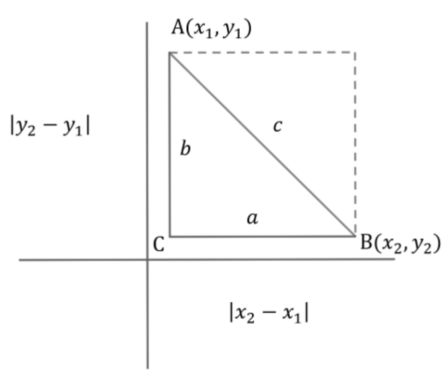


This is true for either an obtuse or an acute angle. Let us try it with oblique triangles.



The coordinates of A are $(b\cos\text{C}, b\sin\text{C})$ and the coordinates of B are $(a, 0)$. Let us review the distance formula so we can apply it to the Law of Cosines.

The Pythagorean Theorem states that $a^2 + b^2 = c^2$ in a right triangle.



The horizontal distance from B to C is $|x_2 - x_1|$, so the length of a is $|x_2 - x_1|$. The vertical distance from C to A is $|y_2 - y_1|$, so the length of b is $|y_2 - y_1|$. Substituting $x_2 - x_1$ for a and $y_2 - y_1$ for b in the Pythagorean Theorem gives us $c^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.

To find the length of c (when you know two ordered pairs of vertices of a triangle on the coordinate grid, but not the length of the sides), take the square root of both sides of the equation.

$$\sqrt{c^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$c = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This is called the distance formula. Here is another version of the distance formula that you learned about in Section 6.13:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

We will use this formula to derive the Law of Cosines.

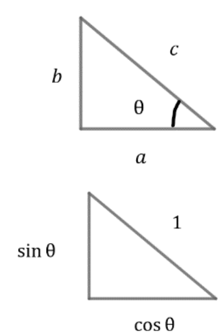
Apply this distance formula to the oblique triangles:

$$c^2 = (b\cos \mathrm{C} - a)^2 + (b\sin \mathrm{C} - 0)^2$$

$$c^2 = b^2\cos^2\mathrm{C} - 2ab\cos \mathrm{C} + a^2 + b^2\sin^2\mathrm{C}$$

$$c^2 = a^2 + b^2(\cos^2\mathrm{C} + \sin^2\mathrm{C}) - 2ab\cos \mathrm{C}$$

In the unit circle, the radius is 1; therefore, $\sin^2\theta + \cos^2\theta = 1$. Substituting 1 into the parenthesis in the above equation yields the following formula for the Law of Cosines:



$$c^2 = a^2 + b^2 - 2ab\cos \mathrm{C}$$

In summary, regarding the Law of Cosines for any triangle ABC:

$$a^2 = b^2 + c^2 - 2bc \cos \mathrm{A}$$

$$b^2 = a^2 + c^2 - 2ac \cos \mathrm{B}$$

$$c^2 = a^2 + b^2 - 2ab \cos \mathrm{C}$$

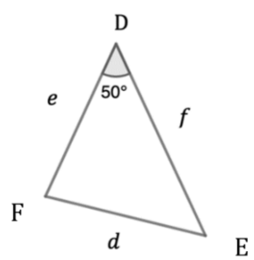
If the three sides of a triangle are known (side-side-side) or two sides and the included angle of a triangle are known (side-angle-side), then the Law of Cosines may be used to find missing sides or angles.

When using the Law of Cosines when three sides are known (SSS), find the largest angle first that is opposite the longest side. Then use the Law of Sines to find the second angle. If the cosine of the largest angle is positive, it is acute. If the cosine of the largest angle is negative, then the angle is obtuse. If the cosine of the largest angle is zero, then it is a right angle. Once the largest angle is found, the other two will be acute.

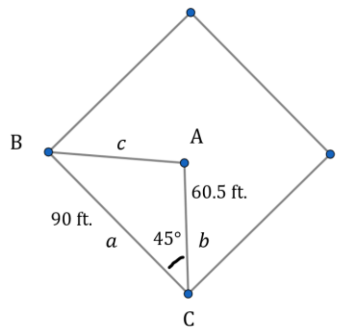
If the largest angle is found using the Law of Sines instead, then the Pythagorean Theorem must be used to determine whether the angle is acute or obtuse, which creates an extra step. This is because the sine is positive for both an acute first quadrant angle and an obtuse second quadrant angle.

<u>Looking Ahead 8.2</u>

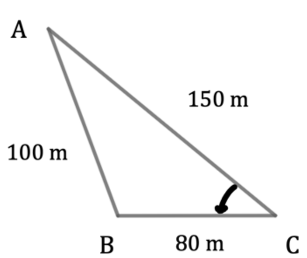
Example 1: In triangle DEF, $e = 8$ and $f = 10$. Find the length of d. Let d, e, and f replace a, b, and c in the formula for the Law of Cosines.



Example 2: A baseball diamond is a square that is 90 feet on each side. The pitcher's mound is 60.5 feet from home plate. How far does a pitcher throw if he catches a ball on the mound and throws it to the third baseman (who is on the base)?



Example 3: A triangular-shaped pool at a resort has side lengths of 80 m, 100 m, and 150 m. Find the angle of C rounded to the nearest tenth degree.



Section 8.3 Fundamental Trigonometric Identities

Looking Back 8.3

The trigonometric identities on the final problems of the previous section are called the Pythagorean Identities:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta$$

$$\sin^2\theta = 1 - \cos^2\theta$$

There are two others that are similar:

$$1 + \tan^2\theta = \sec^2\theta$$

And

$$1 + \cot^2\theta = \csc^2\theta$$

The reciprocal identities can be used in conjunction with these to simplify trigonometric identities.

$\sin\theta = \frac{1}{\csc\theta}$	$\sin\theta\csc\theta = 1$	$\csc\theta = \frac{1}{\sin\theta}$
$\cos\theta = \frac{1}{\sec\theta}$	$\cos\theta\sec\theta = 1$	$\sec\theta = \frac{1}{\cos\theta}$
$\tan\theta = \frac{1}{\cot\theta}$	$\tan\theta\cot\theta = 1$	$\cot\theta = \frac{1}{\tan\theta}$
$\tan\theta = \frac{\sin\theta}{\cos\theta}$		$\cot\theta = \frac{\cos\theta}{\sin\theta}$

There are also <u>Co</u>function Identities. Sine and <u>Co</u>sine are <u>co</u>mplements and <u>co</u>functions.

$\sin(\frac{\pi}{2} - x) = \cos x$		$\cos(\frac{\pi}{2} - x) = \sin x$
Or	And	Or
$\sin(90° - x) = \cos x$		$\cos(90° - x) = \sin x$

Complementary angles add up to 90°. If $m\angle A = 60°$ then the complement of angle A is $(90° - 60°)$. If $m\angle A = x°$ then the complement of angle A is $90 - x°$.

Example 1: Demonstrate that the value of cosine is equal to the value of the complement of sine.

Tangent and Cotangent are complements and cofunctions.

$\tan(\frac{\pi}{2} - x) = \cot x$ And $\cot(\frac{\pi}{2} - x) = \tan x$

Or Or

$\tan(90° - x) = \cot x$ $\cot(90° - x) = \tan x$

Secant and Cosecant are complements and cofunctions.

$\sec(\frac{\pi}{2} - x) = \csc x$ And $\csc(\frac{\pi}{2} - x) = \sec x$

Or Or

$\sec(90° - x) = \csc x$ $\csc(90° - x) = \sec x$

Looking Ahead 8.3

Example 2: Simplify sec θ so it is in terms of sin θ. To do so, use identities to substitute. Look for factoring and combining like terms and multiplying each side of an equation by an identity equal to 1 in order to substitute.

Example 3:	Prove that $\sec\theta\csc\theta = \tan\theta + \cot\theta$. Write them both in terms of $\sin\theta$ and $\cos\theta$.

Example 4:	Prove that $\csc\theta - \cos\theta\cot\theta = \sin\theta$.

Remember, cosine and sine are defined for all real numbers. There are restrictions on the domain of some functions when n is an integer:

$\tan x$ and $\sec x$: not defined for $\frac{\pi}{2} + \pi n$

$\cot x$ and $\csc x$: not defined for πn

So, we are proving these identities for all real numbers except where they are not defined when n is an integer.

Section 8.4 Sum and Difference Formulas

Looking Back 8.4

There are several addition and subtraction identities for trigonometric functions. Remember, a subtraction problem can also be written as an addition problem, and sums can be calculated. We will call these the sum and difference formulas. These formulas allow you to find exact values for non-familiar sines, cosines, and tangents that are not common on unit circle.

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$$

These can be proven using the unit circle's central angles, the distance formula, and the Pythagorean Identities. You might assume that $\sin(\alpha+\beta) = \sin\alpha + \sin\beta$; however, this is not true.

$$\sin 75° \approx 0.965926$$

$$\sin 75° = \sin(30° + 45°)$$

However, $\sin 30° + \sin 45° = \frac{1}{2} + \frac{\sqrt{2}}{2} = \frac{1+\sqrt{2}}{2} \approx 1.20711$

Because $0.965925 \neq 1.20711$, $\sin 75° \neq \sin 30° + \sin 45°$

Looking Ahead 8.4

Example 1: Find the exact value of $\sin 75°$ using the sum formula for sine. This can be found using the values known from the unit circle for angles of $30°$ and $45°$.

Example 2:	Find the exact value of $\cos 105°$. Use the difference formula for cosine.

Example 3:	Find the exact value of $\tan 225°$. Use the sum formula for tangent.

This one should be familiar to you because it is on the unit circle, but it is important that you see the formula also works for what you already have learned using another method.

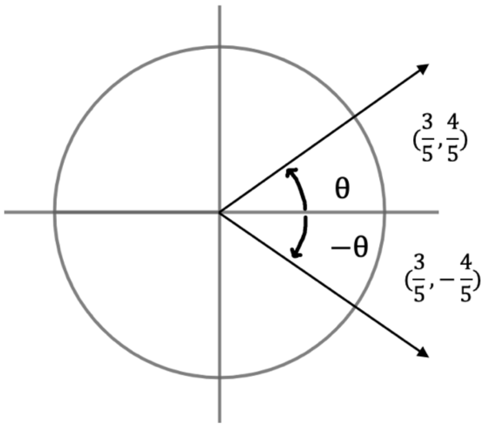
Example 4:	What is $\cos(90°)$ and $\cos(-90°)$?

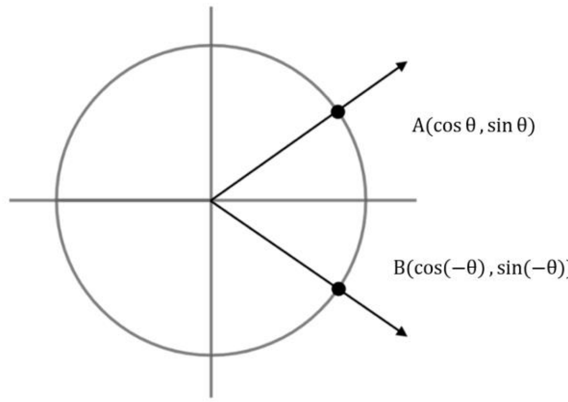
Section 8.5 More Trigonometric Theorems

Looking Back 8.5

Let us investigate other trigonometric theorems, which are useful for simplifying trigonometric expressions or solving trigonometric equations.

Firstly, we will explore the Theorem of Opposites.





Notice that for θ and $-\theta$ the x-coordinates ($\cos\theta$) are the same. Therefore, $\cos(-\theta) = \cos\theta$.

Also notice that for θ and $-\theta$, the y-coordinates ($\sin\theta$) are opposites. Therefore, $\sin(-\theta) = -\sin\theta$.

Because we know that $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)}$.

Using the substitution property, $\tan(-\theta) = \frac{-\sin\theta}{\cos\theta}$, $\tan(-\theta) = -(\frac{\sin\theta}{\cos\theta})$, and $\tan(-\theta) = -\tan\theta$.

For even functions, $f(-x) = f(x)$ for all numbers x and are symmetric about the y-axis. Therefore, $\cos\theta$ is an even function.

For odd functions, $f(-x) = -f(x)$ for all numbers x and are symmetric about the origin. Therefore, $\sin\theta$ and $\tan\theta$ are odd functions. You can see this in the graphs above.

You have already learned about the Complements Theorem (which states that if two angles are complementary to the same angle then the two angles are congruent) and Cofunctions (which are functions that are equal on complementary angles). Here are three more theorems related to these:

For every θ…

$\sin(\pi - \theta) = \sin\theta$ or $\sin(180° - \theta) = \sin\theta$

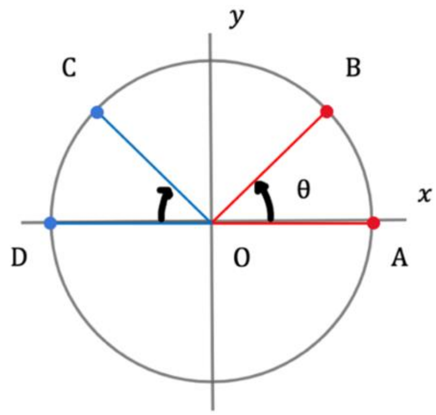
And

$\cos(\pi - \theta) = -\cos\theta$ or $\cos(180° - \theta) = -\cos\theta$

And

$\tan(\pi - \theta) = -\tan\theta$ or $\tan(180° - \theta) = -\tan\theta$

Supplementary angles add up to $180°$. If $m\angle A = 120°$, then the supplement of angle A is $180° - 120°$.
If $m\angle A = x$, then the supplement of angle A is $180° - x$. Let us use the unit circle to prove these.



Point B is reflected over the y-axis to point C. So, $m\angle AOB = m\angle DOC$; $m\angle AOD = \pi = 180°$.

Because $m\angle AOC = m\angle AOD - m\angle DOC$, $m\angle AOC = \pi - \theta$. Therefore, C has coordiantes

$(\cos \pi - \theta, \sin \pi - \theta)$ using the terminal side of the angle; again, because $\mathrm{C} = \mathrm{C}$ by the reflexive property.

$$\cos(\pi - \theta) = -\cos\theta$$

And

$$\sin(\pi - \theta) = \sin\theta$$

Also

$$\tan(\pi - \theta) = \frac{\sin\theta}{-\cos\theta}$$

$$\tan(\pi - \theta) = -(\frac{\sin\theta}{\cos\theta})$$

$$\tan(\pi - \theta) = -\tan\theta$$

Supplementary angles have the same sines, but opposite cosines and tangents.

<u>Looking Ahead 8.5</u>

Example 1: Suppose $\sin\theta = 0.57$. Find the following without a calculator; use the previously learned theorems.

a) $\sin(-\theta)$

b) $\sin(\pi - \theta)$

Example 2:	Verify that $\sin(\pi - \frac{\pi}{2}) = \sin\frac{\pi}{2}$.

Example 3:	Use a calculator to verify that $\cos(\pi - x) = -\cos x$.

Example 4:	Use the Pythagorean Identity to find $\cos\theta$ if $\sin\theta = \frac{4}{5}$.

Section 8.6 Double-Angle Formulas

Looking Back 8.6

To derive the Double-Angle Formula, use the Sum and Difference Formula derived in the previous practice problems section.

Sine of twice an angle or double an angle is $\sin 2\theta = \sin(\theta + \theta)$.

Using the Sum Formula for $\sin(\alpha + \beta)$…

$$\sin(\theta + \theta) = \sin\theta\cos\theta + \cos\theta\ \sin\theta$$

$$= \sin\theta\cos\theta + \sin\theta\cos\theta$$

$$= 2\sin\theta\cos\theta$$

Cosine of twice an angle or double an angle is $\cos 2\theta = \cos(\theta + \theta)$.

Using the Sum Formula for $\cos(\alpha + \beta)$…

$$\cos(\theta + \theta) = \cos\theta\ \cos\theta - \sin\theta\sin\theta$$

$$= \cos^2\theta - \sin^2\theta$$

Two other formulas can be derived for Cosine Double-Angle Formulas using the Pythagorean Identities.

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos 2\theta = (1 - \sin^2\theta) - \sin^2\theta$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos 2\theta = \cos^2\theta - (1 - \cos^2\theta)$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

The Double-Angle Formula for Tangent is $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$.

This will be derived in the next section once the Half-Angle Formulas are learned.

Looking Ahead 8.6

Example 1:	If $\cos\theta = -\frac{4}{5}$, find $\sin 2\theta$ using the Double-Angle Formulas.

Example 2:	If $\cos\theta = -\frac{4}{5}$, find $\cos 2\theta$ using the Double-Angle Formulas.

Hipparchus of Nicaea lived around 190-120 B.C. He was an astronomer who used tables along with the Half-Angle Theorem and the Supplements Angle Theorem to calculate the length of seasons and the length of the year.

Hipparchus of Nicaea is considered the Father of Trigonometry because he introduced a table of chords as a method of solving triangles. He also introduced latitude and longitude and a circle of $360°$.

Hipparchus of Nicaea had a great influence on Claudius Ptolemy (a Greek mathematician and astronomer, among other things) who considered him a lover of truth, as he observed a geocentric system of planets rotating around the sun. Hipparchus observed the motion was not circular. He gives an account of the rising and setting of constellations and a list of bright night stars to determine time.

Claudius Ptolemy's estimated life span is from 90 A.D. to 168 A.D., and he lived in Alexandria. He used Hipparchus' tables and geometric models to calculate the position of planets. Like Hipparchus, very little is known about Ptolemy's life. The following is one of Ptolemy's epigrams:

"Well, do I know that I am mortal,

A creature of one day.

But if my mind follows the winding path of the stars

Then my feet no longer rest on earth,

But standing by Zeus himself I

Take my fill of ambrosia,

The divine dish."

(Zeus is the god of the sky and thunder in ancient Greek mythology)

Many years later, Johannes Kepler (a German mathematician and astrologer) stated three laws, the first of which describes how the planets orbit around the sun in an elliptical pattern. This is called the Law of Ellipses.

The second law is called the Law of Equal Area and states that a line drawn to the sun sweeps out equal areas of space in equal amounts of time. A planet moves fastest when closer to the sun and slowest when farther away from the sun.

The third law is called the Law of Harmonics and states that the ratio of the square of the time required for a planet to orbit the sun (period) to the cubes of the average distances from the sun is the same for every planet.

$$\mathrm{K} = \frac{s^2}{m^3}$$

(where s is the period, m is the average distance, and K is Kepler's constant)

The precision of these laws and their application in nature was no surprise to Kepler because he knew we have a precise Maker.

Kepler argued that because the Son of God was the center of the Christian faith, the sun ought to be the center of the universe. He believed in the systematically heliocentric astronomy of the planets; however, he did not place his faith in astrology, but in a precise Creator.

Section 8.7 Half-Angle Formulas

Looking Back 8.7

To derive the Half-Angle Formulas, use the Double-Angle Formulas we learned in the previous section.

Two Double-Angle Formulas for cosine can be used to derive these new formulas:

$\cos 2\theta = 1 - 2\sin^2\theta$ and $\cos 2\theta = 2\cos^2\theta - 1$

Replace θ with $(\frac{\theta}{2})$ in these two new formulas:

$$\cos 2(\tfrac{\theta}{2}) = 1 - 2\sin^2(\tfrac{\theta}{2}) \quad \text{and} \quad \cos 2(\tfrac{\theta}{2}) = 2\cos^2\left(\tfrac{\theta}{2}\right) - 1$$

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2} \qquad \cos\theta = 2\cos^2\frac{\theta}{2} - 1$$

$$2\sin^2\frac{\theta}{2} = 1 - \cos\theta \qquad -2\cos^2\frac{\theta}{2} = -1 - \cos\theta$$

$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2} \qquad \cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$$

$$\sqrt{(\sin\tfrac{\theta}{2})^2} = \pm\sqrt{\frac{1-\cos\theta}{2}} \qquad \sqrt{(\cos\tfrac{\theta}{2})^2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} \qquad \cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

Both of these have plus and minus signs. The quadrant in which $\frac{\theta}{2}$ is located will determine which sign to use.

The Half-Angle Formulas for sine and cosine can be used to derive the Half-Angle Formula for tangent because $\tan\theta = \frac{\sin\theta}{\cos\theta}$…

$$\tan\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \frac{\pm\sqrt{\frac{1+\cos\theta}{2}}}{\pm\sqrt{\frac{1-\cos\theta}{2}}} = \pm\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}$$

Because $2\cos^2\theta = 1 + \cos 2\theta$ and $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2}{2}\cdot\frac{\sin\theta}{\cos\theta}\cdot\frac{\cos\theta}{\cos\theta} = \frac{2\sin\theta\cos\theta}{2\cos^2\theta} = \frac{\sin 2\theta}{1+\cos 2\theta}$

Replacing θ with $\frac{\theta}{2}$ gives us $\frac{\sin 2(\frac{\theta}{2})}{1+\cos 2(\frac{\theta}{2})} = \frac{\sin\theta}{1+\cos\theta} = \tan\frac{\theta}{2}$

There is one more Half-Angle Tangent Formula, which will be derived in the practice problems section.

Looking Ahead 8.7

Example 1:	Find the exact value of $\sin\frac{5\pi}{8}$ and $\cos\frac{5\pi}{8}$.

Example 2:	Let $180° < \theta < 360°$ and $\cos\theta = -\frac{11}{13}$; find $\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2}$.

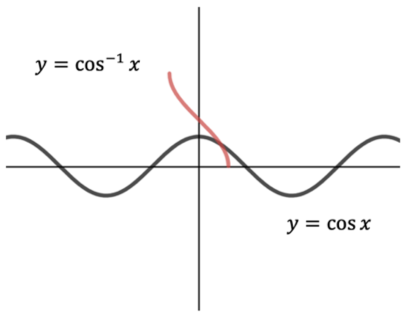
Example 3:	Find the exact value of $\tan\frac{7\pi}{12}$.

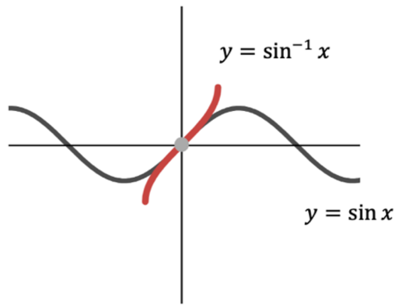
Section 8.8 Restrictions on the Trigonometric Functions

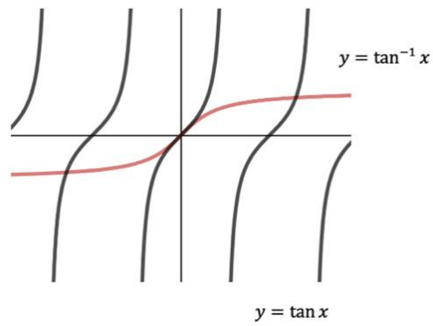
Looking Back 8.8

Observing the sine, cosine, and tangent function, one sees that the inverse (a reflection over the line of symmetry) yields graphs that are not functions.

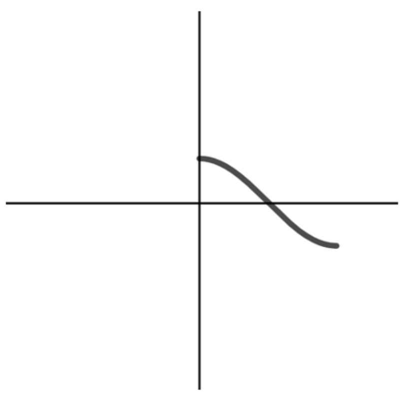
Much like the square root graph, which is the inverse of the quadratic function- in order for the inverse to be a function, the domain must be restricted.





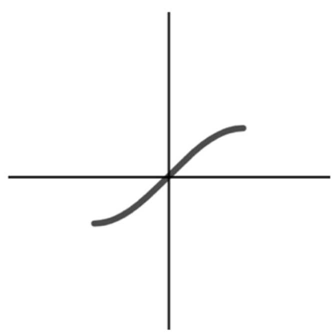


When restricting the domain, as the graphing calculator does, the angles between 0 and $\frac{\pi}{2}$ are included because they are all acute angles in a right triangle. The values of the range that the domain takes are included, and the function is still continuous.



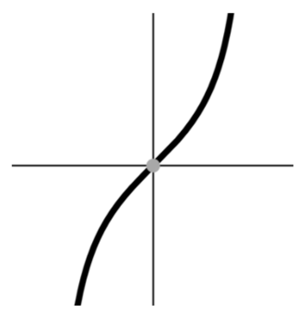
$y = \cos x$

Restrictions: $0 \le x \le \pi$



$y = \sin x$

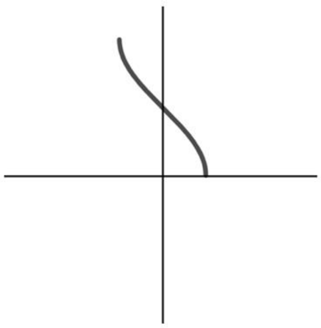
Restrictions: $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$



$y = \tan x$

Restrictions: $-\frac{\pi}{2} < x < \frac{\pi}{2}$

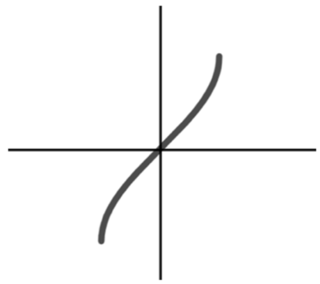
These inverse trigonometric or inverse periodic functions can occur when the circular functions with restrictions are reflected through the line of symmetry, $y = x$.



$y = \cos^{-1} x$

Domain: $-1 \leq x \leq 1$

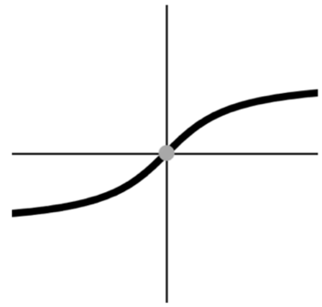
Range: $0 \leq y \leq \pi$



$y = \sin^{-1} x$

Domain: $-1 \leq x \leq 1$

Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$



$y = \tan^{-1} x$

Domain: all real numbers

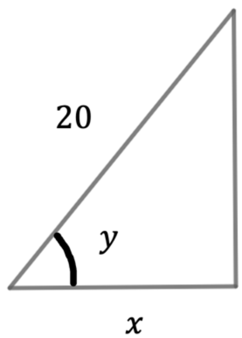
Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$

These periodic functions must be evaluated within these restrictions. Remember, these functions involve reverse thinking. If $x = \cos^{-1}(\frac{1}{2})$, then $\cos x = \frac{1}{2}$. This is the original function.

<u>Looking Ahead 8.8</u>

Example 1: Evaluate $\cos^{-1}(\frac{1}{2}) = y$.

Example 2: A ladder that is 20 feet long leans against a garage. Let y be the angle the ladder makes with the ground and x be the distance from the bottom of the ladder to the garage.



a) Express x as a function of y.

b) What are the domain restrictions and the range of the function?

In this section, we are reviewing the restrictions that will help us solve trigonometric equations in the next section. Then, for the remainder of the module, we will study conic sections (this can be called Analytic Geometry); circles, parabolas, ellipses, and hyperbolas will be studied algebraically as functions.

Section 8.9 Solving Trigonometric Equations

Looking Back 8.9

Trigonometric equations, unlike trigonometric expressions, have equal signs. These can be further simplified by combining like terms after simplifying using trigonometric identities (from the previous sections). However, we are still not done. Trigonometric equations can be even further simplified for the variable using the domain restrictions applied in the last section. The inverse trigonometric functions are also used to solve trigonometric functions.

Example 1: Find the solution of $\cos\theta = 0.8$.

Looking Ahead 8.9

Example 2: Using the Opposites Theorem, find another solution of $\cos\theta = 0.8$ from Example 1, this time between 0 and 2π $(0 \leq \theta \leq 2\pi)$.

Example 3: Find the general solution for $\cos\theta = 0.8$.

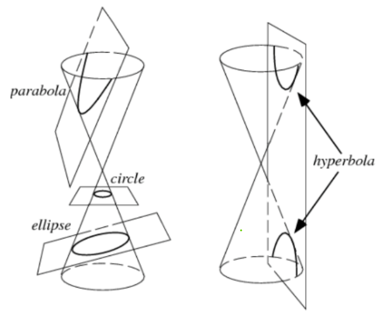
The general solutions include all real solutions. The periodicity for cosine is 2π. Adding or subtracting multiples of 2π gives all solutions.

Example 4: The equation for the electrical voltage E of a circuit is $E = 10\ \cos 2\pi t$ where t is the time in seconds and $t > 0$. At what time does E equal 8 during the first two seconds?

Section 8.10 Conic Sections and Circles

Looking Back 8.10

You have studied about circles earlier in Geometry. Analytic Geometry involves the study of cross sections of two 3-dimensional cones and the shapes or figures that are created when intersected by a plane. Analytic Geometry uses algebra to explore and analyze these intersections or cross sections. These cross sections are called conic sections. The four conic sections we will investigate are the circle, the parabola, the ellipse, and the hyperbola (shown in the diagram below).



We will first watch a brief video for how conic sections are formed. Then we will review the formulas from Analytic Geometry that apply to the circle and other conic sections.

The Distance Formula

The distance between any two points A(x_1, y_1) and B(x_2, y_2) is AB $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

The Midpoint Formula

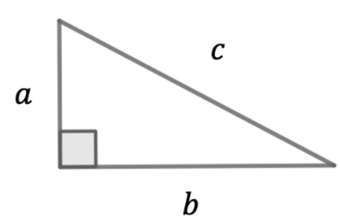
The midpoint M of the line segment AB has coordinates $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$.

The first formula incorporates the Pythagorean Theorem.

Pythagorean Theorem

The sum of the of the squares of the legs of a right triangle is equal to the square of the hypotenuse.

$$a^2 + b^2 = c^2$$



<u>Looking Ahead 8.10</u>

The first cross section we will review is the circle that is created when a plane slices through a 3-dimensional cone parallel to the base of the cone.

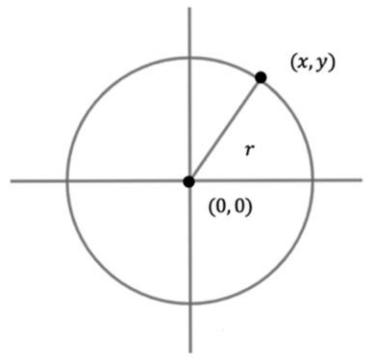
Use the distance formula to find the equation of a circle with center $(0, 0)$ and radius r.

$$\sqrt{(x-0)^2+(y-0)^2}=r$$

$$\sqrt{(x)^2+(y)^2}=r$$

$$(\sqrt{(x^2+y^2)})^2=r^2$$

$$x^2+y^2=r^2$$



This is the parent function of a circle. You learned about transformations of equations in Algebra 2. The h represents a horizontal translation on the x-axis and the k represents a vertical translation on the y-axis.

<u>Equation of a Circle</u>

The equation of a circle with center (h, k) and radius r is as follows:

$$(x-h)^2+(y-k)^2=r^2$$

Example 1: Find the equation of a circle that has center $(1, -5)$ and a radius of 4.

Example 2: Given the equation $(x+5)^2+(y-3)^2=9$, sketch the graph of the circle.

Example 3: Find the equation of a circle that has a radius of 2, has its center in the second quadrant, and is tangent to the x-axis at $(-4, 0)$. Sketch its graph.

We have studied Blaise Pascal (1623-1662) in Algebra 1. His love for geometry is probably what led to his interest in conic sections. Much of what we know today about conic sections is due to Pascal's work.

The construction of a cone begins with a circle, a fixed point not in the plane of the circle, and a straight line through the point that moves around the circle. This generates two cones, or conic surfaces, which stretch to infinity.

Pascal introduced a plane stretching to infinity and cut through the cone. He made the cut six different ways, which produced six different sections: a point, a straight line, an angle, a circle, or an ellipse (a circle is a special case of an ellipse), a parabola, and a hyperbola. We will explore the last three conic sections in the upcoming sections of this module.

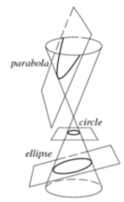
Not much of his work with conic sections survives today, but Pascal's theorems were used extensively by other mathematicians. We do have his work *Pensées*, in which he writes of his religious convictions and faith. In it he says, "Truth is so obscure in these times, and falsehood so established, that, unless we love the truth, we cannot know it."

Pascal pursued truth in mathematics, physics, and philosophy. His theorems are propositions, statements, or formulas; he used these theorems to prove other theorems. Like many mathematicians, Pascal built on the work of others (such as Euclid, Ptolemy, and Hipparchus), and other mathematicians have built on Pascal's work.

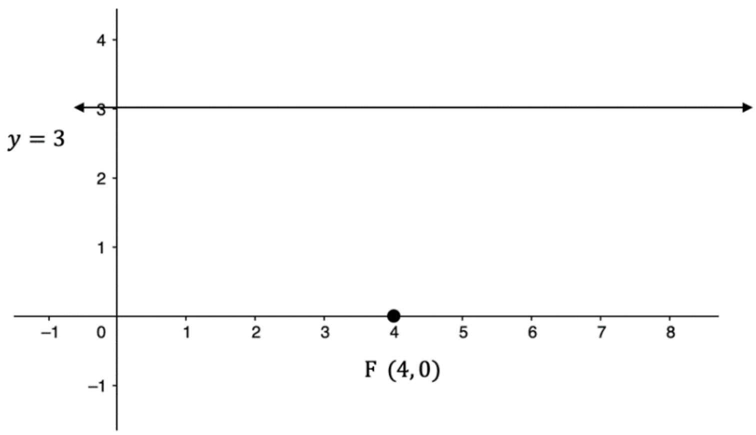
Section 8.11 Conic Sections and Parabolas

Looking Back 8.11

The next conic section we will study is the parabola, which forms when a plane slices through one of two 3-dimensional cones but is neither parallel nor perpendicular to the base.



We have studied parabolas in Algebra 2 when we explored quadratic equations. A parabola is the set of all points equidistant from a fixed line- called a directrix, and a fixed point- called a focus (which is not on the line).



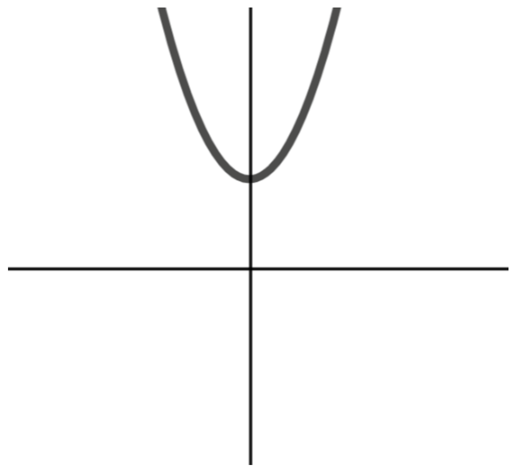
If we draw a line from the points D_1, D_2, and D_3, on the directrix, perpendicular to the x-axis, and draw lines from a focus $(4, 0)$ the same distance to the lines from the directrix, a parabola is formed. Let a point D_3 on the directrix be $(x, 3)$ and let another point on the parabola be $A(x, y)$. According to the definition of a parabola, the distance AF is equal to AD_3 . The vertex is equidistant to the directrix and the focus and is between them.

Example 1:	Use the distance formula with $AF = AD_3$ to find the equation of the parabola.

The form of this equation is $y = a(x - h)^2 + k$, which is the graphing form of an equation that you learned in Algebra 2 when quadratic equations were studied.

Looking Ahead 8.11

The vertex of the parabola is halfway between the directrix and the focus. The focus and vertex lie on the axis of symmetry. The parabola opens towards the focus.



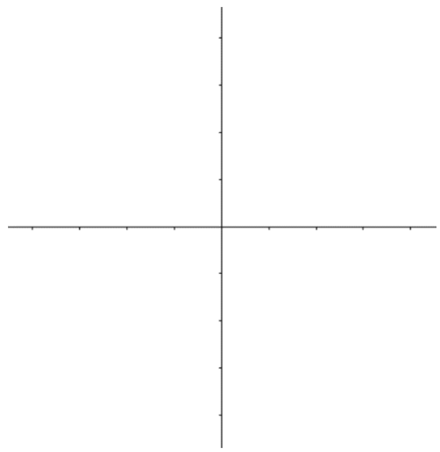
Example 2: A parabola has a vertex $(-3, -1)$ and the directrix is the line $y = 5$. What is the focus of this parabola?

Example 3: Let the directrix be $x = -2$ and the focus be $\mathrm{F}(0, -6)$. Draw the parabola. Label the vertex V, the focus F, and the directrix D, and then find the axis of symmetry. Find the equation of the parabola.

Let any point not on the line be $\mathrm{A}(x, y)$. On the parabola, let $\mathrm{AF} = \mathrm{AD}$. The vertex goes through the axis of symmetry. It is halfway between the directrix and focus. The axis of symmetry is $y = -6$ and the vertex is $\mathrm{V}(-1, -6)$.

$\text{AF} = \text{AD}$ and AD is the perpendicular distance from point A to the directrix.

Let point D on the directrix be $\text{D}(-2, \text{y})$.



This is the form $x = a(y-k)^2 + h$ with vertex (h,k) and the parabola opens right or left. The parabola of the form $y = a(x-h)^2 + k$ has vertex (h,k) as well, but the parabola opens up or down.

Let the distance between the focus and the vertex be some constant $|c|$. Then $a = \frac{1}{4c}$.

In the equation $x = a(y-k)^2 + h$, the parabola opens right if $a > 0$, and left if $a < 0$. The focus is $\text{F}(h+c, k)$ and the directrix is $x = h - c$. The axis of symmetry is $y = k$. The vertex is $\text{V}(h,k)$.

In the equation $y = a(x-h)^2 + k$, the parabola opens upward if $a > 0$, and downward if $a < 0$. The focus is $\text{F}(h, k+c)$ and the directrix is $y = k - c$. The axis of symmetry is $x = h$. The vertex is $\text{V}(h,k)$.

Example 4: Given the equation $x = -\frac{1}{4}(y+2)^2 - 3$, find the vertex $\text{V}(h,k)$ and $|c|$ (the distance between the vertex and focus). Find the focus, directrix, and the axis of symmetry. Tell whether the graph opens right, left, up, or down.

Archimedes was another mathematician we learned about in Algebra 1. He found the area of a section of a parabola by adding the areas of a sequence of triangles that he made smaller and smaller to completely fill the section more and more each time.

If b is the base of the parabolic section and h is the perpendicular line from the base to a tangent line on the parabola, and the tangent line is parallel to the base, then the area of the section is $\text{A}_s = \frac{2}{3}bh$.

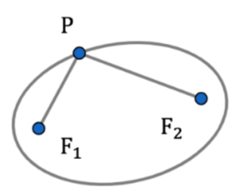
Section 8.12 Ellipses and Their Transformations

Looking Back 8.12

Another conic section is an ellipse, which is formed when a plane slices through a 2-dimensional cone and is neither perpendicular nor parallel to the base of the cone, but does cut through opposite parts of the cone at an angle.

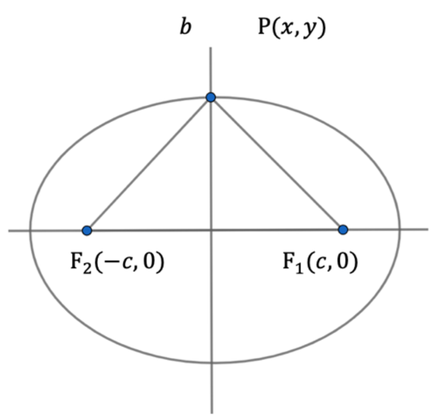
An ellipse is the set of all points in a plane such that the sum of the distance from two fixed points to any point on the ellipse is a given constant.

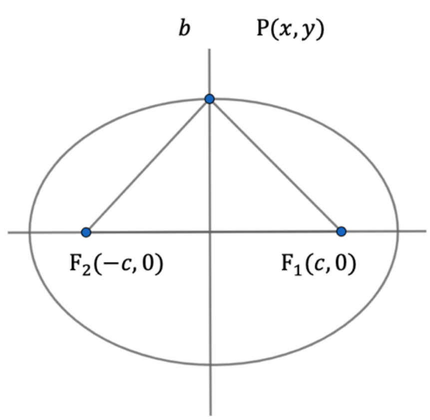
Each fixed point is a focus; for example, F_1 and F_2 in the diagram below. The plural for focus is foci so both points together are called foci. From any point P on the ellipse, the focal radii (PF_1 and PF_2) have a sum that stays constant. The distance $PF_1 + PF_2$ is called the focal constant.



Example 1: Let $P(x, y)$ be some point on the ellipse. Let $k = 2a$, $F_1 = (c, 0)$ and $F_2 = (-c, 0)$. By definition, $PF_1 + PF_2 = k$, so $PF_1 + PF_2 = 2a$.

Use the distance formula and the Addition Property of Equality so that square roots are on both sides of the equation. Note that $k \geq F_1F_2$.




b
P(x, y)
F₂(−c, 0)
F₁(c, 0)

Looking Ahead 8.12

Example 2: Find a and b, and the foci ($(-c, 0)$ and $(c, 0)$), and any extreme points on the ellipse. Sketch the graph of the ellipse given the following equation:

$$\frac{x^2}{36} + \frac{y^2}{16} = 1$$

If $a^2 > b^2$ then the major axis is horizontal and if $b^2 > a^2$ then the major axis is vertical.

Example 3: Write the equation for an ellipse that has $a = 9$ and $b = 5$. Sketch the graph.

Example 4: Graph the ellipse given the following equation:

$$16x^2 + 4y^2 = 64$$

Example 5: Find the horizontal extreme points and sketch the horizontal and vertical shifts of the ellipse given the following equation:

$$\frac{(x-2)^2}{49}+\frac{(y+4)^2}{9}=1$$

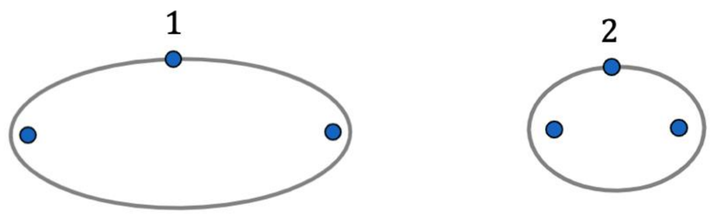
The center $(0,0)$ now becomes (h,k). If $a^2>b^2$ then the foci are $(h-c,k)$ and $(h+c,k)$. If $b^2>a^2$ then the foci are $(h,k-c)$ and $(h,k+c)$.

Example 6: Find the foci and extreme points and sketch the horizontal and vertical shifts of the ellipse given the following equation:

$$\frac{(x-2)^2}{9}+\frac{(y+4)^2}{49}=1$$

Johann Kepler said the sun is one focus of the ellipse created by the orbits of the planets. Kepler built on the work of the Greek mathematicians Ptolemy and Hipparchus, who lived about 2,000 years before him. Newton confirmed Kepler's work, saying he stood on the shoulders of giants (speaking of himself and Ptolemy, Hipparchus, and Kepler, among others, as the giants).

The equation of an ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and its eccentricity, defined as $e=\frac{c}{a}$ for $c=\sqrt{a^2-b^2}$, gives us an idea of the shape of the ellipse. The greater the eccentricity, the more elongated the shape of the ellipse: $e_1 > e_2$ below.



The eccentricity, e, is between 0 and 1 because $0 < \sqrt{a^2+b^2} < a$, $0 < c < a$, and $0 < \frac{c}{a} < 1$. Therefore, $0 < e < 1$.

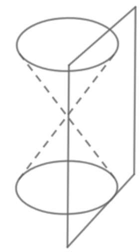
A planet's mean distance from the sun is half the length of the major axis of the orbit. This is the average of the planet's minimum and maximum distance from the sun measured in astronomical units (au). For Pluto, $a = 39.44$, which is the mean distance from the sun. The eccentricity is $e = 0.250$.

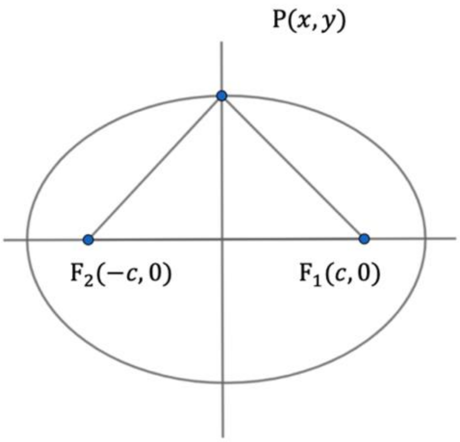
Example 7:	Find the equation for the orbit of Pluto.

Section 8.13 Hyperbolas and Their Transformations

Looking Back 8.13

The final conic section that we study is the hyperbola, which forms when a plane slices through two 3-dimensional cones perpendicular to the base of the cone.





Let us review the properties of ellipses before discussing hyperbolas. The ellipse with local foci $(-c, 0)$ and $(c, 0)$ and a focal constant of $2a$ has the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $b^2 = c^2 - a^2$. So $(a,0)$ and $(-a,0)$ are on the ellipse and $|x| \le a$. The points (0, b) and $(0, -b)$ are also on the ellipse and $|y| \le b$. That means a and b would be the extreme points of the ellipse. By definition $PF_1 + PF_2 = k$, so $PF_1 + PF_2 = 2a$ since $k = 2a$. In the diagram:

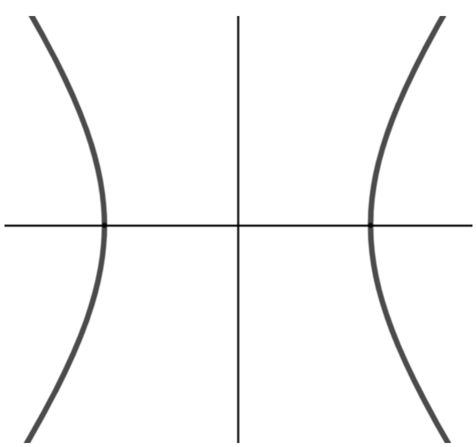
$$PF_1 = PF_2$$

$$PF_1 + PF_1 = 2a$$

$$2PF_1 = 2a$$

$$PF_1 = a$$

By the Pythagorean Theorem, $b^2 = c^2 - a^2$. The segment A_1A_2 is longer and called the major axis. The segment B_1B_2 is shorter and called the minor axis. The longer or major axis always contains the foci. If $a = b$, then the ellipse is a circle. The two axes meet at the center of the circle. They lie on the lines of symmetry. Therefore, $2a$ is the horizontal axis and $2b$ is the vertical axis. For $c^2 = |a^2 - b^2|$, the distance of the foci is $2c$.



The local constant k is less than c and is a positive real number. Hyperbolas have the same symmetries as ellipses though they are unbounded.

A hyperbola is the set of all points P in a plane, such that the differences from the two fixed points (foci) from point P is a given constant k. The foci are F_1 and F_2 and $|PF_1 - PF_2| = k$. The focal constant $k \le F_1F_2$ and is a positive real number.

The hyperbola with local foci $(-c, 0)$ and $(c, 0)$ and a focal constant of $2a$ has the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $b^2 = c^2 - a^2$.

The proof is similar to that for the ellipse in standard form: $|\mathrm{PF}_1 - \mathrm{PF}_2| = k$ and $\mathrm{P}(x, y)$ is a point on the hyperbola. The foci are $\mathrm{F}_1(-c, 0)$ and $\mathrm{F}_2(c, 0)$. Because $k = 2a$, then $|\mathrm{PF}_1 - \mathrm{PF}_2| = 2a$.

Example 1: Use the distance formula to find the general formula for a hyperbola.

The last step differs from an ellipse in that it is a difference rather than a sum.

Looking Ahead 8.13

Example 2: Find the equation for a hyperbola using the distance formula and foci $\mathrm{F}_1(-4, 0)$ and $\mathrm{F}_2(4, 0)$ with a focal radii of 6. That means the constant is $k = 6$ or $2a = 6$.

If a hyperbola with center $(0,0)$ has foci $(-c,0)$ and $(c,0)$ and a difference of the focal radii is $2a$, then the equation is as follows:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

(where $b^2=c^2-a^2$)

The hyperbola has vertices at $(-a,0)$ and $(a,0)$ on the x-axis, and $|x|\geq a$. No part of the graph lies between $(-a,a)$ on the x-axis. The graph is unbounded on the y-axis. The equations of the asymptotes are

$$y=\frac{b}{a}x \text{ and } \quad y=-\frac{b}{a}x.$$

If the center of the hyperbola is $(0,0)$ but the foci are $(0,-c)$ and $(0,c)$, then the equation is as follows:

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$$

(where $b^2=c^2-a^2$)

The hyperbola has vertices at $(0,-a)$ and $(0,a)$ on the y-axis and $|y|\geq a$. No part of the graph lies between $(-a,a)$ on the y-axis. The graph is unbounded on the x-axis. The equations of the asymptotes are

$$y=\frac{a}{b}x \text{ and } y=-\frac{a}{b}x.$$

Example 3:	Sketch a graph of the hyperbola with the equation $\frac{y^2}{9}-\frac{x^2}{16}=1$.

Example 4:	Find an equation for a hyperbola with foci $(4,0)$ and $(-4,0)$ and focal radii that equal 4.

Hyperbolas that have a center other than $(0, 0)$ have been transformed. Again, the equations for hyperbolas with center (h, k) are similar to that of ellipses, but with a minus sign in the equation rather than a plus sign.

Horizontal Major Axis:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Foci at $(h-c, k)$ and $(h+c, k)$ where $c^2 = a^2 + b^2$

Vertical Major Axis:

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

Foci at $(h, k-c)$ and $(h, k+c)$ where $c^2 = a^2 + b^2$

Example 5: Find the equation of a hyperbola with foci $(3, -8)$ and $(3, -2)$ and difference of focal radii of 4.

Example 6: Given the hyperbola $\frac{(x+6)^2}{9} - \frac{(y+1)^2}{16} = 1$, find the local foci and the center.