Geometry and Trigonometry Module 6 Triangles

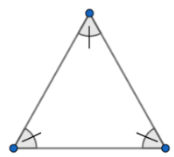
Section 6.1 Introduction to Triangles

Looking Back 6.1

The definition of a triangle is ‘a three-sided polygon that has three angles’ (the word ‘tri’ is derived from the Latin and Greek and means ‘three’). Each pair of sides form an angle. Each side of a triangle is a segment. Two segments meet to form an angle. Since there are three angles in a triangle there are also three vertices in a triangle (‘vertices’ is plural for ‘vertex’).

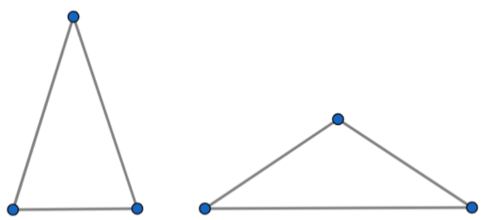
Triangles can be named by their sides or by their angles. Firstly, we will investigate the sides of a triangle.

An equilateral triangle has all three sides equal and is equiangular, which means it has all angles equal. (The Latin word ‘aequs’ means ‘even or level;’ the prefix ‘equi’ is similar to ‘equal;’ the word ‘lateral’ means ‘side.’)

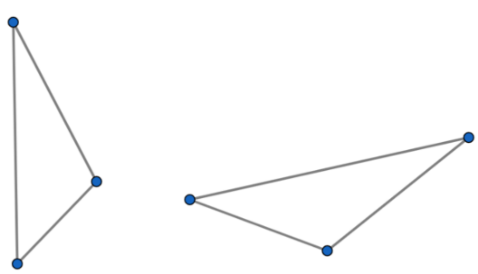


An isosceles triangle has two equal legs. (It comes from the Greek word ‘isos’ (equal) and ‘skelos’ (leg)). You have just learned that a right triangle has two legs that meet at the right angle and are perpendicular. The legs of an isosceles triangle do not have to meet at a 90° angle. (The Indo-European root of ‘isosceles’ means ‘bent’ because each leg is bent relative to the adjoining legs.)

If two sides are equal, what does that say about the third side? It is not equal to the other two, so it is either longer or shorter. The angle formed where the two legs meet is called the vertex angle. The other two angles formed are called base angles.

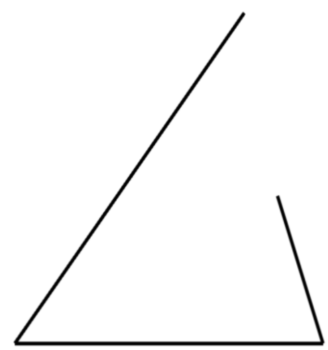


A scalene triangle has no sides equal. (It comes from the Indo-European root ‘skel’ (to cut) and the Greek root ‘shalenos’ (stirred up); when the ground is stirred up the surface becomes uneven.) In a scalene triangle, no two sides are equal.



When sides are equal, they are marked by tick marks. All sides with one tick mark are the same length. All sides with two tick marks are the same length, etc. (This is also the method used to designate equal angles.)

Draw line segments with measures of 3 inches, $1\frac{1}{2}$ inches, and 1 inch. Connect them to form a triangle. What do you notice?

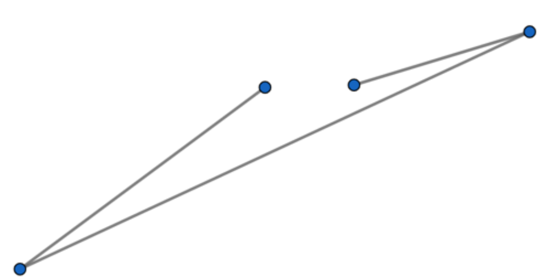


It cannot be done. The two shorter segments will be at either end of the longer segment. To get them to connect the angles must become smaller and smaller. Even if the angles are closed and the two shorter lengths lie on the longer line, they will not meet. They total $2\frac{1}{2}$ inches and the longer segment is 3 inches. As the angles get bigger the line segments would need to increase to meet and form a triangle. This is called the Triangle Inequality Theorem:

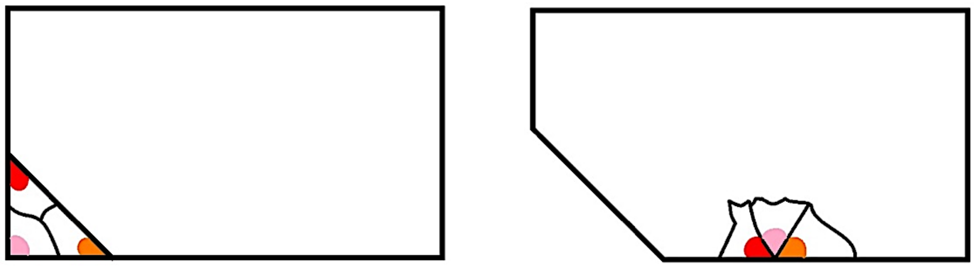
The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

There is a relationship there. To find a ratio, you could start with a 3-inch line and a $1\frac{1}{2}$-inch line and measure how long the third segment would be when the angles are $5°$, $10°$, then $15°$. You could even increase by one degree rather than five each time, but that would be difficult to calibrate and be precise.

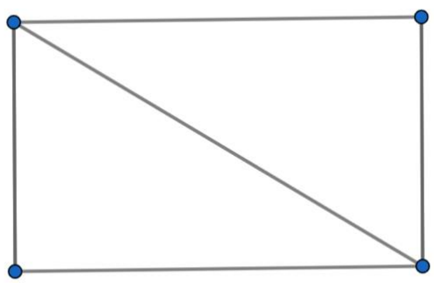
The purpose of this exploration is to demonstrate that not any three lines will form a triangle. It must be a closed, not open, polygon. We also see there is a relationship between the lengths of the sides and the angles. When a triangle has equal sides it also has equal angles. Will the angles always be $60°$ in an equilateral triangle?



The angles of a triangle add up to $180°$. Take an index card and draw a triangle in the corner. Draw lines to form three puzzle-type pieces in the small triangle. Make sure you color each angle of the triangle a different color before cutting the triangle in three pieces. Color the angles that have both rays touching the sides of the card as well. There will be one colored angle in each small puzzle piece. Cut out the three pieces of the small triangle and line up all the angles on the side of the index card to see how they fit along the line with no gaps or overlaps. All the colored angles should be adjacent to one another (touching) along the bottom of the index card. Since we know a straight angle is $180°$ and the three angles of a triangle line up along the straight angle with no gaps or overlaps, then we know the angles of a triangle have a sum of $180°$.



There is another way to demonstrate the angles of a triangle add up to a sum of $180°$. Since a quadrilateral (index card) is four $90°$ angles, the sum of the angles is $4 \cdot 90° = 360°$, the same as that of a circle. If you cut the card in half with a diagonal, you have two congruent triangles. The sum of the angles of each triangle is half the sum of the angles of the entire quadrilateral, or $\frac{1}{2}(360°) = 180°$.



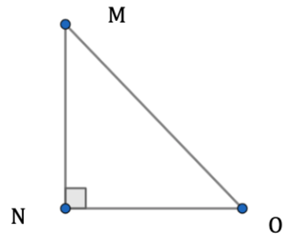
Now we know the angles of a triangle always add up to $180°$; this is called the Triangle Sum Theorem:

The sum of the measures of the angles in every triangle is $180°$.

Now, let us try naming a triangle by its angles. Just as there are three types of angles: acute, obtuse, and right, there are three types of triangles with the same names: acute, obtuse, and right.

You have used right triangles quite a bit when studying the Pythagorean Theorem, so let us begin with a right triangle.

A right triangle has one right angle. A right angle is formed by the two perpendicular legs that meet at a 90° angle.



$m\,\angle MNO = 90°$

Can a right triangle have two 90° angles? It is not possible because $90° + 90° = 180°$ and that leaves 0° for the third angle. Nor could one of the other two angles in a right triangle be obtuse because that angle plus the right angle would be more than 180°.

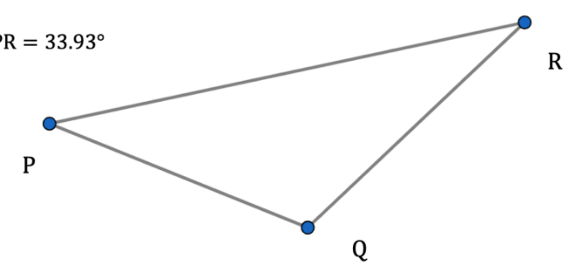
A right triangle has one right angle, and the other two angles must be acute. So, if you know the measurements of two angles in a triangle, you can find the measurement of the third by adding the two angles you know together and subtracting that sum from 180°.

How many obtuse angles do you think an obtuse triangle will have? There is exactly one obtuse angle in an obtuse triangle (for the same reasons there is only one 90° angle in a right triangle), making the other two angles acute again. Remember, an acute angle is greater than 0°, but less than 90° and obtuse angle is greater than 90°, but less than 180°.

An obtuse triangle has one obtuse angle and two acute angles.

$m\,\angle PRQ = 31.41°$

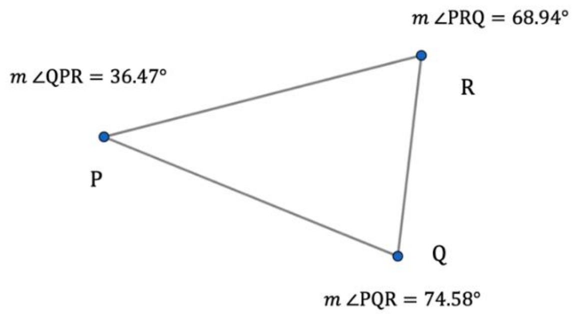
$m\,\angle QPR = 33.93°$



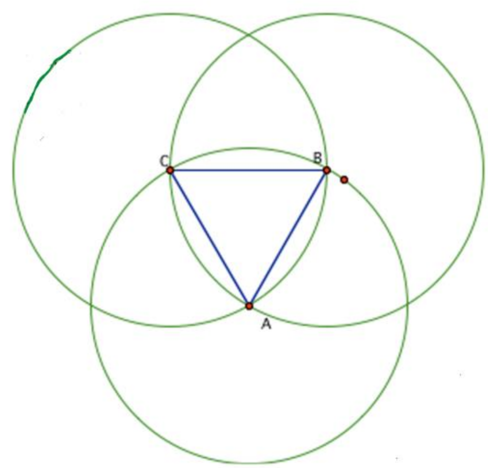
$m\,\angle PQR = 114.67°$

If a right triangle has one right angle and two acute angles, and an obtuse triangle has one obtuse angle and two acute angles, how many acute angles does an acute triangle have? Well, if it has one acute angle and the other two angles are also acute, then an acute triangle has three acute angles. If it has anything else it would not be an acute triangle, but a right triangle or an obtuse triangle.

An acute triangle has all three angles acute.



Let us revisit a question asked previously in this section: Are the angles of an equilateral triangle always $60°$? To construct an equilateral triangle, draw three concentric (overlapping) circles such that each radius is of the same measure. Notice that the radii are all the same length and each form a side of the triangle. Therefore, the three sides of the triangle are equal because they are all equal radii. This means it is an equilateral triangle. Notice also that the angles are all equal to $60°$.



$\text{AB} = 3.49$ cm.

$\text{AC} = 3.49$ cm.

$\text{CB} = 3.49$ cm.

$m\,\angle ACB = 60.00°$

$m\,\angle CBA = 60.00°$

$m\,\angle BAC = 60.00°$

Use your compass to construct several equilateral triangles. Try many different-sized circles so some radii are small, and some are large. Measure the lengths of the sides of the triangle. They should all be equal given they were constructed correctly. Now, measure the angles of the triangle. They should all be equal as well because $180°$ divided into three equal parts is $60°$. This is called the Equilateral and Equiangular Theorem:

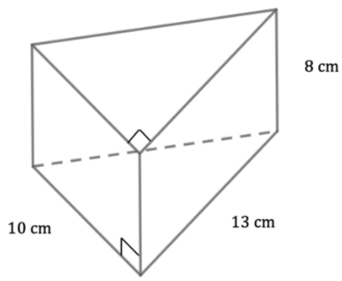
Every equilateral triangle is also equiangular and, conversely, every equiangular triangle is also equilateral. The angles of an equilateral triangle are equal to $60°$.

<u>Looking Ahead 6.1</u>

The area of a triangle is $\text{A} = \frac{1}{2} b \cdot h$ because any rectangle can be divided into two equal triangles by a diagonal (as previously shown).

The volume of a right triangular prism is $\text{V} = \text{BH}$ where B is the base of the triangle ($\frac{1}{2} b \cdot h$), and H is the height of the triangular prism.

Example 1: Find the volume of the right triangular prism.



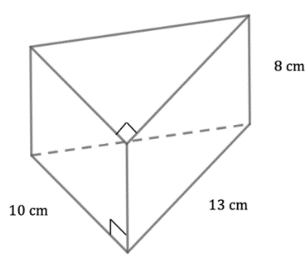
Do you remember when Archimedes sat in the bathtub and noticed how the water rose? This displacement was due to buoyancy created by the volume of his body. He was so excited by his discovery that he jumped from the tub and ran down the street shouting "Eureka! Eureka!" which means "I found it! I found it!" (What was not to be found were his forgotten clothes!)

When you make an ice cream soda you add ice cream (a solid) to cream soda (a liquid), and then the liquid rises. This rising level of liquid is called displacement and is caused by the volume of the object- in this case, the ice cream.

Density is the ratio of mass of matter to its volume. The mass of an object is found by weighing it. Density can be calculated by dividing the mass of an object by its volume:

$$\text{density} = \frac{\text{mass}}{\text{volume}} \qquad d = \frac{m}{v}$$

Example 2: Wyatt drops a piece of metal into the right triangular prism in Example 1 and notices the water level rises 3 centimeters. What is the volume of the amount of water displaced?



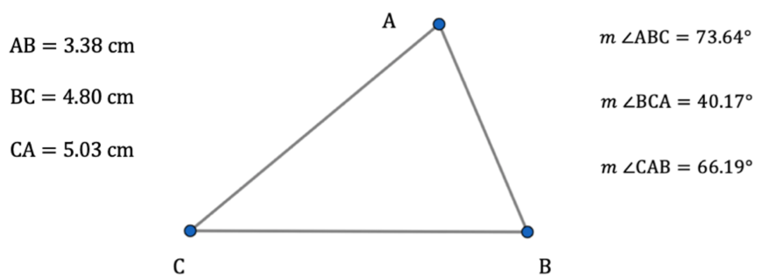
Example 3: The mass of the metal from Example 2 is 1,733.55 grams and looks like it is either lead (with a density of 11.30 g/cm^3) or nickel (with a density of 8.89 g/cm^3). Use the density formula to determine if the metal is nickel or lead.

Section 6.2 Congruence Postulates/Theorems

Looking Back 6.2

If I had a drawing of a triangle and I wanted you to duplicate it without seeing it, I would have to give you some information. There are two things that make a triangle: sides and angles. Therefore, I would definitely tell you the drawing is made up of sides and angles. However, let us suppose that is all I can tell you; I cannot tell you how the sides and angles are oriented.

What do you think is the minimum number of steps I would have to give you in order for you to duplicate this triangle drawing as quickly as possible?

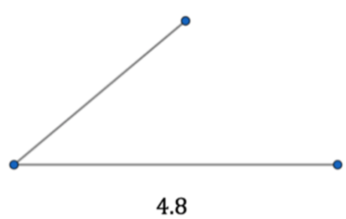


You can use your compass and protractor to draw the angle C (for this purpose, I am going to round angle C to 40° degrees rather than leave it 40.17° degrees). Assume you have not seen the triangle above; I am going to give you the first two steps to draw it:

Step 1: Draw angle C so it measures 40° degrees.

Step 2: Mark a point B on one side of angle C so segment BC measures 4.8 cm.

For Step 1, you must first draw a ray and then use your protractor to find the next ray. For Step 2, open your compass 4.8 cm. and make a mark on the ray on the bottom of your triangle. At this point it will be oriented as above or as follows:

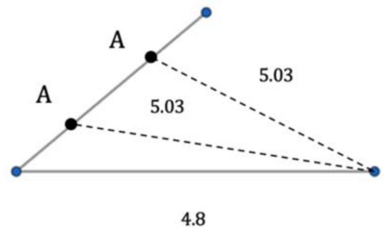


In the end, orientation does not matter. A congruent triangle has all corresponding angles congruent and all corresponding sides congruent.

Could you complete the drawing with one more step, two more steps, etc.? What do you think the next step will be?

If I gave you the measure of the length of side CA, could you complete the drawing accurately? Yes, drawing the side length from C to A would allow only one way to connect A to B to complete the triangle and the measure of angle B would be approximately 74°. Angle C is included between the two known sides of the triangle. This is an example of the Side-Angle-Side (SAS) Congruence Postulate.

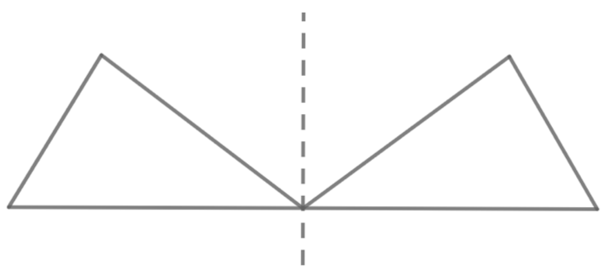
If I gave you the side length of AB, could you complete the drawing accurately? No, because you don't know the angle at B, so the triangle could take on two different possibilities as shown by the dashed lines in the figure below:



If you opened your compass the length 5.03 (approximately 5 cm.) and placed the point at angle B to draw the arcs, it would make a circle that would intersect ray CA in either of the two points at the end of the dashed lines marked A. Therefore, Side-Side-Angle (SSA) is not a Congruence Postulate. Angle B is not included between the two known sides of the triangle.

Congruent polygons have the same size and shape. Congruent transformations of polygons preserve the length and angle measure.

In triangle BAG and triangle FED, $\overline{BG} \cong \overline{FD}$, $\angle B \cong \angle F$, and $\overline{AB} \cong \overline{EF}$.



If you flip triangle BAG over a line of reflection where angle G and angle D meet, you will see that $\angle G \cong \angle D$, $\overline{AG} \cong \overline{ED}$, $\angle A \cong \angle E$, and the two triangles are congruent.

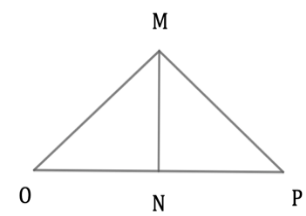
The Side-Angle-Side (SAS) Congruence Theorem can be used to prove that two triangles are congruent:

If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, then the two triangles are congruent.

Congruence Theorems are often called Congruence Postulates because they are assumed to be true. Congruence Postulates come from logical arguments but can be proven by theorems. Congruence Postulates will not be proven here but will be called Congruence Theorems.

We will investigate means to demonstrate that Congruence Theorems can be used to show that two triangles are congruent.

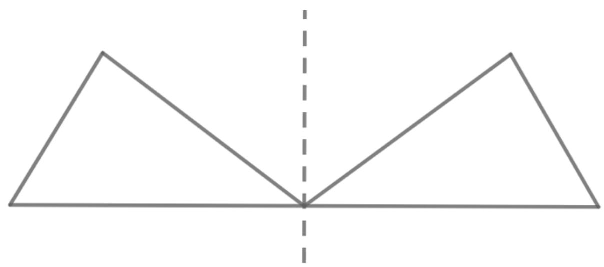
Example 1: Given that beam MN is the perpendicular bisector of beam OP, prove that both sides of a barn roof are congruent using the SAS Congruence Theorem.



Looking Ahead 6.2

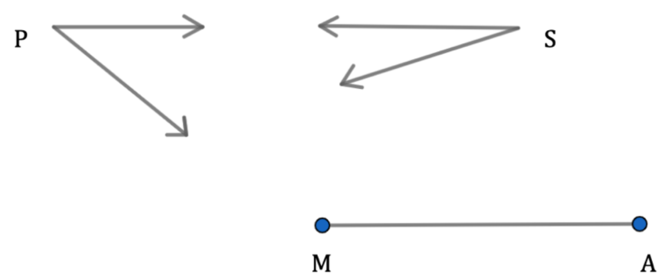
The triangles in the Looking Back 6.2 section have three sides that are also congruent, making $\Delta BAG \cong \Delta FED$. This is an example of the Side-Side-Side (SSS) Congruence Theorem:

If three sides in one triangle are congruent to three sides in another triangle, then the two triangles are congruent.



We know that Side-Angle-Side can be used to show that two triangles are congruent. Is Angle-Side-Angle (ASA) a congruence theorem? Can ASA be used to show that two triangles are congruent?

How many triangles can be constructed using a given side length and two angles?



Given a side length and two angles, try constructing triangles with your compass. Copy the line segment first, then copy angle P on the left side at point M and copy angle S on the right side at point A. Each time you construct the angles, label the point of intersection of the rays to form a triangle. Measure the sides of the triangle. What do you notice?

Example 2: Write the Angle-Side-Angle (ASA) Congruence Theorem.

Thus far, we have investigated three Congruence Theorems that can be used to prove that two triangles are congruent:

Side-Angle-Side (SAS)

Side-Side-Side (SSS)

Angle-Side-Angle (ASA)

We will now investigate the remaining two possibilities to see if they are also Congruence Theorems:

Angle-Angle-Angle (AAA)

Side-Angle-Angle (SAA)

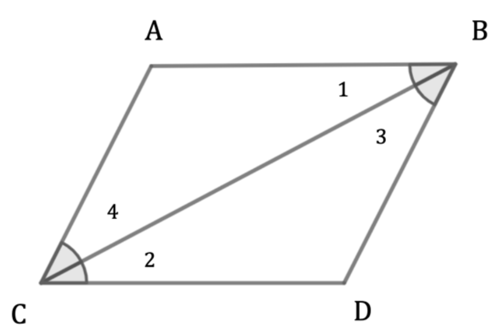
Example 3: Explain why Angle-Angle-Angle Theorem (AAA) is not a valid theorem for proving that triangles are congruent.

These theorems are helpful in two-column proofs where the statements on the left are supported by reasons on the right, which are comprised of axioms, postulates, definitions, properties, and other theorems.

Example 4: Complete the proof to prove that $\Delta ABC \cong \Delta DCB$ given that ABCD is a parallelogram. How does this proof relate to the Side-Angle-Angle (SAA) Congruence Theorem?

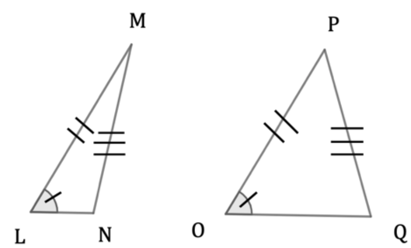
Side-Angle-Angle (SAA) Congruence Theorem:

If two angles of a triangle and a side that is not included between the angles are congruent to two angles of another triangle and the side that is not included between them, the angles are congruent.



Statement	Reason
1. ABCD is a parallelogram	1. ______________________
2. $\overline{AB} \parallel$ ________	2. Definition of a Parallelogram
3. $\overline{AC} \parallel$ ________	3. ______________________
4. $\angle 1 \cong \angle 2$	4. ______________________
5. $\overline{BC} \cong \overline{BC}$	5. ______________________
6. $\angle 3 \cong \angle 4$	6. ______________________
7. $\Delta ABC \cong \Delta DCB$	7. ______________________

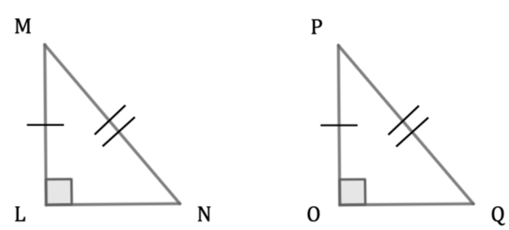
At the beginning of this section, we explored the general case of Side-Side-Angle (SSA) and learned that it does not prove that two angles are congruent. In triangles LMN and OPQ below, $\overline{LM} \cong \overline{OP}$, $\overline{MN} \cong \overline{PQ}$, and $\angle L \cong \angle O$.



However, the triangles do not have the same shape and size, and therefore, are not congruent.

Still, there is a special case where this does not hold true and SSA does demonstrate that the angles are congruent. Triangles LMN and OPQ below are right triangles with right angles L and O. Sides ML and PO are legs of each triangle and sides MN and PQ are the hypotenuses. This is called the Hypotenuse-Leg Congruence Theorem:

If the hypotenuse and leg of one right triangle are congruent to the hypotenuse and leg of another right triangle, then the two right triangles are congruent.



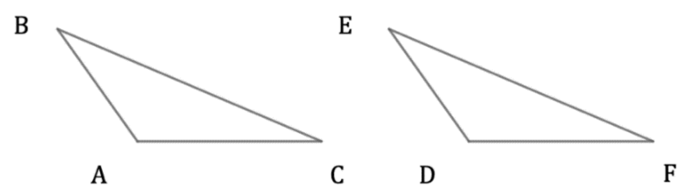
Section 6.3 Corresponding Parts of Congruent Triangles

Looking Back 6.3

If two triangles are congruent then their corresponding parts are congruent. These parts include all segments and angles that make up the triangle.

The definition of congruent triangles actually includes that their corresponding parts are congruent. The letters that make up the acronym CPCTC stand for: Corresponding Parts of Congruent Triangles are Congruent. The acronym helps us remember this.

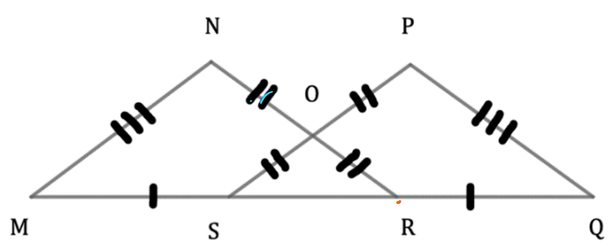
Sometimes, corresponding parts are easy to see. Let $\Delta ABC \cong \Delta DEF$, $\overline{AB} \cong \overline{DE}$, and $\angle B \cong \angle E$.



From the given, we know that acute angle B corresponds with acute angle E. Angle A is obtuse and therefore, corresponds with the only other obtuse angle, which is angle D in the other triangle. That means $\angle C \cong \angle F$ since they are the remaining acute angles in the two triangles. Because side AB corresponds to side DE, $\overline{BC} \cong \overline{EF}$, the longest side of both triangles, and $\overline{AC} \cong \overline{DF}$ since they are the remaining two sides of the two triangles.

If you cut out triangle ABC and slide it to the right over triangle DEF, you can match up the corresponding sides and angles of the congruent triangles. This slide is a rigid motion, which preserves length and angle measure.

Sometimes, corresponding parts are more difficult to see.



Because $\overline{NO} \cong \overline{OR} \cong \overline{PO} \cong \overline{OS}$, then $NO + OR = PO + OS$ by the Segment Addition Postulate. Also, $\overline{SR} \cong \overline{RS}$ by the Reflexive Property and $\overline{MS} \cong \overline{QR}$ is given. This means that $\overline{MR} \cong \overline{QS}$ again by the Segment Addition Postulate.

Because $\overline{MN} \cong \overline{QP}$ is given, then $\Delta MNR \cong \Delta QPS$ by the SSS Congruence Theorem.

Because $\angle M$ and $\angle Q$ are at the base of the corresponding sides MN and QP, which are congruent, then $\angle M \cong \angle Q$ and $\angle N \cong \angle P$ at the top of the corresponding sides. The remaining angles correspond and $\angle NRM \cong \angle PSQ$.

Trace each triangle separately, mark the angles inside each triangle, and color the congruent sides and angles the same color. Then cut out each triangle. Flip ΔMNR over onto ΔSPQ. Congruent sides and angles lay over one another and are aligned. A flip is also a rigid motion that preserves both length and angle measure. We will study this more in depth in an upcoming section.

<u>Looking Ahead 6.3</u>

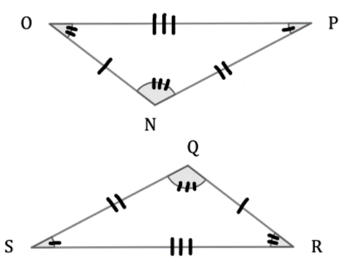
The same properties that are true for congruent segments and angles also hold true for congruent triangles.

Reflexive: $\Delta RST \cong \Delta RST$

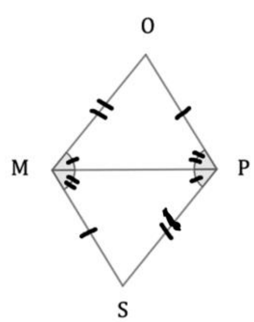
Symmetric: If $\Delta RST \cong \Delta UVW$, then $\Delta UVW \cong \Delta RST$

Transitive: If $\Delta RST \cong \Delta UVW$ and $\Delta UVW \cong \Delta XYZ$, then $\Delta RST \cong \Delta XYZ$

Example 1: Identify the corresponding parts that are congruent in the congruent triangles. What rigid motion will preserve the length and angle measures and will make corresponding parts easier to visualize.

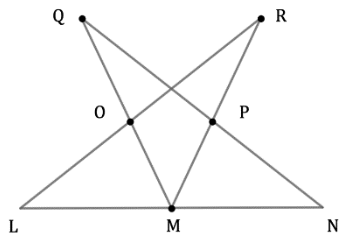


Example 2: How could you use the ASA or SAS Congruence Theorem to prove that $\Delta MOP \cong \Delta PSM$.

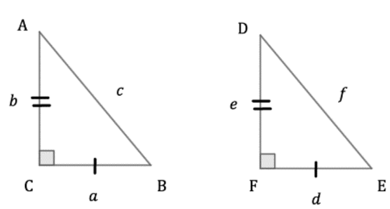


Remember, the equal symbol is used when working with numbers and the congruent symbol is used when working with figures.

Example 3: Given M is the midpoint of $\overline{LN}$, O is the midpoint of $\overline{QM}$, and P is the midpoint of $\overline{RM}$, and $\Delta LRM \cong \Delta NQM$, can you prove that $\Delta LOM \cong \Delta NPM$?



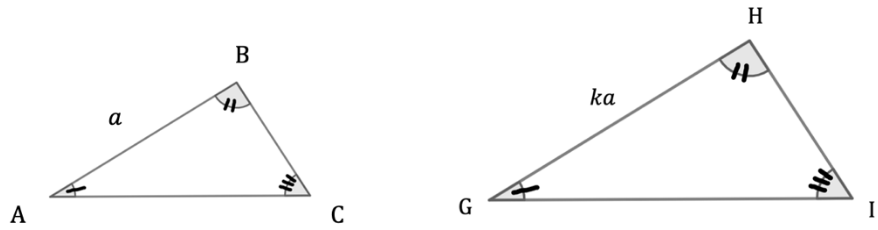
Example 4: Given that side a is congruent to side d and side b is congruent to side e in right triangles ABC and DEF, prove that $\Delta ABC \cong \Delta DEF$ using the SSS Congruence Theorem. What other Congruence Theorem is supported by this proof?



Section 6.4 Similar Triangles

Looking Back 6.4

When polygons are similar, their corresponding angles are congruent, and their corresponding sides are proportional.

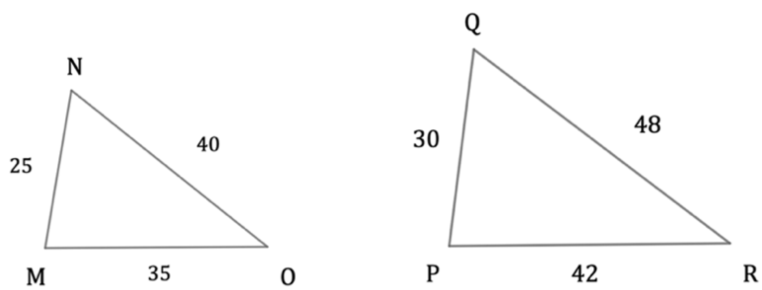


If ΔABC is dilated by a scale factor of k it will be the same size and shape as ΔGHI. The ratio of the corresponding side lengths in the scale factor of k is shown by the following:

$$\frac{\mathrm{HG}}{\mathrm{BA}} = \frac{\mathrm{IH}}{\mathrm{CB}} = \frac{\mathrm{IG}}{\mathrm{CA}} = k$$

"Triangle ABC is similar to triangle GHI" is written "ΔABC~ΔGHI." The wavy line is called a tilde. The scale factor enlarges or shrinks side lengths, which are proportional. The order in which they are named or written is important. Corresponding angles, which are congruent, occupy the same position in each figure and the same place in the order of similarity.

Example 1:	What is the scale factor, k, for the similar triangles?



ΔMNO~ΔPQR

The corresponding sides are the following ratios:

$$\frac{MN}{PQ} = \frac{NO}{QR} = \frac{MO}{PR}$$

The congruent angles are the following:

$$\angle M \cong \angle P$$

$$\angle N \cong \angle Q$$

$$\angle O \cong \angle R$$

<u>Looking Ahead 6.4</u>

Just as there are Congruence Postulates/Theorems, there are Similarity Conjectures/Postulates/Theorems. We have already seen that the Angle-Angle-Angle (AAA) Congruence Theorem does not exist because all equilateral triangles are equiangular, but two triangles may have all three angles of 60° and yet the side lengths are not the same. Each triangle is equiangular, but each side length is proportional.

Therefore, all equiangular triangles are similar, but may not be congruent. We know that if two angles of a triangle are congruent, then the third angle will be congruent.

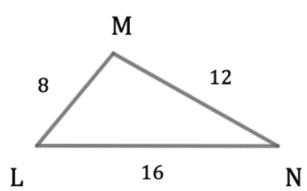
The Angle-Angle (AA) Similarity Theorem states:

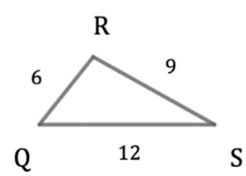
"If two angles of one triangle are congruent to two angles of a second triangle, then the two triangles are similar."

Because this theorem exists, there is no need to investigate the ASA or SAA Similarity Theorem because AA is enough. The Triangle Sum and Third Angle Sum address the remaining angle.

Just as you can use the congruent corresponding angles to show that two triangles are similar, you can use proportional corresponding side lengths to show that two triangles are similar.

Example 2:	Compare the ratios of the sides to see if the triangles are similar.



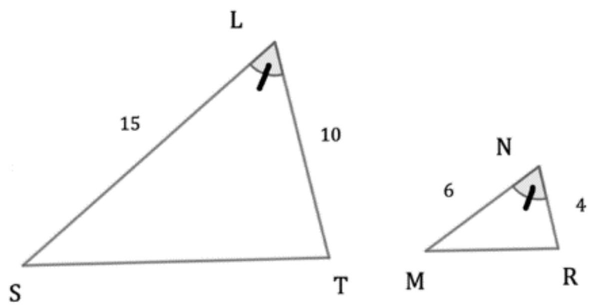


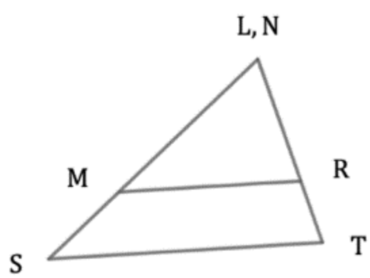
The Side-Side-Side (SSS) Similarity Theorem states:

"If the side lengths of one triangle are proportional to the corresponding side lengths of a second triangle, then the triangles are similar."

Let us investigate the final Similarity Theorem, Side-Angle-Side (SAS) to see if it applies as a Similarity Theorem.

Example 3: Use the diagram to show that one angle of triangle LST is congruent to one angle of triangle NMR and that the side lengths on both sides of the angle between them are proportional.





The Side-Angle-Side (SAS) Similarity Theorem states the following:

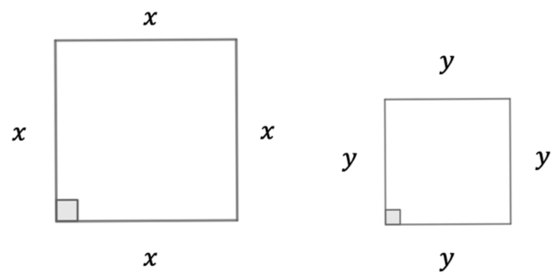
"If an angle of one triangle is congruent to an angle of a second triangle and the lengths of the sides on either side of the included angle are proportional, then the triangles are similar."

Constantine the Great ruled the Roman Empire years and years ago. His personal conversion to Christianity thwarted the persecution of Christians and allowed them to practice their beliefs. The city of Constantinople is present day Istanbul and was the wealthiest city on the boundary between Europe and Asia. Constantinople was the "New Rome" and the seat of the Byzantine Empire for over a thousand years. This magnificent empire was a real-world example of something built using a scale map. Such immense structures that made up the city were built using similarity and proportion (Serpent Column, Walls of Constantinople, etc.).

Section 6.5 Transformations of Triangles

Looking Back 6.5

Just as all equilateral triangles are similar, all squares are similar because the scale factor is the ratio of corresponding sides and all angles are congruent.

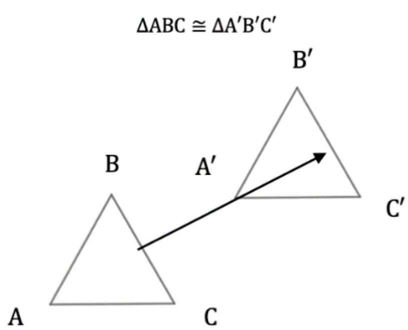


You have been tracing figures and using slides (translations), rotations, and flips (reflections) to show that triangles are congruent. These three transformations can be used with any polygon to show they are congruent because these specific transformations preserve both length and angle measure. Dilations show that triangles are similar. Dilations preserve angle measure but not length measure. We will investigate these four transformations in this section.

Looking Ahead 6.5

A rigid transformation that shows all parts of a copy of a triangle congruent to its original triangle is called an isometry.

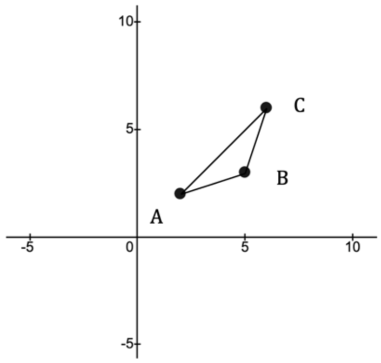
You have worked with these rigid transformations in a previous course. You might recall that the original polygon is a pre-image, and its copy is called the image.



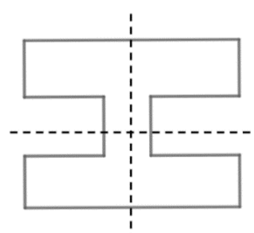
The image $\Delta A'B'C'$ is read "A prime, B prime, C prime." The translation is a slide along a straight path horizontally, vertically, or diagonally. A slide diagonally is a vector that has both horizontal and vertical components, which you will learn more about in Pre-Calculus, but will be introduced to here.

The ordered pairs $(x, y) \longrightarrow (x + h, y + k)$ translates the ordered pair h units horizontally and k units vertically. The vector is named by its horizontal and vertical components and is written "$\langle h, k \rangle$."

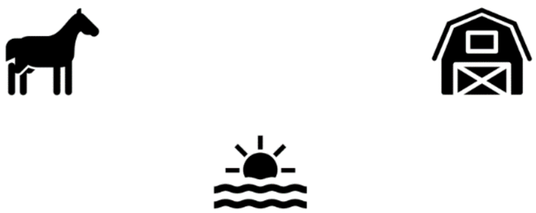
Example 1: The points of the pre-image of ΔABC are: $A(2,2)$, $B(5,3)$, and $C(6,6)$. Apply the translation $(x,y) \rightarrow (x+3, y-6)$ to pre-image ΔABC and find the image $ΔA'B'C'$. Draw the triangle image and label the coordinates.



A reflection produces a mirror image of the pre-image or original figure. In Module 3, we used Miras® as mirrors that reflected through rather than back. The letter “I” has reflectional symmetry. If you fold through center the entire letter folds onto itself. What other alphabetical letters have reflectional symmetry?



Example 2: A horse is in the field and wants to get a drink of water at the creek. After getting a drink the horse walks to the barn to eat. He is tired and wants to lay down and rest after he eats. What is the minimal path he needs to walk so that the distance is as little as possible to the creek and then the barn?

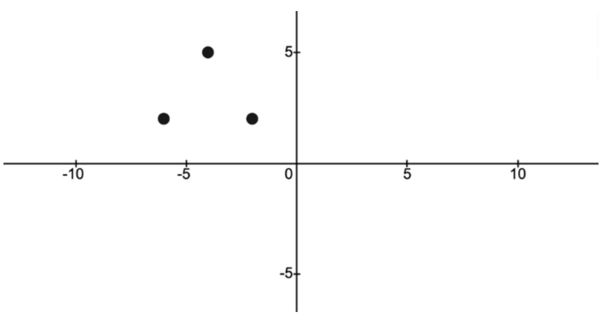


A glide reflection is two transformations. It combines a translation with a reflection. Imagine walking with the line of reflection between your two feet.

In a previous course, you investigated Fibonacci numbers and phyllotaxis. The way leaves wind around and move up a step is a rotation with a translation if the stem is the line of reflection.

Applying one transformation to a figure followed by another is called a composition of transformations.

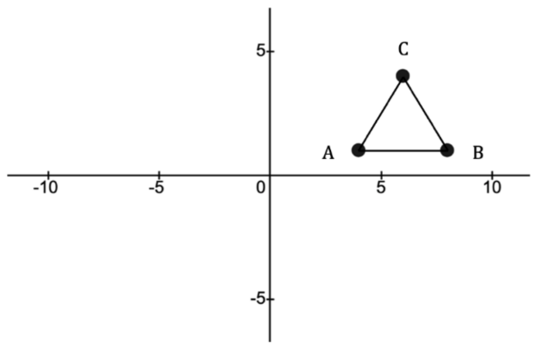
Example 3: Triangle ABC has vertices $A(-2, 2)$, $B(-6, 2)$, and $C(-4, 5)$. It has the coordinate translation $(x, y) \longrightarrow (x - 5, y - 1)$ followed by another coordinate translation $(x, y) \longrightarrow (x + 1, y - 3)$. Draw triangle $A'B'C'$ for the first translation and triangle A"B"C" for the second translation. What single translation would produce triangle A"B"C" given triangle ABC?



Reflecting an object over two parallel lines is a composition of transformations. It is the same as a single translation whose distance from any given point to the second image is twice the distance between the parallel lines.

A final rigid transformation is a rotation. All the points in an object rotate an identical number of degrees counterclockwise, unless otherwise specified, and all points are rotated from a central point. A rotation is defined by its center point and number of degrees rotated.

Example 4: Rotate triangle ABC clockwise $90°$ from center point C. The coordinates are $A(4,1)$, $B(8,1)$, and $C(6,4)$. The center of rotation is point C and the angle of rotation is $90°$.



For a point rotated counterclockwise about the origin, the following holds true:

Given a rotation of 90°, $(a, b) \rightarrow (-b, a)$.

Given a rotation of 180°, $(a, b) \rightarrow (-a, -b)$.

Given a rotation of 270°, $(a, b) \rightarrow (b, -a)$.

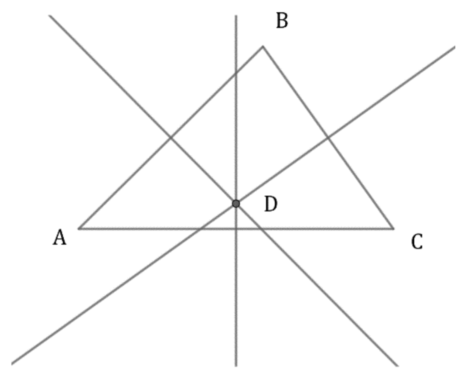
We have investigated dilations when we worked with similar triangles in the previous section. A dilation enlarges or shrinks a figure according to the scale factor. Dilations are not rigid transformations. Dilations do preserve angle measure because similar triangles have congruent angles, but they do not preserve length measure. The length measure of corresponding sides of similar triangles are not congruent; they are proportional. Therefore, dilations are not isometries.

There are countless examples of symmetry in God's expansive creation, such as the wings of a butterfly, the shells of a turtle, or the body of animals or humans. We should not be surprised we are symmetrical as we are created in God's image.

Section 6.6 Circumcenter and Incenter

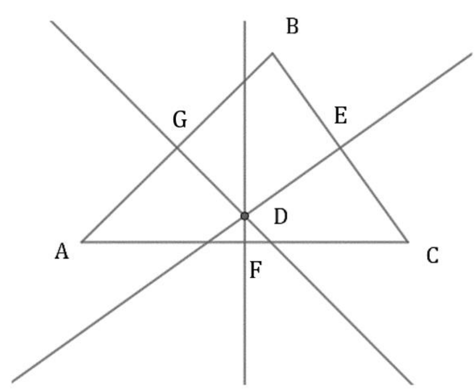
Looking Back 6.6

Use a geometer's utility to sketch acute triangle ABC. Construct perpendicular bisectors of the three sides. When line segments, rays, or lines meet, they are said to be concurrent, and the point at which they meet is the point of concurrency. Label the point of concurrency point D. Measure the distance from the point of concurrency to each of the vertices. What do you notice?



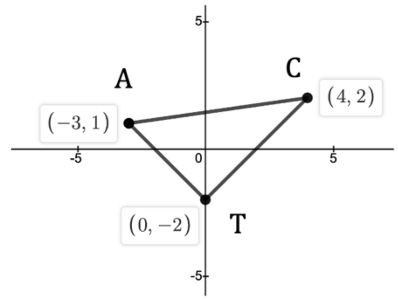
This point of concurrency is specifically called the circumcenter of the triangle. Point D is the point of concurrency where the three perpendicular bisectors of the sides meet.

Draw points where the perpendicular bisector intersects each side. Measure the two segments on both sides of the bisector to verify this is true. Drag any vertex of the triangle and the constructed perpendicular bisector of each side will stay in the middle of each side. The circumcenter changes and the distance from the circumcenter to each vertex changes with it, but all three distances remain equal.



To find the coordinates of the circumcenter, write the equation for two of the sides of the triangle and then find the point where the bisectors intersect.

Example 1: Find the coordinates of the circumcenter of triangle ACT.



1. Find the midpoint of $\overline{AT}$.

2. Find the slope of $\overline{AT}$.

3. Find the slope of the perpendicular bisector of $\overline{AT}$.

4. Use the midpoint and slope to find the y-intercept and equation of the perpendicular bisector of $\overline{AT}$.

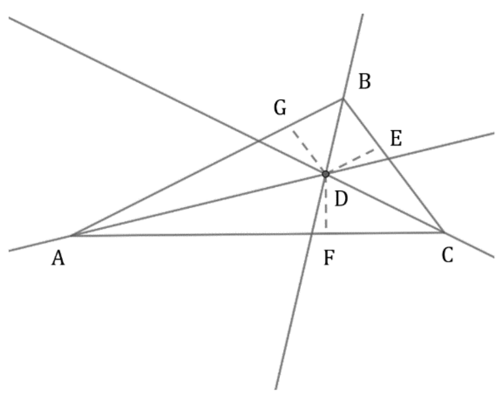
5. Use the same steps to find the equation of the line that is the perpendicular bisector of $\overline{CT}$.

6. Since the intersection of the two lines is the point of concurrency, find the circumcenter by solving a system of the two equations.

<u>Looking Ahead 6.6</u>

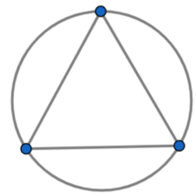
Using a geometer's utility, sketch acute triangle ABC. Construct the angle bisectors at each of the three vertices. The angle bisectors will also be concurrent, and this point of concurrency is the incenter of the triangle; label this point D.

The shortest distance from a point to a line is the measure of the perpendicular segment to a line. Measure the distance from the incenter perpendicular to each side. What do you notice?

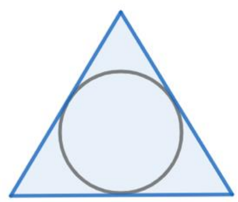


The perpendicular segment from the incenter, point D, to the sides of the triangle is equidistant: $\mathrm{DE} = \mathrm{DF} = \mathrm{DG}$.

If each vertex of a triangle touches the circumference of the circle, then the circle is circumscribed about the triangle. The triangle is inside so it is inscribed in the circle. The center of a circle that circumscribes a triangle is the circumcenter.



If a circle touches each side of the triangle at exactly one point then the circle is inscribed in the triangle, and the triangle is circumscribed about the circle. The center of a circle that is inscribed in a triangle is the incenter.



In the practice problems, you will investigate a circle inside the triangle using the incenter as the center of the circle.

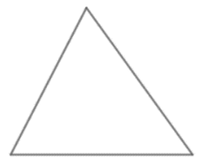
Section 6.7 Orthocenter and Centroid

Looking Back 6.7

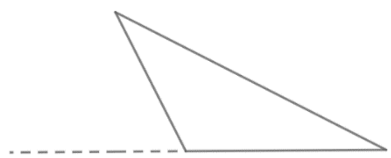
You have already investigated the shortest distance between a point and a line. This distance is the measure of the perpendicular distance between the two.

The altitude of a triangle is the perpendicular segment from any vertex to the opposite side or on the line extended from the opposite side.

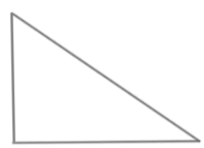
The altitude may be inside the triangle such as in the following acute triangle:



The altitude may be outside the triangle such as in the following obtuse triangle:



The altitude may be on the side of the triangle such as in the following right triangle:



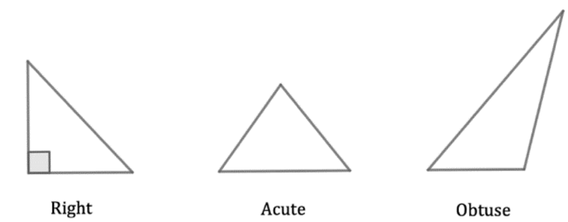
The three lines that represent the altitude (or contain the altitude) from the vertices of a triangle are also concurrent in a point. This point of concurrency is called the orthocenter.

Just as you found the coordinates of the circumcenter of a triangle by finding the equations of the perpendicular bisectors and solving the system of equations, you can also find the orthocenter by finding the equation for the lines of two altitudes and solving the system of equations.

Example 1:	Find the orthocenter of the triangle with coordinates $A(-1,-1)$, $B(3,1)$, $C(-2,3)$.

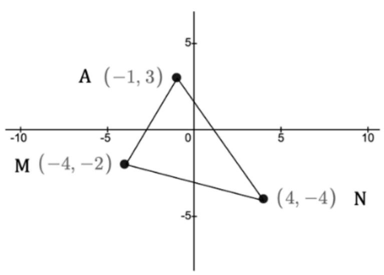
So, now we have investigated the concurrency of the perpendicular bisectors of the sides of a triangle, the angle bisectors of the angles of a triangle, and the altitudes of a triangle. We have one more area of exploration concerning concurrency and that is the medians of a triangle.

Example 2: Find the midpoints of each side of each triangle. Draw a line from each midpoint to the opposite vertex. What do you notice?



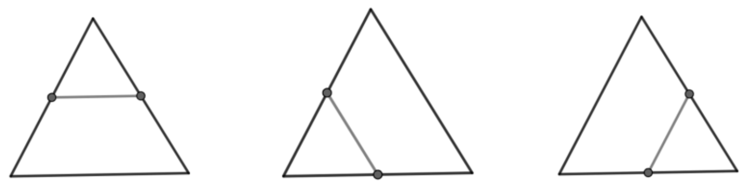
There are three medians in each triangle above. The medians connect the midpoint of a side to the opposite vertex. There are three sides, three vertices, three midpoints, and three medians in a triangle. The point of concurrency of the three medians of a triangle is called a centroid.

Example 3: Find the coordinates of the centroid of ΔMAN that has coordinates $\mathrm{M}(-4,-2)$, $\mathrm{A}(-1,3)$, and $\mathrm{N}(4,-4)$. Find the equations for the two lines containing medians and find their point of intersection by solving a system of equations.



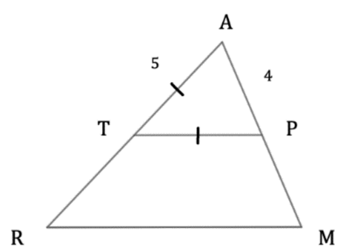
Check the point of intersection using a graphing utility.

The line that connects a midpoint of a side of a triangle to the midpoint on the opposite side of a triangle is called a midsegment.

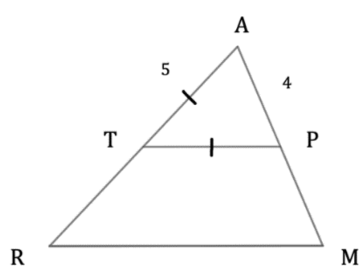


A midsegment of a triangle is parallel to the third side and half its length.

Example 4:	What is the perimeter of ΔRAM if $\overline{TP}$ is a midsegment?



Draw the other two midsegments for the triangle. Let $m\angle ATP = 48°$ and $m\angle AMR = 66°$. Find the angle measures of each of the four inner triangles. What do you notice?



Section 6.8 Napoleon's Theorem

Looking Back 6.8

Napoleon Bonaparte was born in 1769 in Corsica, France. His parents sent him to military school when he was 10 years old, where he also learned strategies for war and public speaking. When he was only 16 years old, he graduated. He was small in stature, only 5′2″, but quickly rose through the military ranks of the French army from captain to commander and then to general in only 10 years. He led campaigns in Italy, Austria, Netherlands, Egypt, and Syria. By age 30, Napoleon Bonaparte had appointed himself Emperor.

As Emperor, he did many good things for France, such as establishing a public banking system and restoring public parks. Napoleon was wise but very arrogant, however, and also did things that angered people, such as appointing his young son Emperor of Rome and appointing other family members to govern entire countries.

In 1814, Napoleon invaded Russia with the largest troupe the world had ever seen and never lost a battle there. However, he did lose many troops to freezing temperatures, food shortages, and Russian attacks. Once he returned home, Napoleon was banished; he spent the last seven years of his life in exile on the island of St. Helena's with General Baron Gourgard. Like Alexander the Great, who conquered and ruled most of the known world by age 32, Napoleon was considered a military genius and master of military tactics. In his talks with General Gourgard, who kept a detailed journal (*Helena*) of his time in exile, Napoleon gave testimony to Christ as a god and not a man. He said there was no connecting him with other rulers.

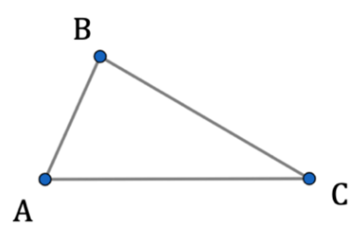
Napoleon was also a mathematician and said the most intellectual mathematicians, Descartes and Newton, Leibnitz and Pascal, were devout and practicing Christians because the very nature of Christ is astonishing: His Spirit is awesome and He confounds other religions of man and leaders of man; Napoleon further stated that between Christ and himself and all other leaders, there is no possible comparison; Christ is in a class all by Himself, a being alone; His gospel, His march across ages and realms and the mightiest of difficulties is unmatched and mysterious; He commanded more people and kingdoms than any and yet never wielded a sword, but conquered through love and peace.

Looking Ahead 6.8

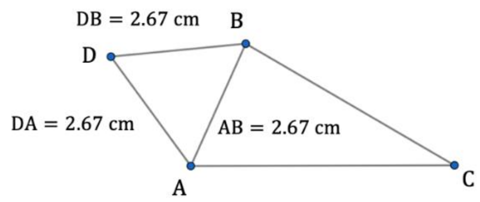
Napoleon Bonaparte's work as a mathematician led him to a theorem that is now called Napoleon's Theorem (in his honor).

Unlike the Pythagorean Theorem, where squares are drawn on the side of a triangle, we will be drawing equilateral triangles on each side of a triangle.

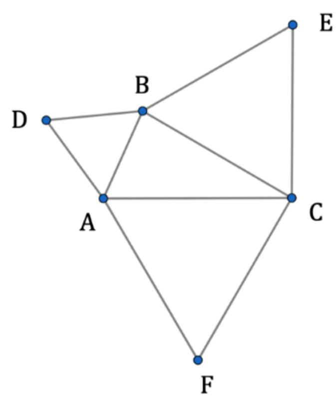
Step 1: Draw triangle ABC.



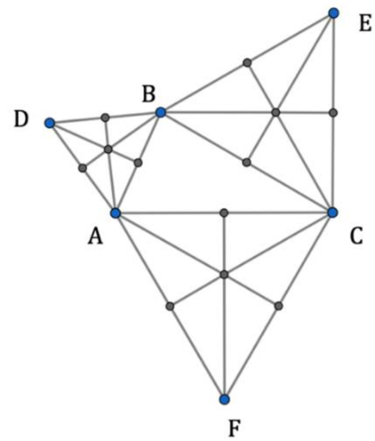
Step 2: Open your compass the length of AB. Make an arc above A the length of AB and make an arc from B that same length until the two arcs intersect. Draw segment AD from point A to the intercepting arc and draw segment BD from point B to the intercepting arc. Measure the length of AB, AD, and BD to assure they are all the same length and you have created an equilateral triangle on side AB of the triangle ABC.



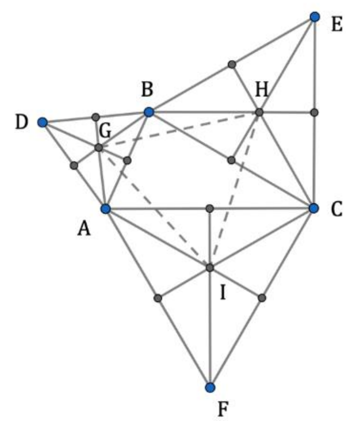
Step 3: Follow the directions in Step 1 to create an equilateral triangle on side BC of triangle ABC. Label the point created by the intercepting arcs E. Follow the same steps to draw an equilateral triangle on side AC of triangle ABC. Label the point created by the intercepting arcs F.



Step 4: Locate the centroid of each of the equilateral triangles. As you learned in the last few sections, the centroid is the point of concurrency where the three medians of the three sides of the triangle meet. The median connects a vertex to the midpoint of the opposite side. In an equilateral triangle the median is perpendicular to the opposite side and therefore, is also the altitude.



Step 5: Connect the centroids of each of the equilateral triangles and measure the sides. What do you get? You get another equilateral triangle. Using a geometer's utility (as shown here), if you drag the sides of ABC and make them longer or shorter, the sides of the equilateral triangle get bigger and smaller but always stay equal. Using paper and pencil, you can draw smaller or larger versions of triangle ABC and follow the steps outlined to see that an equilateral triangle is formed by connecting the centroids of equilateral triangles drawn on the sides of a triangle.



The dashed lines make up what is called the outer Napoleon triangle because the dashed segments extend outside of the sides of the original triangle ABC. Napoleon's Theorem states that the outer triangle will always be equilateral.

In the practice problems section, you will investigate variations of Napoleon's Theorem.

Section 6.9 Fractals and the Sierpinski Triangle

Looking Back 6.9

In Algebra courses, you learned about arithmetic sequences and geometric sequences. Algorithms were determined to find a series of numbers or series of shapes. The figurate numbers you have learned about are both algebraic and geometric representations of something that iterates (iterates means something occurs repeatedly). You found each term of the arithmetic sequence or geometric sequence by looking at the previous term in the series and applying a rule to it to get the next term. You can create a fractal by finding a rule to iterate a geometric shape. The design that is achieved is self-similar. If you look closely at any part of the design, it resembles the whole. This type of design was named a “fractal” by Benoit Mandelbrot in 1924. It opened up a whole new field of mathematical research where high-speed computers were used to generate a design called the Mandelbrot set.

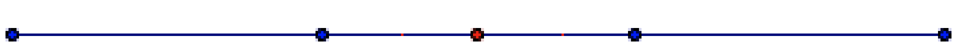
Before Benoit Mandelbrot coined the term fractal for these geometric iterations, Koch von Helge explored these patterns and designs that showed proportional similarity between the parts and the whole. Below is a fractal called the Koch Curve. Each step of the process is shown. Follow along and try it for yourself.

Looking Ahead 6.9

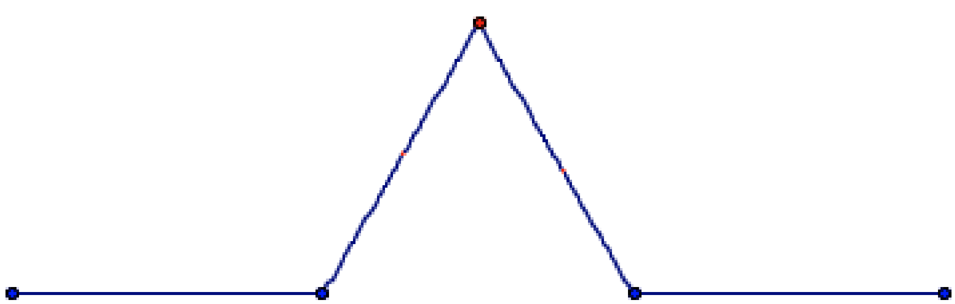
Step 1: Draw a straight line:

Step 2: Find and mark the midpoint that divides the line into two equal parts:

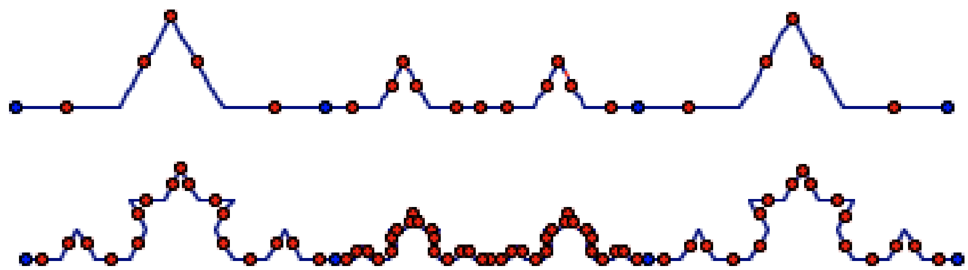
Step 3: Divide the whole line into three equal parts.

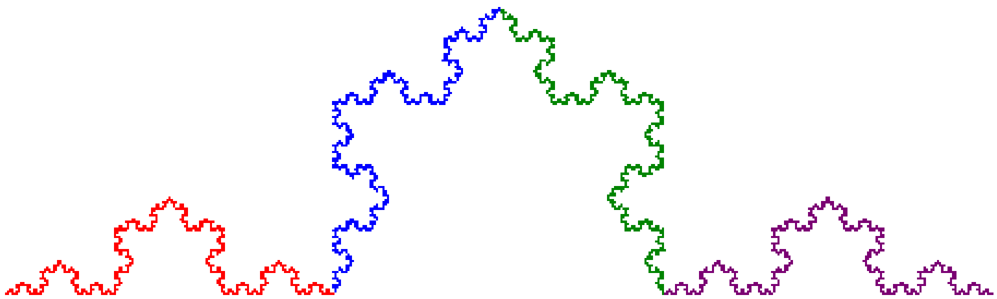


Step 4: Remove the middle piece and move the middle point up to form an equilateral triangle whose sides are as long as the missing piece:

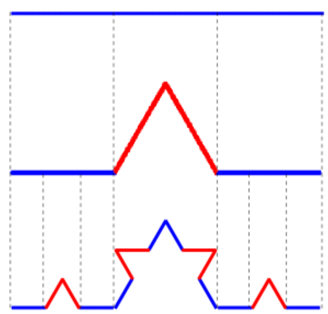


Step 5: Keep repeating this process on each side until you form a Koch snowflake:

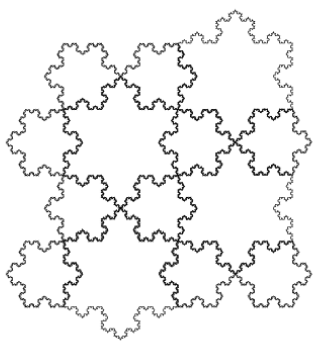




This process could be repeated indefinitely:



The length of each side is broken into three equal parts. The dashed lines above demonstrate this 1:3 ratio. This ratio for two sizes of Koch snowflakes can tile a plane. You learned more about tiling planes when tessellations and transformations were explored.



In the practice problems section, you will investigate another famous fractal called the Sierpinski triangle.

Section 6.10 Right Triangles and the Pythagorean Theorem

Looking Back 6.10

You have learned about Pythagoras and his secret society (in what is now present-day Italy) in Pre-Algebra. He and his society discovered irrational numbers and the five plutonic solids. Pythagoras is often called the first "pure mathematician" yet none of his written work survives.

Though he was attributed with the Pythagorean Theorem, it was later discovered that the Babylonians and Chinese had used it over a thousand years earlier. The Pythagorean Theorem states that the sums of the areas of the squares formed by the legs of a right triangle are equal to the area of the squares formed by the hypotenuse of the right triangle. Algebraically, if the legs are a and b and the hypotenuse is length c, then $a^2 + b^2 = c^2$. The Pythagorean Board video (in this video lesson) demonstrates this theorem.

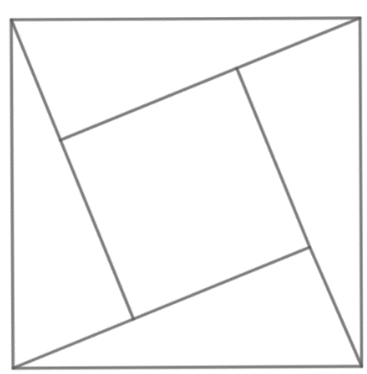
Do you think the Pythagorean Theorem applies to acute and obtuse triangles? Make a conjecture now and we will return to this question in the upcoming sections.

Looking Ahead 6.10

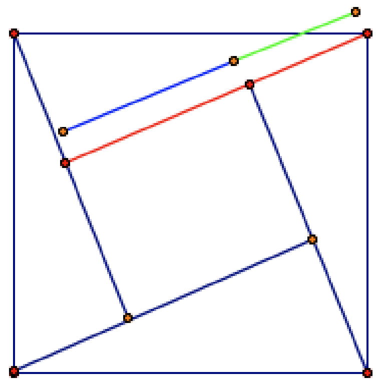
Bhaskara was a mathematician who lived in India from the year 1114 to the year 1185 and continued Brahmagupta's work with numbers (you learned about Brahmagupta in Pre-Algebra). Bhaskara used number systems to solve equations algebraically. He was the one who introduced the idea that "minus times minus equals plus," etc. Through his work, Bhaskara gained some understanding of dividing by 0, but thought it was an infinite quantity. He also studied polygons with up to 384 sides!

Moreover, Bhaskara wrote many math books; one of which is titled "Seed Counting." He also wrote astronomical books and was head of the observatory at Ujjain. His astronomy was less than perfect, however, and in one account that has been passed down, his astronomical meanderings and an unfortunate twist of fate cost his daughter a marriage. In order to console her, Bhaskara wrote "The Beautiful."

Lastly, Bhaskara has become very famous for writing a proof of the Pythagorean Theorem called the "Behold Proof." The story goes that he drew the diagram (shown below) and said "Behold" as proof of the Pythagorean Theorem.



Let us add some detail and see if we can complete the proof.



Let us label the length of the entire red line b. Let us label the length of the green line a, then label the length of the blue line, which is the side of the small square, $b - a$. The area of the large square is the area of the four triangles plus the area of the small square in the middle.

Let us call the side of the large square c. That means the area of the large square is c^2.

The area of the small square inside the big square is the length of the side squared, or $(b - a)^2$.

The legs of the right triangles are the length of the green line and the length of the red line. The area for a triangle is $\frac{1}{2}(\text{base} \cdot \text{height})$. This is $\frac{1}{2}(\text{green} \cdot \text{red})$ or $\frac{1}{2}(ab)$. So, the area of all four triangles is $4 \cdot \frac{1}{2}(ab)$ or $2ab$. The area of the four triangles plus the small square is $2ab + (b - a)^2$.

$c^2 = 2ab + (b - a)^2$

$c^2 = 2ab + (b - a)(b - a)$

$c^2 = 2ab + b^2 - ab - ab + a^2$

$c^2 = 2ab + b^2 - 2ab + a^2$

$c^2 = b^2 + a^2$

$c^2 = a^2 + b^2$

There it is! The Pythagorean Theorem again!

Section 6.11 The Converse of the Pythagorean Theorem

Looking Back 6.11

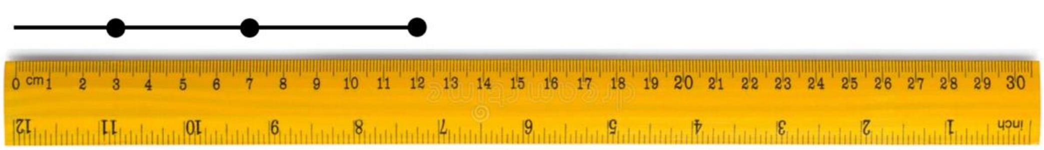
The Pythagorean Theorem states that the sum of the squares of the length of the legs in a right triangle is equal to the square of the length of the hypotenuse. This is the algebraic formula. The sum of each represents the areas geometrically. Is the converse of this true? In other words, if the three sides a, b, and c of a triangle satisfy the algorithm $a^2 + b^2 = c^2$, then does that mean the triangle is a right triangle?

A set of three numbers that work in the Pythagorean Theorem is called a Pythagorean Triple. You solved these in the previous practice problems section. The first set is $3 - 4 - 5$ because $3^2 + 4^2 = 5^2$ is a true statement. That means $6 - 8 - 10$ is also a Pythagorean Triple because it is similar to a $3 - 4 - 5$ triangle. The sides are just double the length, which means they are proportional. The scale factor is 2. Therefore, a $6 - 8 - 10$ triangle is also a right triangle. A $9 - 12 - 15$ triangle, which is triple the lengths of a $3 - 4 - 5$ triangle, is also a Pythagorean Triple; a $12 - 16 - 20$ triangle is quadruple a $3 - 4 - 5$ triangle, and also a Pythagorean Triple. You could multiply $3 - 4 - 5$ by 5, 6, and 7, etc. and get an infinite number of Pythagorean Triples from that one set.

There are other sets that are Pythagorean Triples that are not multiples of $3 - 4 - 5$. An example of this is $5 - 12 - 13$. What is the Pythagorean Triple that is proportional to $5 - 12 - 13$ with a scale factor of 2? If you said $10 - 24 - 26$, you are correct. The Babylonians made a clay tablet of Pythagorean Triples called the Plimpton 322 tablet, which dates sometime before 1600 B.C. There is a large one that is a $1679 - 2400 - 2929$ triangle, which was calculated without the use of calculators!

In addition, you have also learned in Algebra about how the Egyptians used knotted rope to find square corners of fields. Using the Egyptians' method, you tied a knot in yarn, then folded it over and tied another knot the same distance and kept folding it over and tying another knot the same distance until you had an equal size for 12 knots. You also learned how the Egyptians walked out the $3 - 4 - 5$ right triangle to give a square corner. We are going to try to duplicate the Egyptian's method with the Pythagorean Triples using string and a centimeter ruler to see they all form right triangles. In Egypt, the Nile would flood every year and the land boundaries would have to be marked again. Knowing the Pythagorean Triples would have been a great help to these farmers in marking accurate boundaries.

Looking Ahead 6.11



Put a piece of string along a metric ruler. At 3 centimeters, tie off a knot; count out 4 more centimeters and tie off another knot; then count out 5 more centimeters and tie off another knot. Put a clear piece of tape over the string from the start to the first knot. Pull the string straight up from the first knot to the second knot so that it is perpendicular to the ruler. Use a protractor to make sure the angle is $90°$ before putting a clear piece of tape over the vertical string. The remaining piece of string should reach all the way back to the beginning of the ruler. Now, a right triangle has been formed with side lengths 3 cm., 4 cm., and 5 cm.

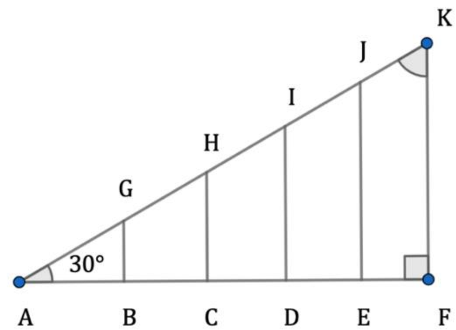
Now try it with a $5 - 12 - 13$ Pythagorean Triple. As you get into larger numbers, such as with a $16 - 30 - 34$ Pythagorean Triple, it might be easier to use the millimeter marks on the metric ruler to save string and then check geometrically that a right triangle is formed. The sides have to complete the circuit with no gaps or overlaps.

Section 6.12 Special Ratios in Right Triangles

Looking Back 6.12

There are some special ratios in right triangles. You investigated two of these in previous practice problems. When the angles are $30° - 60° - 90°$ (read, "thirty, sixty, ninety") the hypotenuse is double the length of the short leg, and the long leg is $\sqrt{3}$ times the length of the short leg. In a right triangle that is isosceles with angles $45° - 45° - 90°$ ("forty-five, forty-five, ninety"), the hypotenuse is $\sqrt{2}$ times the length of either leg. You will explore these special right triangles further in the next module.

Let us look at the $30° - 60° - 90°$ right triangle once again and see if there are any special relationships between its angles and sides.



In triangle AKF, ∠KAF is 30°, ∠AKF is 60°, and ∠KFA is 90°. The line segment that is a leg of the triangle and a side of ∠KAF is the adjacent side of the 30° angle.

Therefore, AB, AC, AD, AE, and AF are adjacent sides to ∠KAF. Side KF is the opposite side of ∠KAF as it is across from it and does not touch it. Likewise, GB, HC, ID, JE, and KF are all opposite sides of ∠KAF.

The hypotenuse of the right triangle is AK, which includes AG, AH, AI, and AJ.

Example 1: Find the ratios of the lengths of the sides opposite of each of the five triangles in ΔAKF to the length of the hypotenuse of each triangle. Measure in millimeters.

Ratio	$\frac{GB}{AG}$	$\frac{HC}{AH}$	$\frac{ID}{AI}$	$\frac{JE}{AJ}$	$\frac{KF}{AK}$
Value					

What stands out to you about the table?

We call this the special ratio the sine of an angle that is $30°$. We could extend sides AF and AK and the ratio would stay the same. This works for any angle θ.

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$

Example 2: Find the ratios of the lengths of the adjacent side of each triangle in ΔAKF (there are five including ΔAKF) to the length of the hypotenuse.

Ratio	$\frac{\text{AB}}{\text{AG}}$	$\frac{\text{AC}}{\text{AH}}$	$\frac{\text{AD}}{\text{AI}}$	$\frac{\text{AE}}{\text{AJ}}$	$\frac{\text{AF}}{\text{AK}}$
Value					

What stands out to you about the ratios? What can you conclude about ΔAGB, ΔAHC, ΔAID, ΔAJE, and ΔAKF?

The ratios of the lengths of the adjacent sides to the length of the hypotenuse are all the same. We call this the cosine of the $30°$ angle, which is ∠KAF. This is true for any angle θ.

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

The five triangles are all similar. They share the $30°$ angle. They are all right triangles (which means they all have a $90°$ angle). That leaves the third angle in each triangle to be $60°$. Therefore, the angles are all congruent and the sides are proportional.

This similarity and ratio relationship probably helped the Egyptians build the pyramids. These ratios were recorded in clay tablets called the _____________________ _____________________ as early as ___________ in _____________________. Angular cuneiform script was used.

For years, these tablets could be found in tables at the back of trigonometry and geometry books. Now, they can be found stored in graphing calculators. In the graphing calculator, go to “Settings” and make sure you are in “Degree” mode, and find the sin function under the “Trig” key.

$$\sin 30° = \frac{1}{2}$$

This is when the calculation mode is "Auto" or "Exact." If you change it to "Approximate," you will get a decimal approximation:

$$\sin 30° = 0.5$$

These are examples of patterns that God has given us to discover and explore. It makes sense that triangles are such strong structures for building bridges and other architectural wonders, such as the pyramids!

There is one more ratio we would like to investigate. This is called the tangent function. It is the ratio of the length of the opposite side of any angle of a right triangle to its adjacent side.

$$\tan \theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

Example 3: Complete the table below and make a conclusion about the tangent function.

Ratio	$\frac{GB}{AB}$	$\frac{HC}{AC}$	$\frac{ID}{AD}$	$\frac{JE}{AE}$	$\frac{KF}{AF}$
Value					

The ratios are equivalent.

Looking Ahead 6.12

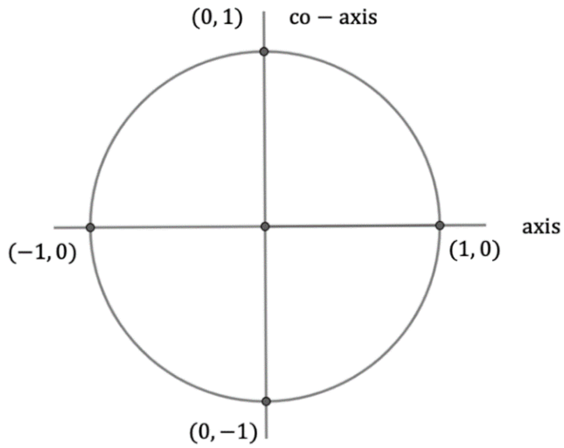
There are three more foundational trigonometric functions. These are cosecant, secant, and cotangent. These are called the reciprocal functions, which means that these ratios are reciprocals of the previous three trigonometric functions.

$$\text{cosecant} = \frac{\text{hypotenuse}}{\text{opposite side}} = \frac{1}{\sin \theta}$$

$$\text{secant} = \frac{\text{hypotenuse}}{\text{adjacent side}} = \frac{1}{\cos \theta}$$

$$\text{cotangent} = \frac{\text{adjacent side}}{\text{opposite side}} = \frac{1}{\tan \theta}$$

Example 4: The unit circle has a radius of $r = 1$ unit. If the right triangle that is $30° - 60° - 90°$ is inscribed in the unit circle, then the radius becomes the hypotenuse. Draw the segment in the unit circle and answer the questions below.



a) Draw a secant line, which includes the radius at an angle of approximately $30°$ with the horizontal axis and call it "theta." Label the point "O" where the secant line intersects the unit circle. Color the radius green.

b) Draw the sine of theta in blue.

c) The complementary angle of θ is co-θ. Label it on the unit circle.

d) Draw the tangent line to the unit circle at $(1, 0)$. Color it red. Why is the line segment from the horizontal axis to the secant line the tangent of the right triangle?

e) Draw the tangent to point $(0, 1)$ from the co-axis to the secant line extended. What function is it in relation to co-θ? Color it accordingly.

Notice that the sine of co-theta is the cosine of θ and the tangent of co-theta is the cotangent of θ. Moreover, the secant of co-theta is the cosecant of θ.

Section 6.13 Oblique Triangles

Looking Back 6.13

A conditional statement takes the form: "If A, then B." The first part of the statement is A and the last part of the statement is B. Therefore, the converse would be: "If B, then A." Let us look at an example using the Pythagorean Theorem since we know the Pythagorean Theorem states: "In a right triangle, the sum of the squares of the lengths of the legs equals the square of the hypotenuse."

"If the triangle is a right triangle" is A; "Then the sum of the square of the lengths of the legs equals the square of the hypotenuse" is B. As we learned in the previous section, the converse is also true. "If B, then A" gives us the following statement: "If the sum of the squares of the lengths of the legs equals the square of the hypotenuse in a triangle, then the triangle is a right triangle."

When inductive reasoning is used to form a conclusion, it is called a conjecture. When a demonstration is performed, then the conjecture has been proven and you have a theorem.

The converse of a conditional statement is not always true. Since we have already talked about the Nile River in Egypt flooding the plains each year, let us use that for an example.

Conditional Statement: If you live in Egypt near the Nile River, you will experience flooding.

Converse Statement: If you experience flooding, then you live in Egypt near the Nile River.

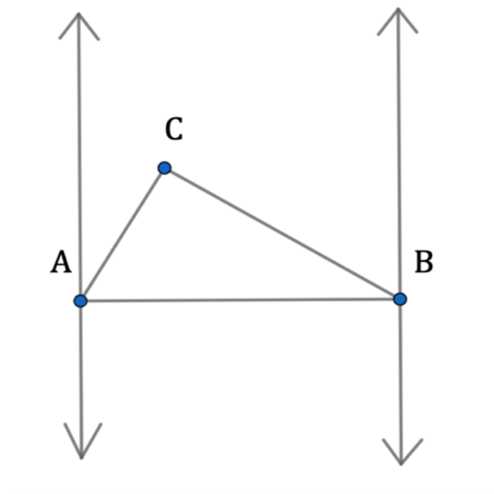
The converse is not always true. There are floods all over the world. Many of them occur in the Philippines each year and they are not near Egypt or the Nile River. However, no matter how many floods there are, we know we will never experience world-wide flooding again because of God's covenant in Genesis.

So, if we know the Pythagorean Theorem is true, then we have a right triangle. Sometimes, the Pythagorean Theorem does not work for a triangle. In this case, the triangle must be either obtuse or acute.

Is there a way we can tell by looking at the sides of a triangle whether or not it is acute or obtuse? There is a way, and we will explore that now.

In $a^2 + b^2 = c^2$, we can subtract c^2 from the other side and get $a^2 + b^2 - c^2 = 0$. This could also be written as "$c^2 - (a^2 + b^2) = 0$," which is when a^2 and b^2 are subtracted from the other side, or subtracted from c^2, and this is the method we will use in our exploration. If $c^2 - (a^2 + b^2)$ does equal 0, then we know we have a right triangle, and this is another form of the Pythagorean Theorem.

In the diagram below, as angle C (called ACB below) gets closer and closer to 90°, the difference of the square of the hypotenuse and the square of the sum of the other two sides gets closer and closer to 0°. In our investigation, the side representing the hypotenuse will be side AB, which is between the two parallel lines and touches the parallel lines. It is referred to as side c in the Pythagorean Theorem; it is opposite the angle representing our right, obtuse, or acute angle in the following explorations, which is at point C.



m ∠ACB	AB	CA	BC
90.07°	10.65 cm.	5.41 cm.	9.17 cm.

$$\text{AB} \cdot \text{AB} - (\text{CA} \cdot \text{CA} + \text{BC} \cdot \text{BC}) = 0.12 \text{ cm.}^2$$

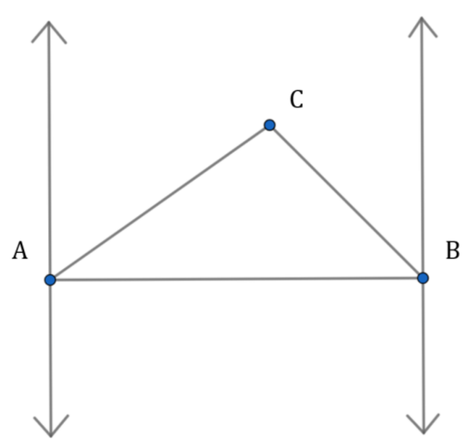
$$m\angle\text{ACB} = 90.07°$$

$$\text{AB} = 10.65 \text{ cm.}$$

$$\text{CA} = 5.41 \text{ cm.}$$

$$\text{BC} = 9.17 \text{ cm.}$$

Let us look at the difference between the diagram above when angle C gets closer and closer to 90° and when angle C gets more obtuse. In an obtuse triangle, the longest side of the triangle is opposite the obtuse angle. In these cases, the longest side is AB, referred to in the Pythagorean Theorem as c.



m ∠ACB	AB	CA	BC
100.06°	10.65 cm.	7.68 cm.	6.16 cm.

$$\text{AB} \cdot \text{AB} - (\text{CA} \cdot \text{CA} + \text{BC} \cdot \text{BC}) = 16.53 \text{ cm.}^2$$

$$m\angle\text{ACB} = 100.06°$$

$$\text{AB} = 10.65 \text{ cm.}$$

$$\text{CA} = 7.68 \text{ cm.}$$

$$\text{BC} = 6.16 \text{ cm.}$$

When C is obtuse the difference of the square of side c and the square of the other two sides is not 0, but a positive number. This must mean that the square of the side opposite the obtuse angle is longer than the square of the sum of the other two sides. That makes sense because an obtuse angle is big and the bigger the angle gets, the longer the side opposite it gets as well.

Therefore, using the letters from the Pythagorean Theorem for the sides, we have the equation $c^2 - (a^2 + b^2) > 0$. On the other hand, what is written in parenthesis could be added to the other side to give us $c^2 > (a^2 + b^2)$. Writing this in the standard form of the Pythagorean Theorem gives us the following equation:

$$a^2 + b^2 < c^2$$

In an obtuse triangle, two angles are acute and only one is obtuse, and the longest side is opposite that one obtuse angle. In an obtuse triangle, the square of the side opposite the obtuse angle is greater than the sum of the squares of the other two sides.

There is a conjecture that fits nicely here and explains part of what is taking place; it was originally called the Side-Angle Inequality Conjecture, but is now called the Side-Angle Inequality Theorem because it has been formally proven:

In a triangle, if one side is longer than another side, then the angle opposite the longer side is larger than the angle opposite the shorter side.

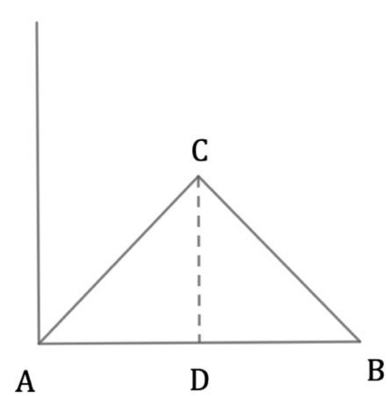
Do you want to make any conjectures as to the theorem when the triangle is acute? I hope so because that is what you will be doing in the first part of this Practice Problems section.

Looking Ahead 6.13

Triangles that are not right triangles are called oblique triangles. There is a way to find the angles and side measures of these triangles using the Law of Sines and the Law of Cosines.

In Pre-Calculus and Calculus, we will use α (alpha) for angle A, β (beta) for angle B, and γ (gamma) for angle C. Alpha is the beginning of the Greek alphabet and Omega is the end. God calls himself the Alpha (α) and Omega (ω) because He is the beginning and the end.

Let triangle ABC be an acute triangle. Point A is at the origin and point B lies on the positive x-axis at point $(c, 0)$; side a is across from ∠A, side b is across from ∠B, and side c is across from ∠C.



The height, h, of ΔACB is the altitude because it is perpendicular to $\overline{\text{BA}}$. From ΔBCD and ΔACD…

$$\sin\alpha = \frac{h}{b} \qquad \sin\beta = \frac{h}{a}$$

$$b\sin\alpha = h \qquad a\sin\beta = h$$

$$b\sin\alpha = a\sin\beta$$

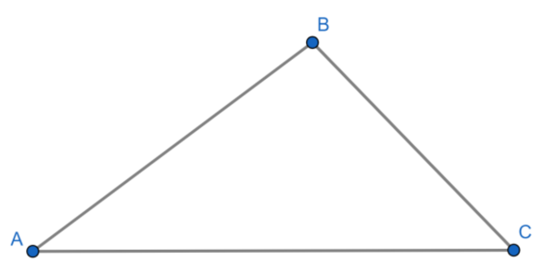
$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b}$$

The Law of Sines states that $\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$ for any triangle ABC. It may also be written $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ when each angle is across from the side of the same letter. The third ratio is found by repositioning the triangle.

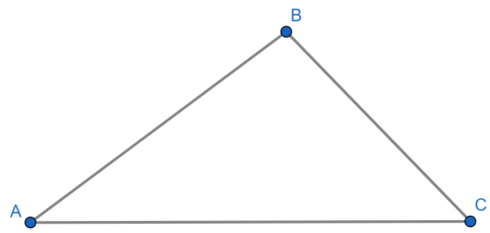
The Law of Sines cannot be used to find the sides of a triangle when only the three sides are known (side-side-side).

If you know two angles and one side of a triangle, you can use the Law of Sines to find the length of the other two sides.

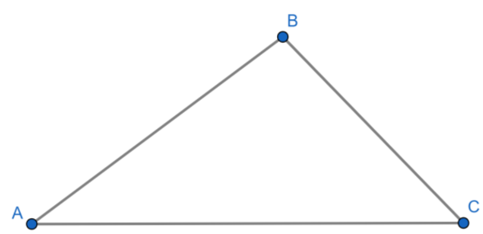
Let us investigate angle-angle-side firstly. Find the missing side that is across from a known angle. The third angle may be found using the Sums of the Angles of a Triangle Theorem, and then third side may be found by using the Law of Sines a second time.



Next, let us investigate angle-side-angle. In this case, the third angle may be found firstly followed by the two missing sides using the Law of Sines.



The Law of Sines works when the given angle is opposite one of the two given sides. If it were the included angle between the two given sides, the Law of Sines would not work.

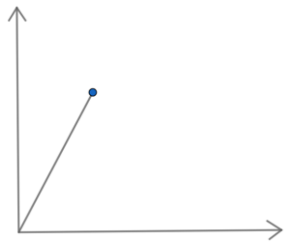


There is not enough information given, leaving the proportion in the Law of Sines Formula with two unknown variables.

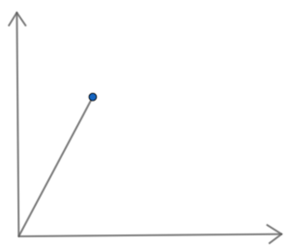
If two sides and an angle opposite one of them are given (side-side-angle), there are three possibilities: no triangles are formed, one right triangle or one unique triangle is formed, and in the ambiguous case, two triangles are formed.

Let us suppose angle β is given and two sides a and b are given but angle β is not the included angle between them.

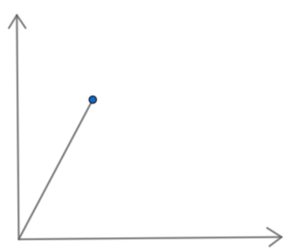
a) If side b is too short, then a triangle cannot be made.



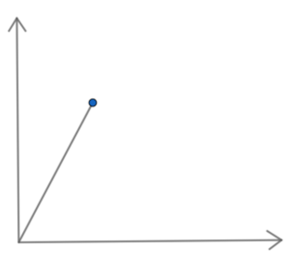
b) If $b = h$ and both are the altitude (the perpendicular distance from C to the x-axis) then a right triangle is made.



c) If side b is longer than h, but shorter than side a, then two triangles can be made.

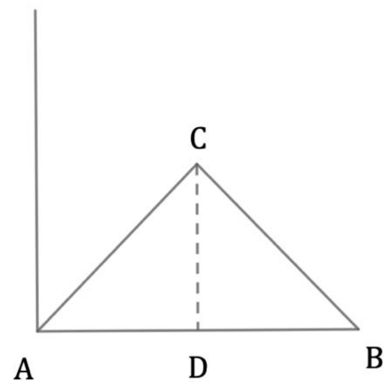


d) If side b is longer than h and as long or longer than side a, then only one unique triangle can be made.



In the Practice Problems section, we will investigate whether or not it is possible to use the Law of Sines to form a triangle. In Module 8, we will apply the use of the above formula.

For now, we will introduce the Law of Cosines. This can be used to find the missing angles of an oblique triangle when the three sides are known (side-side-side) or when two sides and the angle between them are known (side-angle-side). Let ΔABC have the vertex A at the origin and B on the positive x-axis at point $(c, 0)$. Let $\overline{\text{CD}}$ be perpendicular to $\overline{\text{AB}}$ at point D such that $\overline{\text{CD}}$ is the altitude h and point C is the ordered pair (d, h).



Looking at triangle ACD

$\sin\alpha = \frac{h}{b}$ and $\cos\alpha = \frac{d}{b}$

$b\sin\alpha = h$ $b\cos\alpha = d$

If we plug these values into the Pythagorean Theorem for triangle BCD, we get the following equation:

$$a^2 = (c-d)^2 + h^2$$

Substitute in the values for h and d:

$$a^2 = (c - (b\cos\alpha))^2 + (b\sin\alpha)^2$$

$$a^2 = c^2 - 2bc\cos\alpha + b^2\cos^2\alpha + b^2\sin^2\alpha$$

$$a^2 = c^2 - 2bc\cos\alpha + b^2(\cos^2\alpha + \sin^2\alpha)$$

$$a^2 = c^2 - 2bc\cos\alpha + b^2(1)$$

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

By repositioning the triangle, as is the case in the Law of Sines, we are able to get the two additional equations shown below:

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

And

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

These three formulas make up the Law of Cosines.

For the Law of Sines, when two sides and the angle measure opposite one of those two given sides are given, then it is possible to get no triangle, one triangle, or two triangles. When the length of two sides and the measure of the included angle between them are given, there is one unique triangle, and the Law of Cosines can be used to find the length of the remaining side and the measure of the other two angles.

We can also find the area of ΔABC using three formulas depending on which two side lengths and which angle between them are known:

$$\boldsymbol{A} = \frac{1}{2}bc\sin\alpha$$

$$\boldsymbol{A} = \frac{1}{2}ac\sin\beta$$

$$\boldsymbol{A} = \frac{1}{2}ab\sin\gamma$$

One-half of the perimeter of ΔABC is called the semiperimeter s where $s = \frac{1}{2}(a + b + c)$.

Heron's Formula is another way to find the area of ΔABC when the sides are known:

$$\boldsymbol{A} = \sqrt{s(s-a)(s-b)(s-c)}$$

We will practice using these area formulas in the Practice Problems section but will wait until Module 8 to apply the Law of Cosines.