Pre-Calculus and Calculus Module 1 Circular and Periodic Functions

Section 1.1 The Sine Function

Looking Back 1.1

In Geometry and Trigonometry, you were introduced to the trigonometric functions, their transformations, and their applications.

In Pre-Calculus and Calculus, we will review the periodic and circular functions and investigate more Trigonometric Identities that help in problem-solving. The periodic functions model many phenomena that occur in God's natural world as He designed these rhythmic processes. The circular functions derive from the unit circle. In Geometry and Trigonometry, you investigated the unit circle using tin cans and Smartees® candy to convert between radians and degrees for these types of periodic or circular functions.

In Pre-Calculus and Calculus, you will learn about geometric structures: vector spaces, the complex number plane, and the polar coordinate plane. You will also learn how they are all related. However, for now we will begin with a review of the trigonometric functions and their occurrences in real-world problems.

Firstly, we will review what we learned in Trigonometry here. If *a*, *b*, *c*, and *d* are positive real numbers, then the following transformations are given from the parent function $y = f(x)$:

$y = af(x)$	This is a vertical stretch of if |*a*| > 1 and a vertical shrink if 0 < |*a*| < 1
$y = f(bx)$	This is a horizontal shrink if |*b*|> 1 and a horizontal stretch if 0 < |*b*| < 1
$y = -f(x)$	This is a reflection in the *x*-axis
$y = f(-x)$	This is a reflection in the *y*-axis
$y = f(x + c)$	This is a horizontal phase shift left *c* units
$y = f(x - c)$	This is a horizontal phase shift right *c* units
$y = f(x) + d$	This is vertical shift up *d* units
$y = f(x) - d$	This is vertical shift down *d* units

Explore these on the graphing calculator using real numbers for *a*, *b*, *c*, and *d*.

Looking Ahead 1.1

The transformations for sinusoidal or periodic functions that are cyclical is of the form

$$y = af(bx + c) + d:$$

and may be written:

$$y = \text{a}\sin\left[b\left(x + \frac{c}{b}\right)\right] + d$$

where |*a*| is the amplitude and $\frac{2\pi}{|b|}$ is the period and $-\frac{c}{b}$ is the horizontal phase shift and *d* is the vertical shift.

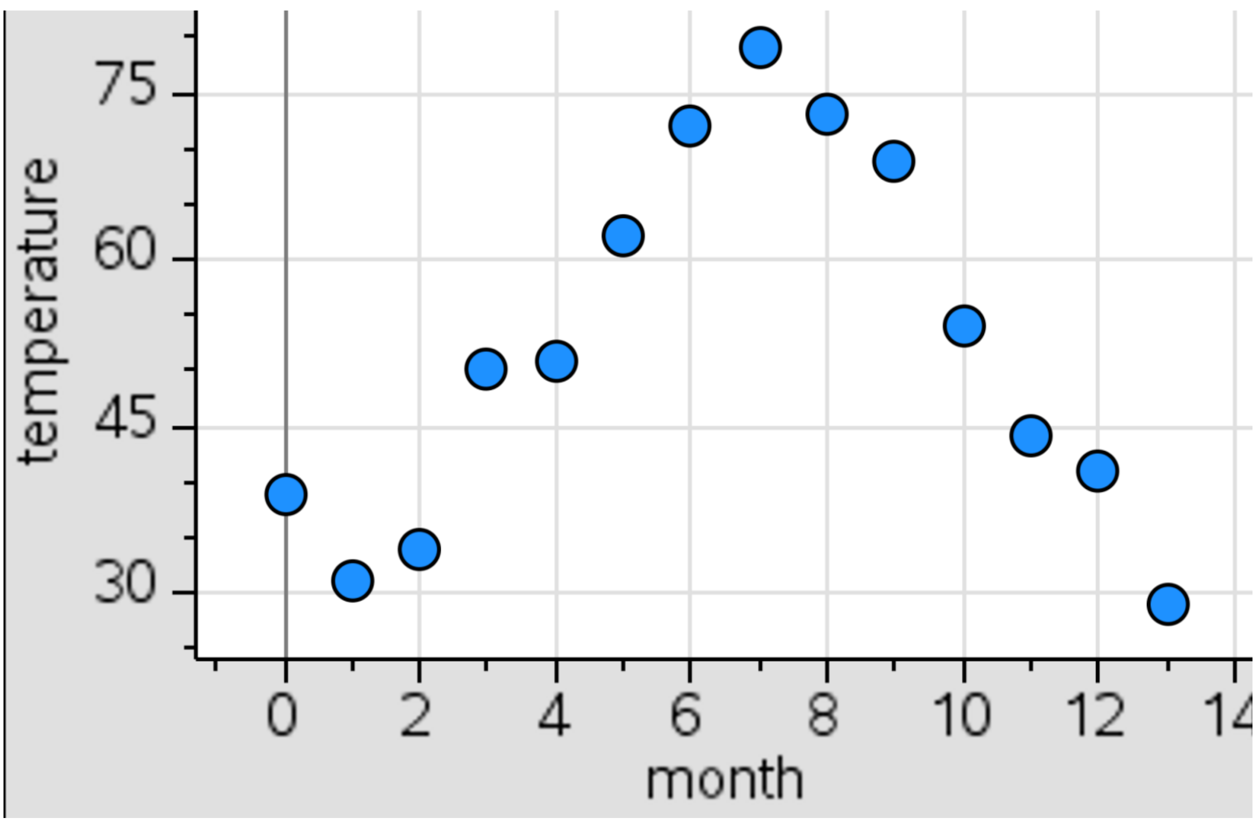
Example 1: Compare $y = 3\sin[(\pi x - 3\pi)] + 1$ to $y = 3\sin[\pi(x - 3)] + 1$

Example 2: Explore $y = \sin(x - 3) + 1$ in terms of $y = asin(x - h) + k$

We will now investigate a real-world application of a periodic function by looking at the temperatures in Dayton, Ohio as reported by Vectren Energy in 2011-2013.

Example 3: Using the table that is a monthly summary of temperatures in Dayton, Ohio from December 2011 to January 2013, find the period, vertical shift, amplitude, and horizontal phase shift and then write the sinusoidal equation that models the data. Using a sinusoidal function to model the data assumes the pattern with repeat in following years. This is a sensible assumption since yearly seasonal changes are cyclical.

Month and Year	**Average Dayton Temperature (F°)**
December 2011	39
January 2012	31
February 2012	34
March 2012	50
April 2012	51
May 2012	62
June 2012	72
July 2012	79
August 2012	73
September 2012	69
October 2012	54
November 2012	44
December 2012	41
January 2013	29



a) What is the period? What is b?

b) What is the vertical shift, d? What is the midline?

c) What is the amplitude, $|a|$?

d) What is the horizontal phase shift, $\frac{c}{b}$?

e) What is the graph of the sine equation that models the graph?

We have seen that the sine wave and cosine wave model phenomena in God's creation, such as lunar phases and ocean tides. Now we have seen that the temperatures also follow the same patterns.

If we were to graph the gas usage in the Dayton area during the same period, we would see that as the temperature goes down, the gas usage goes up, and as the temperature goes up, the gas usage goes down. Therefore, the cosine function is a model for gas usage in Dayton during those same time periods. We will explore the cosine function in the next section. For now, we will further explore the sine function.

In the Practice Problems section there is a unit circle with a radius of 10 cm. (1 decimeter). This circle represents the scale model of a new ride at The Kingdom's Canyon. It is called the Socasinusoid Ferris Wheel. You will be given instructions on what to do with the given unit circle as it pertains to the Socasinusoid Ferris Wheel.

Section 1.2 The Cosine Function

Looking Back 1.2

We have seen that the sine wave and cosine wave model phenomena in God's creation, such as the lunar phases and the ocean's tide.

Only God creates because only He can make something out of nothing. However, man can design intelligently (make something out of something) because man is created in God's image.

Isn't it interesting how the graph of the escape routes of a Ferris wheel looks like a roller coaster? A Ferris wheel goes around and around. The motion is circular, and the vertical distance a seat is from the horizontal platform is periodic. Today, we will investigate the horizontal distances walked on the platform to get to the origin during the escape route.

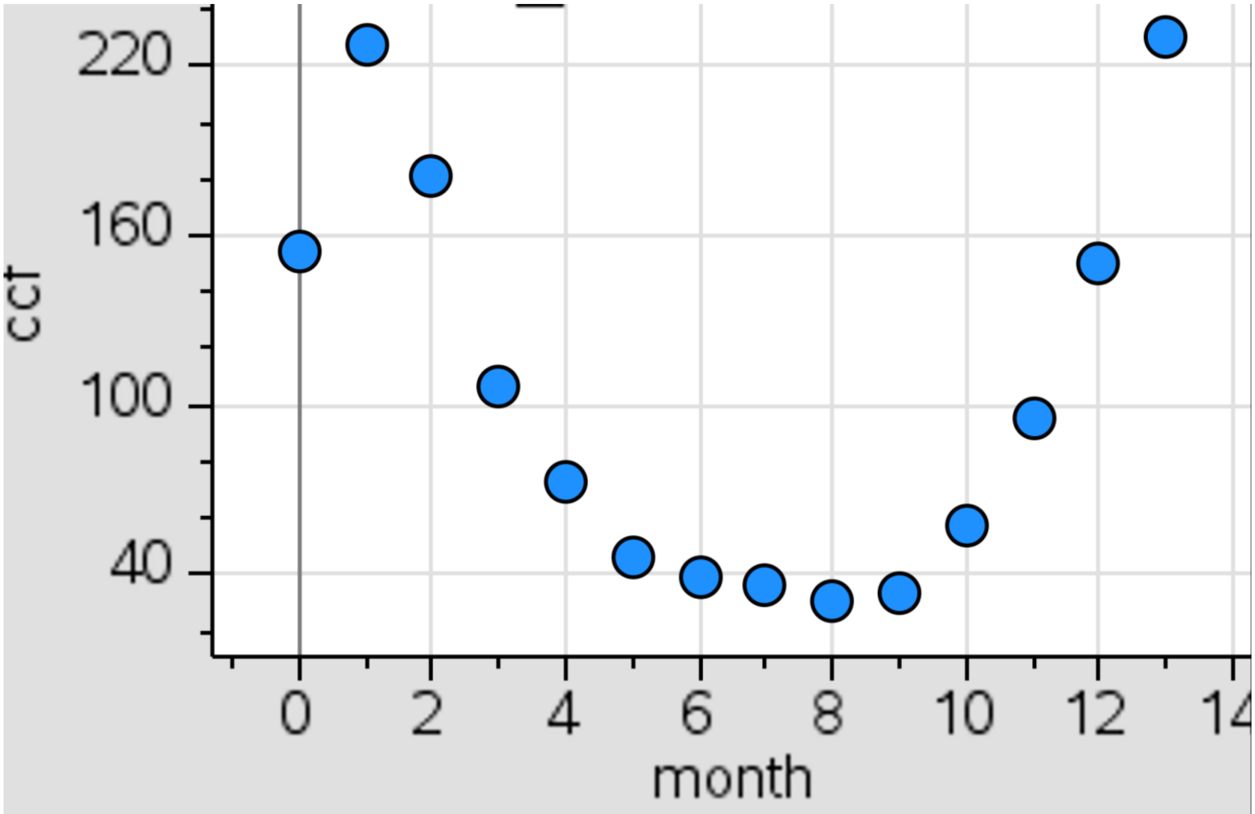
Looking Ahead 1.2

Let us investigate a real-world example of the cosine function using the gas usage in Dayton, Ohio from 2011-2013. A CCF is a volumetric measure of gas use. The natural gas piped to a residence of business is measured in CCF and is 100 cubic feet of gas.

Example 1: Using the table that is a monthly summary of gas usage in Dayton, Ohio from December 2011 to January 2013, find the period, vertical shift, amplitude, and horizontal phase shift and write the sinusoidal equation that models the data. (A cosine equation is also called "sinusoidal" as it is a shift of this function.)

Month and Year	**Average Dayton Gas Usage (CCF)**
December 2011	155
January 2012	228
February 2012	181
March 2012	107
April 2012	73
May 2012	46
June 2012	39
July 2012	37
August 2012	31
September 2012	33
October 2012	58
November 2012	95
December 2012	150
January 2013	230

Below is the graph of the gas usage for Dayton from the table above. The x-axis represents the month and the y-axis represents the amount of CCF gas used during that month. Answer the following questions and find the cosine equation that best models the function.



a) What is the period. What is *b*?

b) What is the vertical shift, *d*? What is the mid-line?

c) What is the amplitude, |*a*|?

d) What is the horizontal phase shift, *c*?

e) What is the equation of the cosine function that models the graph?

You will use the circle you constructed in the previous Practice Problems for today's problems. The scale model of the angle of each seat for the Socasinusoid Ferris Wheel remains the same, but you will measure the horizontal distances walked to the center to exit once the horizontal platform is reached. Complete the chart and graph and answer the questions representing the cosine of the seat angle.

Section 1.3 Socasinusoidal Ferris Wheel

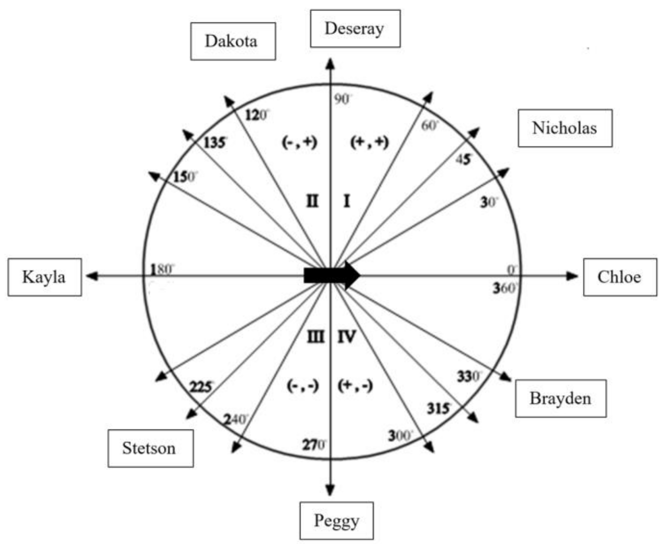
Looking Back 1.3

You have drawn a sine graph to find the distances that riders must walk vertically to get to the ride platform and the centrally located exit in case of an emergency. You have also drawn a cosine graph to find the distances that riders must walk horizontally to get to the centrally located exit in case of an emergency once they reach the platform. These graphs modeled the distance at each seat of a Socasinusoid Ferris wheel measured in degrees counterclockwise from the platform around the Ferris wheel.

Looking Ahead 1.3

In today's Practice Problems section, you will answer real-world questions about what the various locations on the sine graphs, cosine graphs, and Ferris wheel (represented by the unit circle) mean. However, we will first do a brief examination of the riders on the Ferris wheel.

Deseray, Nicholas, Chloe, Brayden, Peggy, Stetson, Kayla, and Dakota are riding the Socasinusoid Ferris Wheel. Imagine the angle markings remain fixed while the riders rotate counterclockwise. For example, if Nicholas rotates to the 60° position, then Deseray will rotate to the 120° position. The 0° position marks where the riders may enter or leave the ride and is currently occupied by Chloe. Use the diagram below to answer the following questions:



a) The ride made three complete circles before stopping at the same spot to let Chloe off. How many degrees did the Ferris wheel rotate?

b) If the ride spins counterclockwise four and five-eighths times before stopping, who would be at the first seat (stop position)?

c) Tell the degree of the seat angle of each person on the ride when Chloe is being seated.

Section 1.4 The Unit Circle

Looking Back 1.4

You first learned about the unit circle in Geometry and Trigonometry when you used tin cans to build it from scratch, so to speak. You have also used a paper circle and Smartees® to measure the circle radius and you were able to convert the arcs of the circle from Smartee® fractions to arcs of the circle measured in Smartee® decimals.

Looking back to the tin can activity, we can see that one paper radius went around the tin can about 6.28 times. This worked for any size tin can. We used this 6.28 radians for the circumference of any circle ($\pi \approx 3.14\ldots$ and $3.14(2) = 6.28$), but this is an approximation. The exact number of radians around the circumference of a circle is 2π radians. Halfway around is π radians, which is the same as 180°.

Using either of the ratios below allows you to convert from radians to degrees:

$$\frac{\text{radians}}{\text{degrees}} \qquad \frac{\pi}{180^\circ} \quad \frac{2\pi}{360^\circ} \qquad \frac{\text{radians}}{\text{degrees}}$$

Looking Ahead 1.4

Example 1:	Convert the degrees to radians.

a) 330°

b) 120°

Example 2:	Convert the radians to degrees.

a) $\frac{\pi}{4}$

b) $\frac{5\pi}{3}$

You have learned about the importance of the 30° - 60° - 90° triangle and the 45° - 45° - 90° triangle in Geometry and Trigonometry. There are many patterns in the unit circle to help you remember the radians of each ordered pair around it.

Below is another method for you to use to find sine and cosine. With this method, you use your hand.

Use your hand to find sine and cosine:

*Cosine on top: alphabetically first (count top fingers)
*Sine on bottom: alphabetically last (count bottom fingers)

Top numbers are radicals
Bottom numbers are 2

Count the fingers above or below the angle represented.

30° is represented by the 2nd finger (ring finger) folded in

Cosine is on top. Three fingers are on top. It is $\frac{\sqrt{3}}{2}$.
Sine is on bottom. One finger is on bottom. It is $\frac{1}{2}$.

45° is represented by the 3rd finger (middle finger) folded in

Cosine is on top. Two fingers are on top. It is $\frac{\sqrt{2}}{2}$.

Sine is on bottom. Two fingers are on bottom. It is $\frac{\sqrt{2}}{2}$.

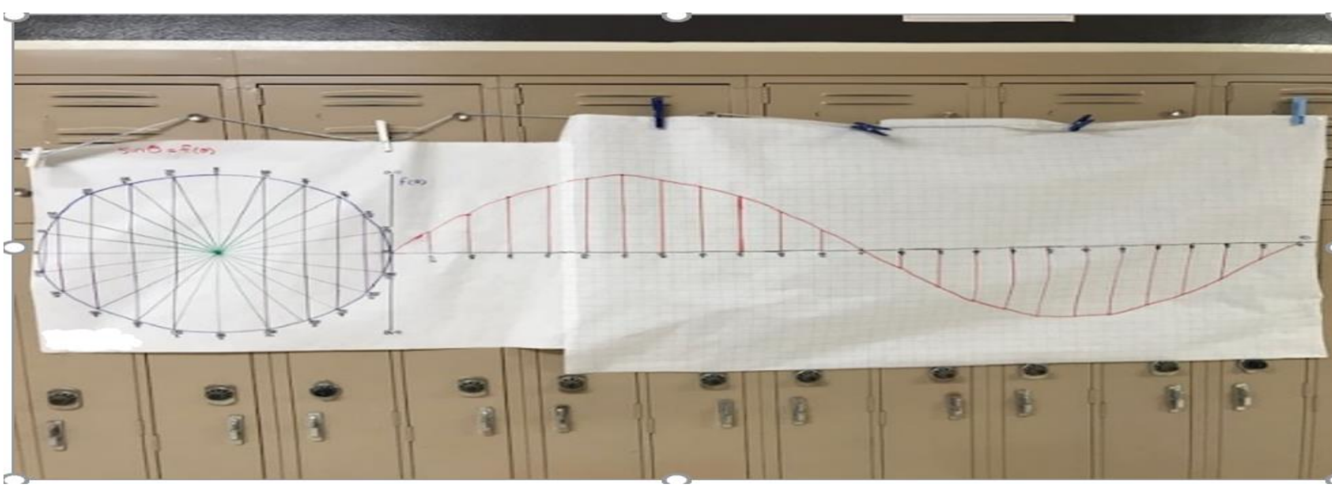
60° is represented by the 4th finger (index finger) folded in

Cosine is on top. One finger is on top. It is $\frac{1}{2}$.

Sine is on bottom. Three fingers are on bottom. It is $\frac{\sqrt{3}}{2}$.

Tangent θ is $\frac{\sin\theta}{\cos\theta}$ ($\tan\theta = \frac{\sin\theta}{\cos\theta}$) so fold your finger in for the angle and then flip your hand over. Radicals are on top and bottom. So, for $\tan 60^{\circ}$, fold in your 4th finger and flip your hand over. There are three fingers on top and one on bottom, so tangent is $\frac{\sqrt{3}}{\sqrt{1}} = \sqrt{3}$: $\tan 60^{\circ} = \sqrt{3}$.

Along with all you have learned previously and all you know about God's patterns in nature as well as this new hand method, complete the unit circle in the Practice Problems section.



Section 1.5 Toothpick Curves

Looking Back 1.5

When you investigated the Socasinusoid Ferris wheel you built the sine and cosine curve to measure vertical and horizontal distances.

The unit circle, sine and cosine, and trigonometric functions are all about ratios. Their principles hold true whether the radius is the radius of one small tin can or the radius of one large tin can. In this section, we will be building the sine and cosine curve using toothpicks. This will show us how the principles of these functions will hold true.

Looking Ahead 1.5

Follow the instructions below on a blank white sheet of paper to build a toothpick sine curve:

1. At one end of the paper, construct a circle on a set of axes, which has a radius of one toothpick length.
2. Use a protractor to measure and mark every $15°$ around the circle.
3. To the right of the circle and adjacent to the center of the circle, draw an x-axis that is about 6.5 toothpicks in length. Draw the y-axis to the left of the start of the x-axis.
4. Using a string, measure the distance from $0°$ to $15°$. Use this distance to mark these intervals on the x-axis from $0°$ to $360°$.
5. Place a toothpick from the origin to the $15°$ mark on the circle. Take another toothpick and break it to measure the vertical distance from the "Hypotenuse" (toothpick radius) to the x-axis. Take this measurement and transfer it to the vertical axis. Place the vertical piece perpendicular to the x-axis above the $15°$ mark and place a dot at the end of the toothpick to record the height of the toothpick.
6. Continue the process from Step 5 until you have gone completely around the circle at each $15°$ mark.
7. Draw a smooth curve to connect your dots.
8. You have now graphed the sine curve. Take out your calculator and graph $y = \sin(x)$. Does it look like the graph you have drawn?
9. Now try to graph the cosine curve. What values do you need to transfer to the axes this time?

Section 1.6 Trigonometric and Parametric Equations

Looking Back 1.6

In the unit circle, there are two things going on at once as the central angle (θ) increases counterclockwise around the circle. The y-value (height of the constructed triangles that have a hypotenuse of one radian) is changing (increasing and decreasing periodically), and the x-values (width of the constructed triangles that have a hypotenuse of one radian) is changing (decreasing and increasing periodically).

If we investigate just the changes in the height of y as θ increases, we get the graph of a sine curve. If we investigate just the changes in the width of x as θ increases, we get the graph of a cosine curve.

What happens if we explore both changes at one time? Let us try it and see!

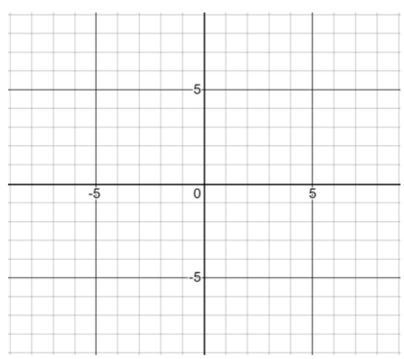
The best way to investigate two things happening at one time is to use a helping parameter, often called t (because t often represents time as the independent variable). This helping parameter is the hallmark of parametric equations. Both x and y become dependent variables because they depend on the helping parameter, t. For our purposes, t does not represent time; it represents our central angle, θ, which is our helping parameter (independent variable). The two dependent variables are $x(t) = \cos(t)$ and $y(t) = \sin(t)$.

In this Practice Problems section, we are going to work backwards from the activity done in the previous section. The graphing calculator will make our work much easier if we use the parametric mode. Follow the instructions in the Practice Problems section to graph the parametric equations and answer the questions. For now, we will investigate simpler equations.

Looking Ahead 1.6

A method to define a relation in x and y is to use functions written as parametric equations in which the input value for each is the helping parameter, t, and the output values are the x and y. This is called parameterizing an equation. In the Practice Problems you will parameterize the unit circle.

Example 1:	Write and graph the two parametric



Example 2:	Write and graph the two parametric equations for $y^2 = x$.

Section 1.7 Periodic Functions

Looking Back 1.7

In the last section, you learned about parametric equations. You will work with these again in upcoming modules when you study vectors and polar equations.

For now, we have discovered that sine and cosine are periodic. A function is periodic if there is a positive real number c, such that $f(x+c) = f(x)$ for all real numbers x. The smallest number c is the period of the function. The function repeats at all intervals of c.

For the sine function, 2π is the smallest number such that $\sin(x+2\pi) = \sin x$ for all numbers x. For the cosine function, 2π is the smallest number such that $\cos(x+2\pi) = \cos x$ for all numbers x. The period of the tangent function is π; therefore, $\tan(x+\pi) = \tan x$ for all numbers x. This can all be written $\tan(x+\pi n) = \tan x$ where n is any integer. Similarly: $\cos x = \cos(x+2\pi n)$ and $\sin x = \sin(x+2\pi n)$, where for any integer n.

Since $x = \theta$ for the central angle, the theorems can be written as follows:

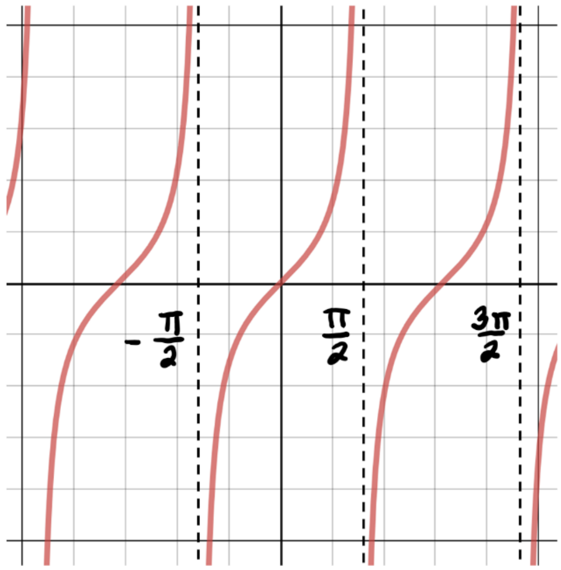
$$\sin(\theta + 2\pi n) = \sin\theta$$
$$\cos(\theta + 2\pi n) = \cos\theta$$
$$\tan(\theta + 2\pi n) = \tan\theta$$

Looking Ahead 1.7

Example 1: If $\cos(0.9991) \approx 0.54159$, find four other values of x where $\cos x \approx 0.54159$.

Example 2: Find two other values where $\tan x = 3$. Solve for x.

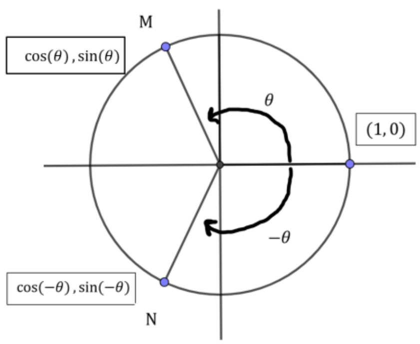
Example 3:	Using the tangent function, what is the equation for the asymptotes of the graph below.



Section 1.8 Trigonometric Identities

Looking Back 1.8

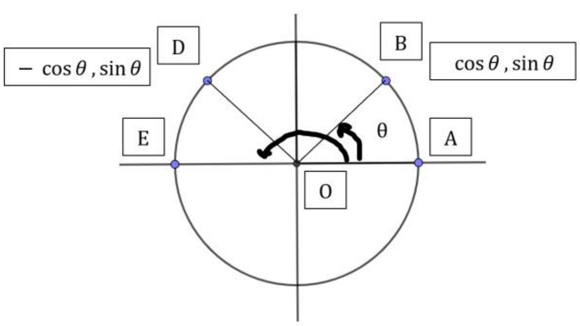
In the last section, the cosine function was the only function that was symmetric over the y-axis. Since $f(-x) = f(x)$ for all values of x in an even function, cosine is an even function! The sine and tangent functions are symmetric about the origin. This makes them odd functions because $f(-x) = -f(x)$ for all values of x.
For every value of θ, $\cos(-\theta) = \cos\theta$, $\sin(-\theta) = -\sin\theta$, and $\tan(-\theta) = -\tan\theta$. This is illustrated on the unit circle below:

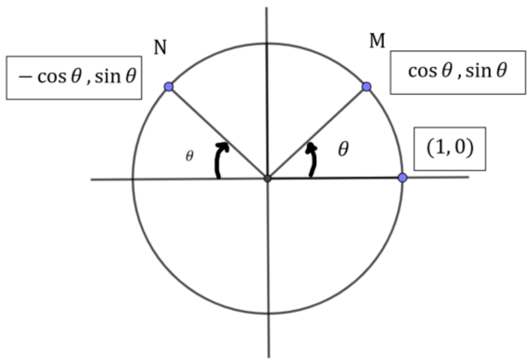


If $(1, 0)$ is the pre-image, then M$(\cos(\theta), \sin(\theta))$ is the image over a rotation of θ, and N$(\cos(-\theta), \sin(-\theta))$ is the image over a rotation of $-\theta$. The points M and N are reflected over the x-axis. The cosines are equal, and the sines are opposites. The x-coordinates are equal (cosines), and the y-coordinates are opposites (sines). Since $\tan\theta$ is equal to $\frac{y}{x}$, which is a ratio of y-coordinates to x-coordinates, the tangents are opposites as well.

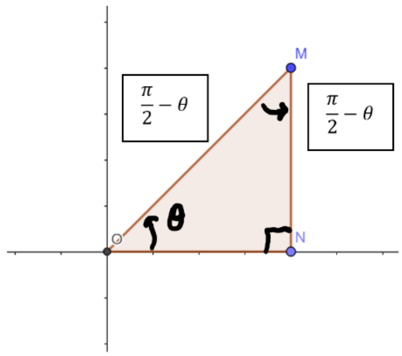
Looking Ahead 1.8

Example 1: Given the first diagram below what can you conclude about how $\sin(\theta)$ and $\cos(\theta)$ relate to $\sin(\pi - \theta)$ and $\cos(\pi - \theta)$?
In the second diagram below, the points M and N which are reflections of each other over the y-axis. We may therefore conclude that if point M is at angular position θ, then N must be at angular position $\pi - \theta$. What can you conclude about how $\tan(\theta)$ relates to $\tan(\pi - \theta)$?





Example 2: Note the diagram below which shows a right triangle with a hypotenuse of length 1. If we designate one acute angle of the triangle θ , then we know the opposite and adjacent legs of the triangle must be of length $\sin(\theta)$ and $\cos(\theta)$, respectively. The other acute angle is known as the complement of θ, and must be of size $\frac{\pi}{2} - \theta$. Given the diagram, what can you conclude about how the sines and cosines of complementary angles relate?



Example 3: Since the Pythagorean Theorem states that $x^2 + y^2 = r^2$, the trigonometric Pythagorean Identity states that $\cos^2\theta + \sin^2\theta = 1$ for every θ in the unit circle. If $\sin\theta = \frac{4}{5}$, find $\cos\theta$.

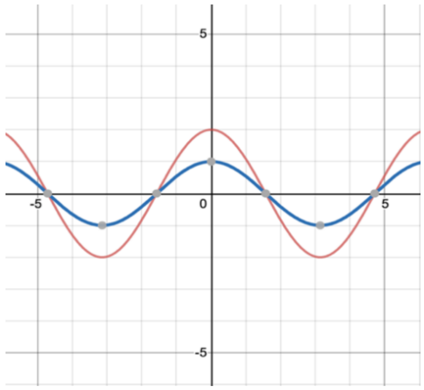
Section 1.9 Transformations of Periodic Functions

Looking Back 1.9

Sound travels in sine waves. The graph for sound waves can be the sine or cosine function after it has been translated or undergone a change. Remember, the amplitude is one-half of the absolute value of the difference between the maximum and minimum values of the wave created by the function.

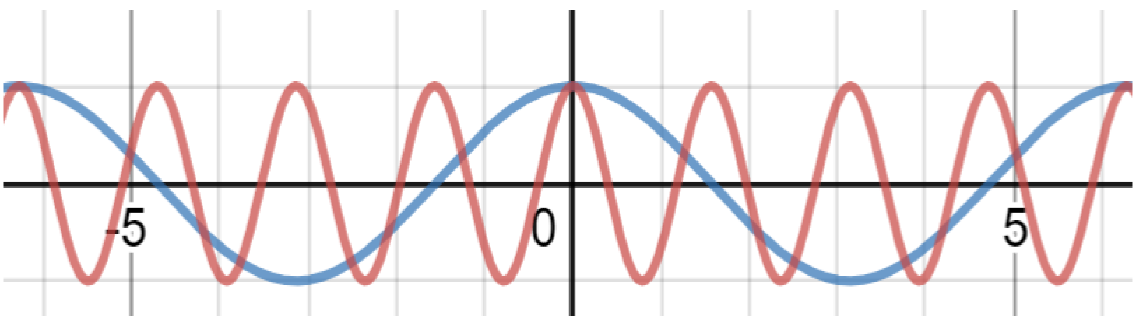
For $y = \cos x$ and $y = \sin x$ the maximum value is 1 and the minimum value is -1. Therefore, the amplitude is also 1, which comes from $\frac{1-(-1)}{2} = \frac{2}{2} = 1$.

When $y = \cos x$ becomes $y = 2\ \cos x$ the amplitude becomes 2 as it is double $\cos x$. Notice, the period of the function does not change.



There has been a vertical stretch from the parent function. Therefore, $y = \frac{1}{2} \cos x$ has an amplitude of $\frac{1}{2}$ and is a vertical compression.

However, if $y = \cos x$ becomes $y = \cos 2x$, the period is now π. This is from $\frac{2\pi}{|2|} = \pi$. Where there was one wave in a period of 2π, now there is one wave in a period of π. Therefore, in 2π there are two full waves. The amplitude has not changed. The period has changed; the frequency of waves has increased over 2π.



Red: $y = \cos 4x$ Blue: $y = \cos x$

In sound waves, the higher pitched the sound, the shorter the period of the wave. The lower pitched the sound, the longer the period of the wave. Thus, the higher a sine wave's frequency, the shorter the period of the sine wave, and the lower a sine wave's frequency, the longer the period of the sine wave.

The frequency represents the number of cycles per units of time. When the frequency of sound waves double it becomes a pitch one octave higher. Therefore, quadrupling the frequency would become a pitch two octaves higher.

The frequency is the reciprocal of the period. One wave represents one cycle before the wave repeats. So, in the graph above, the frequency is 4 occurrences per (2π) period, and the period is $\frac{2\pi}{4}$. The period is the duration of the wave, and the frequency is the rate of occurrence. Frequency is one event per unit of time. The period is the duration of time for one cycle so the period is the reciprocal of the frequency. They are inversely related.

<u>Looking Ahead 1.9</u>

Example 1: In the equation $y = -3\sin(\frac{x}{4})$…

a) Find the amplitude.

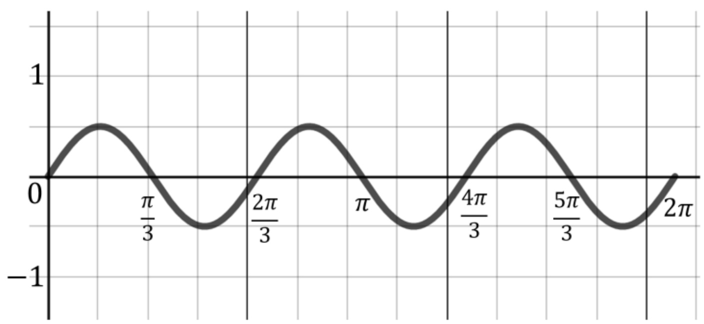
b) Find the period.

c) Find the frequency.

d) Is there a reflection in the x-axis?

e) Draw the graph of $y = -3\sin(\frac{x}{4})$.

Example 2: The graph below is a transformation of the graph of $y = \sin x$.



a) Find the amplitude.

b) Find the period.

c) Find the frequency.

d) Is there a reflection in the x-axis given the graph models a sine function?

e) Is there a horizontal phase shift?

f) What is the equation for the graph?

The functions defined by $y = a\sin bx$ and $y = a\cos bx$ have an amplitude of $|a|$ and a period of $\frac{2\pi}{|b|}$. The functions defined by $y = a\sin(\frac{x}{b})$ and $y = a\cos(\frac{x}{b})$ have an amplitude of $|a|$ and a period of $2\pi \cdot |b|$. Functions defined by $y = a\sin[\,b(x-c)]$ or $y = a\sin[b(x+c)]$ have a right and left phase shift of c respectively, as we have previously seen. Functions defined by $y = a\cos bx + d$ or $y = a\cos bx - d$ have an up and down vertical shift of d respectively, as we have also previously seen.

Section 1.10 Compositions of Trigonometric Functions

Looking Back 1.10

If you have two functions, f and g, and they are composed together, they become a new function. This composition of functions is a composite of f and g, which is written "$f \circ g$."

$$(f \circ g)(x) = f(g(x))$$

The set of all possible values x may take constitutes the domain of the function $g(x)$. As $g(x)$ now serves as the input of the function f, the range of the function $g(x)$ also becomes the domain of the function $f(g(x))$. The rule is to work innermost to outermost.

If $f(x) = x^2 + 3$ and $g(x) = x - 4$ then $f\big(g(x)\big) = f(x - 4)$.

$$f(x-4) = (x-4)^2 + 3$$
$$f(x-4) = x^2 - 8x + 16 + 3$$
$$f(x-4) = x^2 - 8x + 19$$

$$g\big(f(x)\big) = g(x^2 + 3)$$
$$g(x^2+3) = (x^2+3) - 4$$
$$g(x^2+3) = x^2 - 4$$

Looking Ahead 1.10

Let's investigate compositions of trigonometric functions.

Example 1: Let $f(x) = \sin x$ and $g(x) = \cos x$. Is $f\big(g(x)\big)$ equal to $g(f(x))$?

$$f\big(g(x)\big) = f(\cos x)$$
$$f(\cos x) = \_\_\_\_\_\_\_(\_\_\_\_\_\_\_)$$

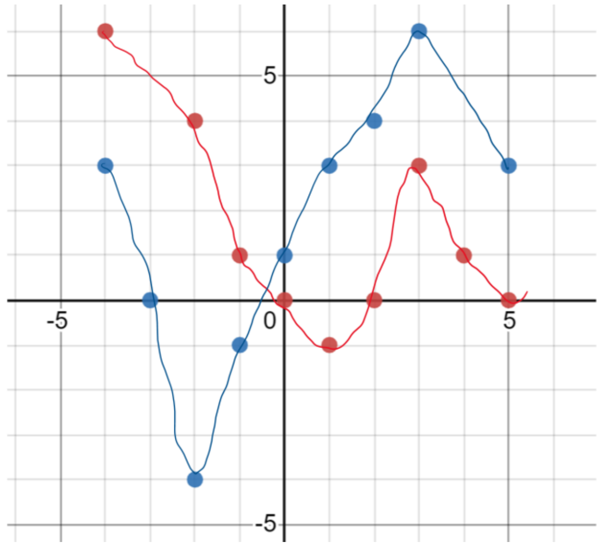
$$g\big(f(x)\big) = g(\sin x)$$
$$g(\sin x) = \cos(\_\_\_\_\_\_\_\_\ x)$$

Example 2: Let $g(x) = \cos x$ and $h(x) = \tan x$. Is $(g \circ h)(x)$ the same as $(h \circ g)(x)$? Use the graphing calculator to draw the graphs of the composite functions.

$$(g \circ h)x = g(\tan x)$$
$$g(\tan x) = \_\_\_\_\_\_\_\_(\_\_\_\_\_\_\_\_)$$

$$(h \circ g)x = h(\cos x)$$
$$h(\cos x) = \_\_\_\_\_\_\_\_(\_\_\_\_\_\_\_\_\ x)$$

Example 3: Use the graph below to find the value of each composition of functions. The red graph is $g(x)$ and the blue graph is $f(x)$.



a) $(f \circ g)(5)$

b) $(g \circ f)(-2)$

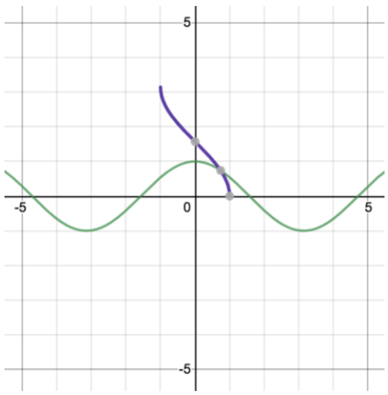
c) $(f \circ g \circ f)(2)$

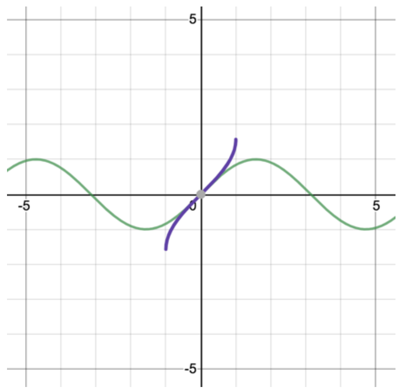
d) $(g \circ g)(-1)$

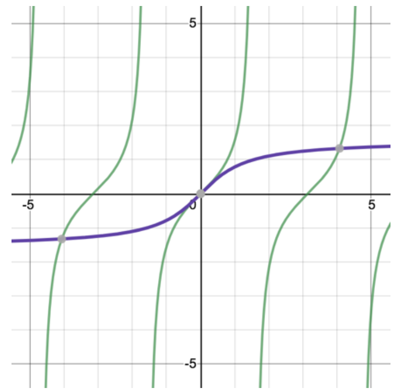
Section 1.11 Inverse Circular Functions

Looking Back 1.11

The inverse of the cosine, sine, and tangent functions are given below by reflecting the points of the graph over the line $y = x$.

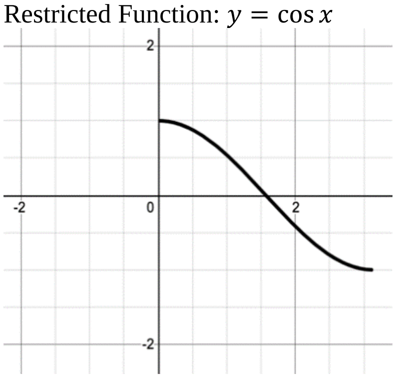






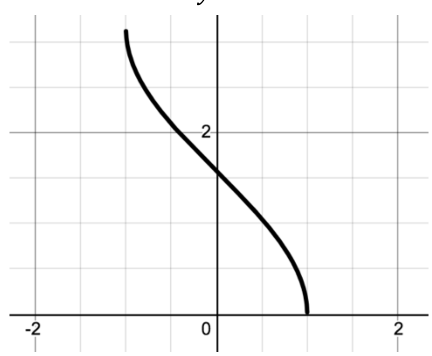
The inverse curves produced are not functions, as they assign multiple outputs to one input. (They do not pass the vertical line test.) We may make them functions by only reflecting a restricted portion of the original trigonometric function to ensure no ambiguity. The inverse trigonometric function keys on your calculator give values according to such restricted functions. The standard restriction for cosine, and the associated inverse function, are shown below.

Restricted Function: $y = \cos x$



Domain: $0 \leq x \leq \pi$
Range: $-1 \leq y \leq 1$

Inverse Function: $y = \cos^{-1} x$

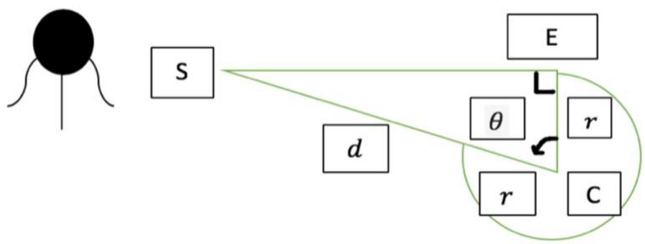


Domain: $-1 \leq x \leq 1$
Range: $0 \leq y \leq \pi$

You will be drawing the graphs of $y = \sin^{-1} x$ and $y = \tan^{-1} x$ in the Practice Problems section.

Looking Ahead 1.11

Example 1: A satellite circling the earth takes pictures of the top portion of the earth, which appears in the view screen. The point C is the center of the earth and r is the radius of the earth. The angle ∠ECS is the angle θ formed from the point on the earth's horizon to the center of the earth to the satellite. If the distance (d) of the satellite above the earth is $1{,}450$ km. and the radius of the earth is approximately $6{,}380$ km., then what is the measure of the angle θ?



Example 2: Use the inverse trigonometric functions and reverse thinking to evaluate $\tan(\sin^{-1}0.6)$.

Section 1.12 Reciprocal Trigonometric Functions

Looking Back 1.12

Reciprocal trigonometric functions are not the same as inverse functions. The reciprocal of $\cos\theta$ is $\frac{1}{\cos\theta}$ and is called secant of θ ($\sec\theta$), where $\cos\theta \neq 0$. The reciprocal of $\sin\theta$ is $\frac{1}{\sin\theta}$ and is called cosecant of θ ($\csc\theta$), where $\sin\theta \neq 0$. The reciprocal of $\tan\theta$ is $\frac{1}{\tan\theta}$ or $\frac{\cos\theta}{\sin\theta}$ and is called cotangent of θ ($\cot\theta$), where $\sin\theta \neq 0$. The symbol θ represents any real number.

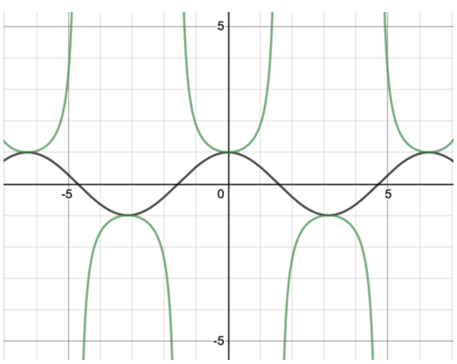
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \text{ and } \frac{1}{\tan\theta} = \frac{1}{\frac{\sin\theta}{\cos\theta}} \text{ or } \frac{\cos\theta}{\sin\theta}, \text{ where } \sin\theta \neq 0 \text{ and } \cos\theta \neq 0$$

Looking Ahead 1.12

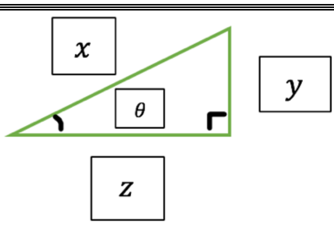
Example 1: Find the exact value of $\sec\frac{\pi}{6}$.

Example 2: Find the approximate value of $\csc\frac{\pi}{3}$.

Example 3: The graph of $y = \cos x$ is in black and the graph of $y = \sec x$ is in green. For $y = \sec x$, identify the domain, the range, the period, and the maximum and the minimum.



Example 4: Use the right triangle to express the trigonometric functions in terms of x, y, and z.



a) $\csc(90° - \theta)$

b) $\sec\theta$

Would it be fair to say that $\csc(90° - \theta) = \sec\ \theta$ for all right triangles?

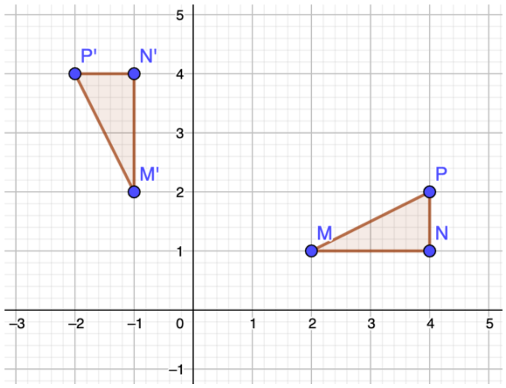
Section 1.13 Matrices and More Trigonometric Identities

Looking Back 1.13

In a rectangular coordinate grid, translations are easier to perform than rotations. In a polar coordinate grid, as you will explore later, rotations are easier than translations.

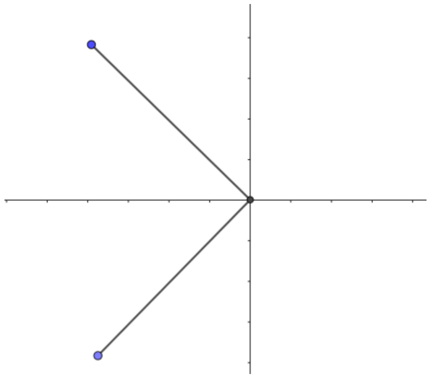
It will be much easier to find points after rotations once you can combine what you have learned about matrices in Algebra 2 with what you have learned in Geometry and Trigonometry.

Example 1: Look at the pre-image of the triangle MNP with the image of $M'N'P'$ after a 90° rotation counterclockwise about the origin. Looking at the points you can see that $(x, y) \to (-y, x)$. Find the matrix for this rotation and apply it to the points.



$M(2, 1)$ $M'(-1, 2)$
$N(4, 1)$ $N'(-1, 4)$
$P(4, 2)$ $P'(-2, 4)$

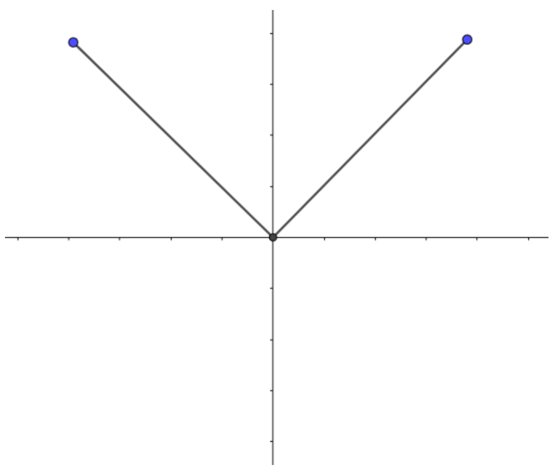
Example 2: Using the graph below, find the matrix for rotating an angle θ counterclockwise 180° and 270° and θ° about the origin.



Example 3: Use the rotating matrix in Example 2 to find the matrix of rotating an angle 90° counterclockwise.

Looking Ahead 1.13

Example 4: Find the matrix of a rotation of α followed by a rotation of β.



Example 5: In problems of mathematics and science, $\sin 2\theta$ and $\cos 2\theta$ occur often. Functions using these are called Double Angle Identities. By applying the sum and difference identities for sine and cosine, derive the double angle identities for $\sin(2\theta)$ and $\cos(2\theta)$.

Example 6: In science and mathematics, the double angle theorem can be used to find the horizontal distance d a ball travels at an initial angle of θ degrees to the ground where v is velocity (m/sec.) and g is acceleration due to gravity (m/sec.^{2}).

How far will a kickball travel with respect to the ground if it is kicked at a $30°$ angle to the ground at a velocity of 28 m/sec.? Use the formula $d = \frac{v^2 \sin 2\theta}{g}$ to solve for the horizontal distance the ball travels.