

**Pre-Calculus and Calculus Module 3 Vectors and Parametric Equations**Section 3.1 Introduction to VectorsLooking Back 3.1

A vector is a quantity that has both magnitude and direction. Magnitude is a number representing quantity, size, or the greatness of something. It is the length of the directed line segment.

Position usually is given on a map using longitude or latitude. We can use North, South, East, West, or a combination of them: Northeast, Southeast, Northwest, Southwest to give directions. The directed line segment represents the vector's direction.

Vectors can be used in navigation to give the bearing (the angle in degrees measured clockwise from due North) a ship is traveling.

Moreover, vectors can be used to describe real-world phenomena such as the direction and force of the wind. They can also be used to describe force, distance, and speed in physics. Vectors are introduced here but can be used later as we study velocity in the Calculus portion of this text.

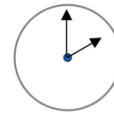
We are told in the scriptures that the magnitude of God's love knows no bounds; it is limitless. In Psalm 103:12, we are told: "As far as the east is from the west, So far has He removed our transgressions from us." If this verse had used "north" and "south" rather than "east" and "west," we would have known it is a defined distance because once one reaches the north pole, you start heading south. However, He used "east" and "west" because it is an undefined distance. How far do we travel east before we are heading west? It is limitless, just as His love and forgiveness are for us.

Looking Ahead 3.1

A scalar is a quantity that has magnitude, but no direction. Examples of scalars are time, temperature, length, mass, miles per hour, and any real number.

Examples of Scalars:

2:00 p.m.



43°F



23.6 cm



6.02 g



65 mph



A vector has both magnitude and direction. Examples of vector quantities are force, displacement, velocity, and acceleration. We will study displacement, velocity, and acceleration in great detail when we get to the Calculus portion of this text. For now, let us look at force (specifically weight); we will compare it to mass.

Example 1: A tennis ball is hollow, made of inflated rubber, and covered with fabric. It weighs about  $2\frac{1}{16}$  ounces on earth, which is about 0.13 pounds. How much does a tennis ball weigh on Mars? Acceleration due to gravity on Mars is  $3.711 \text{ m/s}^2$ , but on earth it is  $9.81 \text{ m/s}^2$ .

Weight is a measure of force. It measures how hard gravity is pulling on an object; therefore, the weight of the tennis ball changes from planet to planet. The weight of the tennis ball is less on Mars so Mars must have less gravity. Weight is a vector quantity, and because it is a measure of force, force is also a vector quantity.

The English unit for mass is the slug, and the English unit for force is  $\frac{\text{slug}\cdot\text{ft}}{\text{sec}^2}$ . The English unit for displacement is foot, and the English unit for time is seconds. When you multiply displacement by mass and divide by a time unit squared, you get force.

You may not have heard the term slug before because  $\frac{\text{slug}\cdot\text{ft}}{\text{sec}^2} = \text{pound}$ , and “pound” is something you have likely heard quite often. So, pound is a measure of weight and weight is a measure of force.

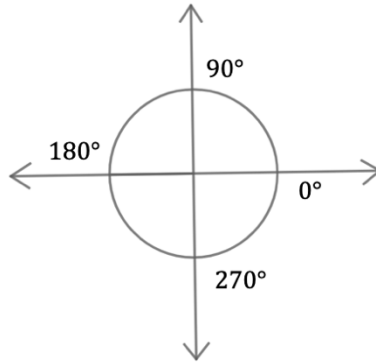
Example 2: You will want to stand up and face your computer the entire time you perform the following activity:

- Step 1 Move right 2 steps
- Step 2 Move back 2 steps
- Step 3 Move left 2 steps
- Step 4 Move forward 2 steps

Are you back where you started?

Let us draw the vectors for each step using a coordinate grid. (Think about the work you did in Geometry and Trigonometry when you placed the unit circle on the coordinate grid centered at  $(0, 0)$ .)

A move right from the origin along the  $x$ -axis is a terminal angle of  $0^\circ$  and a move left from the origin along the  $x$ -axis is a terminal angle of  $180^\circ$ . A move up from the origin along the  $y$ -axis is a terminal angle of  $90^\circ$  and a move down from the origin along the  $y$ -axis is a terminal angle of  $270^\circ$ .



Using vector notation, the first component represents the horizontal movement, and the second component represents the vertical movement. The start does not always begin at the origin, but can begin anywhere on the grid. If you used vectors to represent the movements you made, you would have the following steps:

- Step 1  $\langle 2, 0 \rangle$
- Step 2  $\langle 0, -2 \rangle$
- Step 3  $\langle -2, 0 \rangle$
- Step 4  $\langle 0, 2 \rangle$

The angle could represent the direction you moved. The number of steps (with each step being one unit) could represent the magnitude. Now, you would have the following steps with angle and magnitude shown:

- Step 1  $0^\circ$  and 2
- Step 2  $270^\circ$  and 2
- Step 3  $180^\circ$  and 2
- Step 4  $90^\circ$  and 2

Notice that when you used vectors to write the movements, sometimes 2 was positive and sometimes 2 was negative. That is because vectors have both magnitude and direction. Positive can mean up or right and negative can mean down or left.

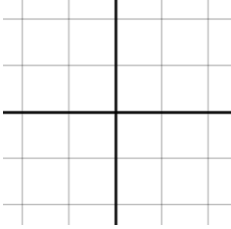
When you wrote the steps as angles and magnitude, then the angles gave the direction and the 2 represented magnitude (the length of the steps).

Example 3: Use Example 2 to answer the following questions:
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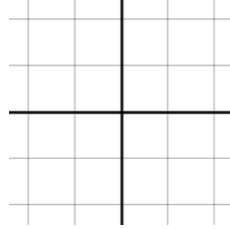
- 1) Which steps had the same magnitude?
  
  
  
  
  
  
  
  
  
  
- 2) Which steps had the same direction?

Example 4: Draw the sketch of each of the vectors on the coordinate grids given:

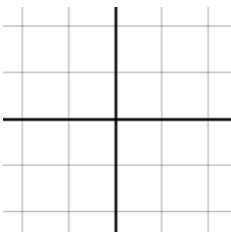
Step 1:



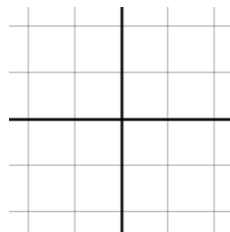
Step 2:



Step 3:



Step 4:



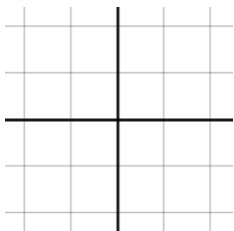
It does not matter where you start. The direction and distance are what is important.

Example 5: 1) Draw the vector for the following directions:  
 Move up 2 and left 2.

2) Write the steps using vector notation.

3) Write the steps using angle and magnitude.

1)



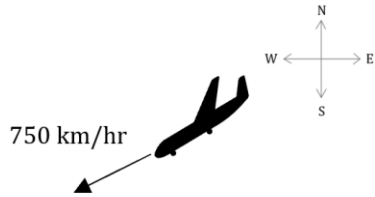
2)

3)

Section 3.2 Vector Notation

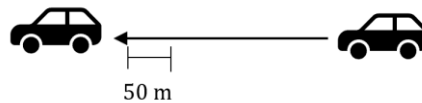
Looking Back 3.2

As we saw in the last section, the graph of a vector can be represented by a ray that has magnitude (size/length) and direction, whose angle is measured from the positive  $x$ -axis. On the other hand, in the case of navigation, the angle is measured from due north.



The velocity of an airplane is given by the plane's speed and direction. For example, we can say a plane is traveling at 750 km/hr southwest.

Therefore, the velocity of the plane is a vector quantity. The arrow represents a length or size of the vector that is proportional to the magnitude of the quantity. This may be indicated by a scale.



The car in the diagram above has a displacement of 250 miles due west.

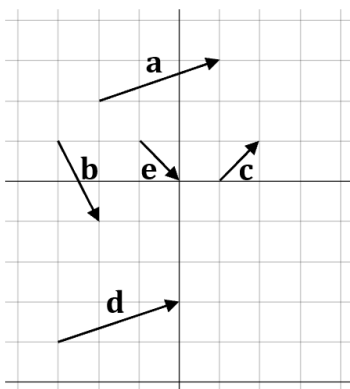
Looking Ahead 3.2

The symbol  $\overrightarrow{HI}$  represents the vector extending from point H to point I. Point H is called the beginning or initial point (tail). Point I is called the endpoint or terminal point (head). A vector can be named by one letter with a bar over it: for example,  $\vec{u}$  or  $\vec{v}$ . If it is two letters, it can be named with an incomplete arrow over it: for example,  $\overrightarrow{HI}$ . If the vector were written  $\overleftarrow{IH}$ , then I is the initial point (tail) and H is the terminal point (head). The incomplete arrow indicates the direction of the vector.

In textbooks, vectors can be written with bold lowercase letters: for example,  $\mathbf{u}$  or  $\mathbf{v}$ . In handwritten work, vectors can be written with a bar or an unfinished arrow over top the letter: for example,  $\vec{u}$  or  $\vec{v}$ .

Example 1:      1) In the graph below, name a vector that is equivalent to vector **a**.

                    2) Draw a vector that is equivalent to **b**. Name it **f**.



Are vector **c** and vector **e** equivalent vectors?

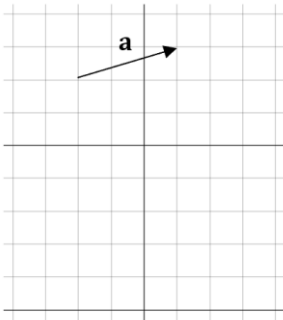
Vectors can also be written in component form as was used in the previous section. The component form for 2 units up is  $\langle 0, 2 \rangle$  (component form is  $\langle x, y \rangle$ ).

The first component of the vector,  $x$ , represents the horizontal direction. The second component of the vector,  $y$ , represents the vertical direction.

The component form of the vector  $\langle 2, 0 \rangle$  means move right 2 because the  $x$ -component is positive 2, and do not move up or down because the  $y$ -component is 0.

The initial point of the vector is called the tail. The terminal point (with the arrow) is called the head. The movement is the horizontal and vertical direction from tail to head.

Example 2: Use component form to write the vectors **a**, **b**, **c**, **d**, **e** from Example 1. The first vector, **a**, is done for you.



$$\mathbf{a} = \langle 3, 1 \rangle$$

This is because you start at the tail; on the left and move 3 units right (+3) and 1 unit up (+1) to get to the head on the right (where the arrow is).

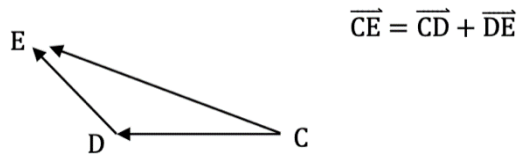
Section 3.3 Resultant Vectors

Looking Back 3.3

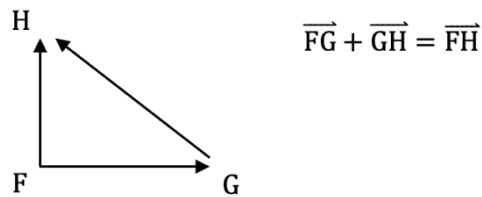
Vectors can be added and subtracted; when they are added or subtracted, we get a resultant vector.

If a plane flies due west from C to D and then northwest from D to E, then the plane could start and end at the same position by flying directly from point C to point E.

The sum of  $\overrightarrow{CD}$  and  $\overrightarrow{DE}$  is  $\overrightarrow{CE}$ .

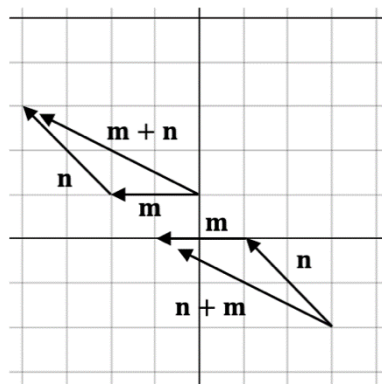


When adding two vectors, the head of the first vector becomes the tail of the second vector. The sum, or resultant vector, extends from the tail of the first vector to the head of the second vector.



We say that vectors go from initial point to terminal point or tail to head. The initial point of one vector can be placed at the terminal point of another vector, which means the tail of one vector can be placed at the head of another vector; they are movable. It does not matter where vectors begin on the grid.

$\mathbf{m} + \mathbf{n} = \mathbf{n} + \mathbf{m}$



The resultant vectors are the same direction and magnitude. Therefore, vector addition is commutative:

$\mathbf{m} + \mathbf{n}$ : up 2 units, left 4 units

$\mathbf{n} + \mathbf{m}$ : up 2 units, left 4 units

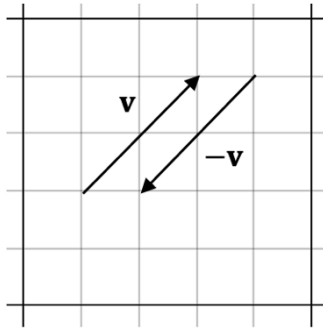
Looking Ahead 3.3

In the last section, you learned that vectors with the same magnitude, but opposite directions are not equivalent, but opposites.

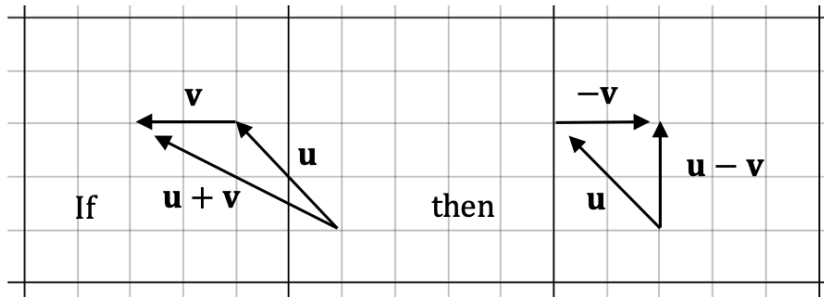
The opposite of north is south; the opposite of southeast is northwest;  $\mathbf{v}$  and  $-\mathbf{v}$  are opposite directions:

$$\mathbf{v} + -\mathbf{v} = \mathbf{0}$$

If a point is represented by a vector, then it has a length of 0. The zero vector is written as a boldface zero:  $\mathbf{0}$ .



To subtract  $\mathbf{v}$  from  $\mathbf{u}$ , change  $\mathbf{u} - \mathbf{v}$  to  $\mathbf{u} + -\mathbf{v}$ . The opposite of  $\mathbf{v}$  goes the opposite direction. It is the same as  $-1(\mathbf{v})$  and is written  $-\mathbf{v}$ .



Therefore, vector subtraction is not commutative.



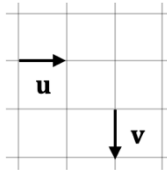
Section 3.4 Scalar MultiplicationLooking Back 3.4

You have seen that vectors can be added and subtracted. The sum is called the resultant. The difference of vectors is found by changing the problem to a sum and adding the opposite. The sum is still called the resultant vector. Therefore, we talk about vector addition and vector subtraction as separate operations.

A vector can also be multiplied by a scalar. A scalar is a real number; it is a quantity. Whereas vectors are fully described by a magnitude and direction, scalars are fully described by a magnitude.

In the previous section, you worked on some problems in which you had to find  $2\mathbf{u}$  and  $3\mathbf{v}$ ;  $2\mathbf{u}$  is 2 times  $\mathbf{u}$  or double the length of  $\mathbf{u}$ ;  $3\mathbf{v}$  is 3 times  $\mathbf{v}$  or triple the length of  $\mathbf{v}$ .

The magnitude (or length) of a vector is called the norm. If the norm is 1, then the vector is called a unit vector.



The symbol for the magnitude is similar to the sign for an absolute value, but has 2 bars on each side of the variable rather than 1:

$$\|\mathbf{u}\| = 1 \quad \text{or} \quad \|\mathbf{v}\| = 1$$

This means the magnitude (or length) of  $\mathbf{u}$  is 1. The magnitude (or length) of  $\mathbf{v}$  is also 1. Therefore, both are unit vectors.

Looking Ahead 3.4

In Algebra 2, you learned about matrices. Vectors can be written in matrix form. A column vector is quantities or numbers written in a column:

$$\begin{pmatrix} 1 \\ -3 \end{pmatrix} \quad \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

These vectors have two components.

A column vector with  $n$  components is written as follows:

$$\mathbf{u} = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \dots \\ \mathbf{u}_n \end{pmatrix}$$

A row vector is quantities or numbers written in a row:

$$(0 \quad 2 \quad -1) \quad (5 \quad 7 \quad 3.2)$$

These vectors have three components.

A row vector with  $n$  components is written as follows:

$$\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \dots \mathbf{v}_n)$$

If the components are equal, the vectors are equal:  $\mathbf{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$  and  $\mathbf{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ ; or  $\mathbf{u} = (3, 4)$  and  $\mathbf{v} = (3, 4)$ ; and the vectors can be added:  $\begin{pmatrix} \mathbf{u}_1 + \mathbf{v}_1 \\ \mathbf{u}_2 + \mathbf{v}_2 \end{pmatrix}$  or  $(\mathbf{u}_1 + \mathbf{v}_1, \mathbf{u}_2 + \mathbf{v}_2)$ . If you add the vectors,  $\mathbf{u} + \mathbf{v} = (3 + 3, 4 + 4) = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$  or,  $\mathbf{u} + \mathbf{v} = (3 + 4, 3 + 4) = (6, 8)$ .

In Linear Algebra, row vectors and column vectors with the same components are called dual vectors rather than equivalent. It is more common to have the vectors in all row spaces or column spaces as with matrices. Also, in Linear Algebra, parentheses are used rather than brackets.

Vectors can be added by adding the following components:

For two component vectors: 
$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix} + \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix}$$

For three component vectors: 
$$\mathbf{u} + \mathbf{v} = (\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3) + (\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)$$

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \text{ because}$$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 + 3 \\ 2 + 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

And

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 + 1 \\ 4 + 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

Therefore, multiplying a number  $n$  times a vector  $\mathbf{v}$  is defined by multiplying  $n$  by the components of  $\mathbf{v}$ .

$$n\mathbf{v} = n \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{pmatrix} = \begin{pmatrix} n\mathbf{v}_1 \\ n\mathbf{v}_2 \\ n\mathbf{v}_3 \end{pmatrix}$$

Example 1: Multiply  $\mathbf{u}$  by the scalar 6 if  $\mathbf{u} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$

Example 2: Find  $-\mathbf{u}$  from Example 1.

Section 3.5 Unit Vectors and Magnitude

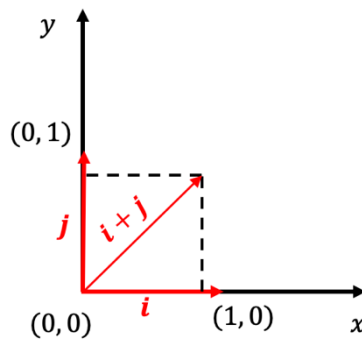
Looking Back 3.5

Previously, you have seen that vectors can be written in component form where the  $x$ -component is the horizontal move from tail to head and the  $y$ -component is the vertical move from initial point to terminal point. Vectors can also be written in component form for the coordinate plane. The components of  $\mathbf{v}$  are the  $x$ -component,  $\mathbf{v}_x$ , and the  $y$ -component,  $\mathbf{v}_y$ , or  $\mathbf{v} = \langle \mathbf{v}_x, \mathbf{v}_y \rangle$ . This is the horizontal and vertical components of the vector written with chevron brackets, or could be written in matrix form as a column  $\mathbf{v} = \begin{pmatrix} \mathbf{v}_x \\ \mathbf{v}_y \end{pmatrix}$ .

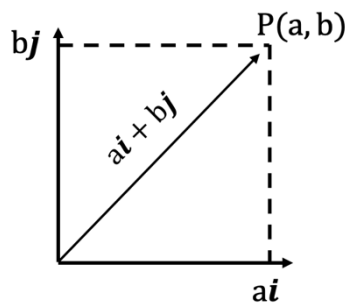
In the coordinate plane,  $\mathbf{i}$  is used to represent the unit vector that is the horizontal vector that has a length of 1 unit. This is the vector  $\mathbf{i} = \langle 1, 0 \rangle$ . It has the tail at point  $(0, 0)$  in the coordinate plane and the head at point  $(1, 0)$  in the coordinate plane.

In the coordinate plane,  $\mathbf{j}$  is used to represent the unit vector that is the vertical vector that has a length of 1 unit. This is the vector  $\mathbf{j} = \langle 0, 1 \rangle$ . It has the tail at point  $(0, 0)$  in the coordinate plane and the head at point  $(0, 1)$  in the coordinate plane.

In the coordinate plane, the resultant unit vector is  $\mathbf{i} + \mathbf{j} = \mathbf{u}$  using the parallelogram method.



If some point  $P$  has the coordinates  $(a, b)$  representing real numbers, then  $\overline{OP}$  is the resultant vector or sum of  $a\mathbf{i} + b\mathbf{j}$ . Therefore,  $\overline{OP} = a\mathbf{i} + b\mathbf{j}$  using the parallelogram method. This is a position vector as it relates to the origin.



Although  $\mathbf{i}$  and  $\mathbf{j}$  are particularly useful unit vectors aligned with the positive  $x$  and  $y$  axes, respectively, any vector of length 1 is also designated a unit vector. Because the unit vector in any given direction is unique, and may be multiplied by any scalar  $a$  to give length  $a$  in that direction, unit vectors are typically taken to be synonymous with a given direction. You may obtain the unit vector parallel to any vector by taking the vector and dividing by its magnitude.

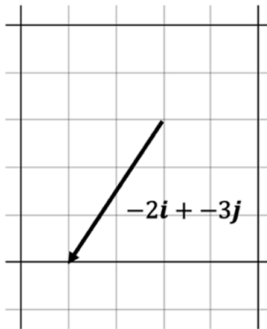
Any vector in the unit plane can be expressed in unit vector form where  $a$  is the  $x$ -component of  $\mathbf{v}$  and  $b$  is the  $y$ -component of  $\mathbf{v}$ ;  $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$  uses the components  $\langle a, b \rangle$  in unit vector form:  $\mathbf{v} = \mathbf{i} + \mathbf{j}$ .

These are scalar multiples of the unit vector. A unit vector is a direction vector, and it must have a magnitude equal to 1. It has a unit length and tells you which direction you are going in space.

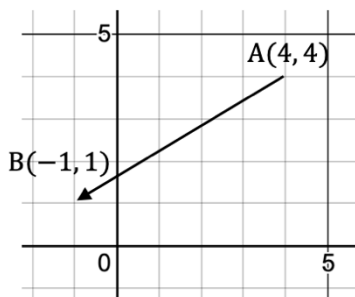
To make any vector a unit vector, divide the unit vector by the magnitude of the vector.

### Looking Ahead 3.5

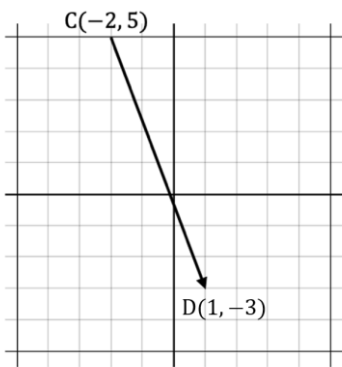
Example 1: Find the magnitude of the vector  $\langle -2, -3 \rangle$ . Write the vector in unit vector form. The  $x$ -component is left 2. The  $y$ -component is down 3. Find the unit vector.



Example 2: Given the position vector below, express  $\overrightarrow{AB}$  in component form and find the magnitude of  $\overrightarrow{AB}$ .



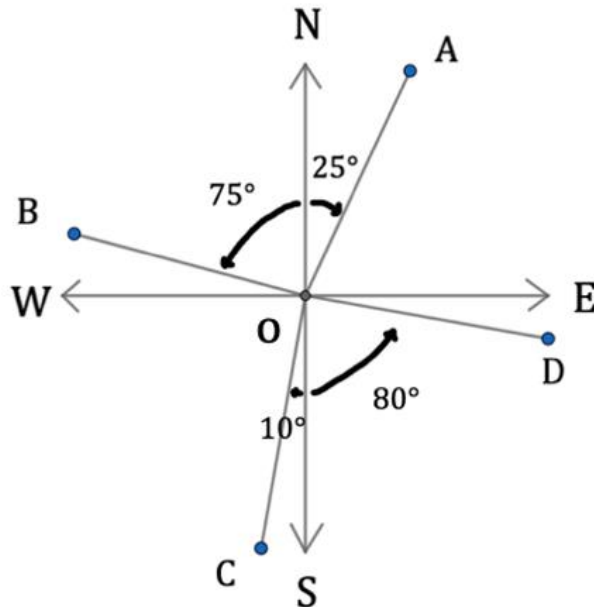
Example 3: Given the points  $C(-2, 5)$  and  $D(1, -3)$ , find the magnitude of  $\overline{CD}$ . Write the resultant vector in component form and unit vector form. Use the triangle method by finding the components or addend vectors that sum to the resultant vector  $\overline{CD}$ .



Section 3.6 Directions and BearingsLooking Back 3.6

Vectors are used practically in many areas. One of these areas is navigation. To help navigate ships from one port to another, directions are given. Directions are given for plane and car travel as well.

Directions to a given point on the plane are stated in terms of degrees east or west of north, or degrees east or west of south.



The  $x$ -axis is East-West and the  $y$ -axis is North-South.

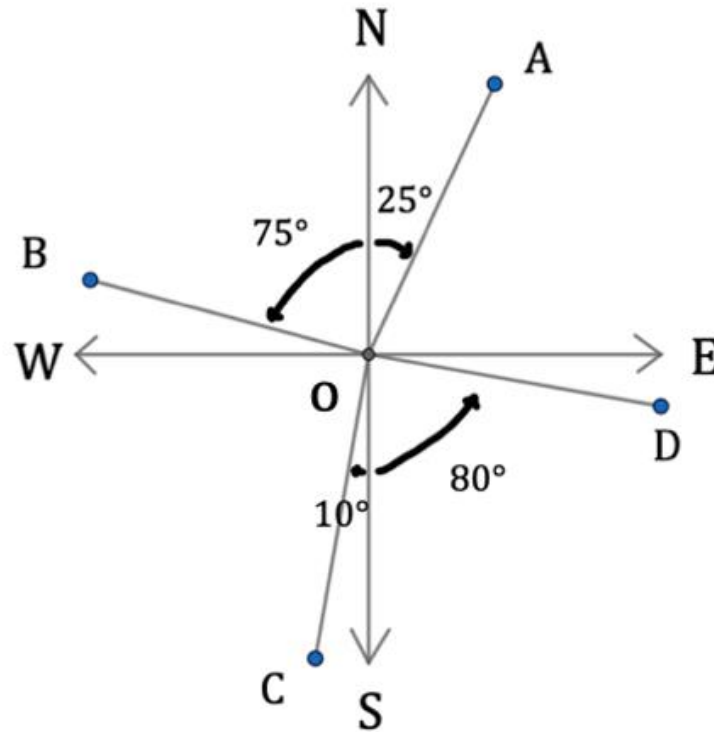
From the origin (o) to point A, travel north and then go  $25^\circ$  east. From o to A, the direction is written: “N $25^\circ$ E.” From o to point B, travel north and then go  $75^\circ$  west; this direction is written: “N $75^\circ$ W.” From o to point C, travel south and then go  $10^\circ$  west; this direction is written: “S $10^\circ$ W.” From o to point D, travel south and then go  $80^\circ$  east; this direction is written: “S $80^\circ$ E.”

There is another way to state these directions. For example, another way to state “S $80^\circ$ E” is: “80 degrees east of south.” Direction is measured from north or south first, then degrees to the east or west to get to the vector.

In the first section of this module when you were first introduced to vectors, you measured in degrees, but the angle was from the positive  $x$ -axis counterclockwise to the vector just as the angles are measured on the unit circle. For bearings, the measurement in navigation is done differently.

Looking Ahead 3.6

Bearings to a given point on a plane are stated in terms of degrees clockwise from the north line only. Therefore, using our previous diagram, the bearing from  $O$  to point  $A$  is still  $25^\circ$ . The bearing from  $O$  to  $B$  changes. To get from the north line to  $\overline{OB}$ , you must go clockwise through three quadrants (I, IV, and III), which is  $270^\circ$  ( $90^\circ + 90^\circ + 90^\circ$ ), and then add  $15^\circ$  more ( $90^\circ - 75^\circ$ ) in Quadrant II;  $270^\circ + 15^\circ = 285^\circ$  so that is the bearing to  $\overline{OB}$  clockwise from the north line.

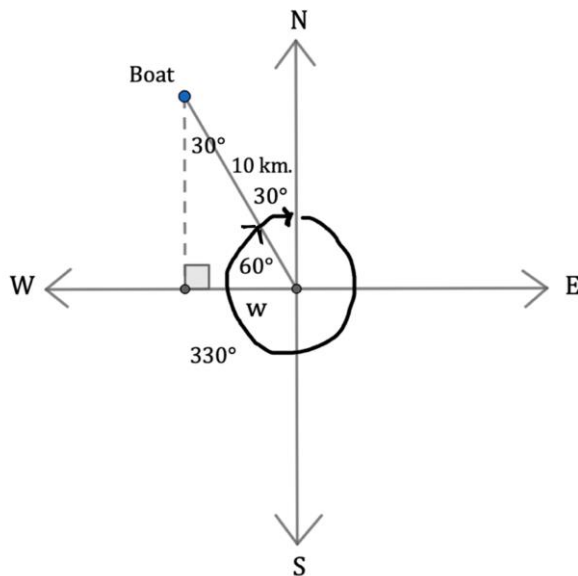


The bearing to  $\overline{OC}$  is  $190^\circ$  from the north line (from  $180^\circ$  in Quadrant I and Quadrant IV and  $10^\circ$  in Quadrant III).

The bearing to  $\overline{OD}$  is  $100^\circ$  from the north line (from  $90^\circ$  in Quadrant I and  $10^\circ$  (from  $90^\circ - 80^\circ$ ) in Quadrant IV).

What you learned in Trigonometry can be used to solve problems in navigation.

Example 1: A boat travels at a bearing of  $330^\circ$ . After traveling 10 km., how far west of the port is the boat?



The boat has traveled in a northwest direction. The direction is  $N30^\circ W$  ( $360^\circ - 330^\circ$ ). Therefore, we already know the base acute angle is  $60^\circ$  and the upper acute angle is  $30^\circ$  from  $180^\circ - (90^\circ + 60^\circ) = 30^\circ$ .

We can use  $w$  as the adjacent side to  $60^\circ$ ...

$$\cos 60^\circ = \frac{w}{10}$$

$$(10) \cos 60^\circ = w$$

$$10 \left( \frac{1}{2} \right) = w$$

$$w = 5 \text{ km.}$$

Or we can use  $w$  as the opposite side to  $30^\circ$ ...

$$\sin 30^\circ = \frac{w}{10}$$

$$(10) \sin 30^\circ = w$$

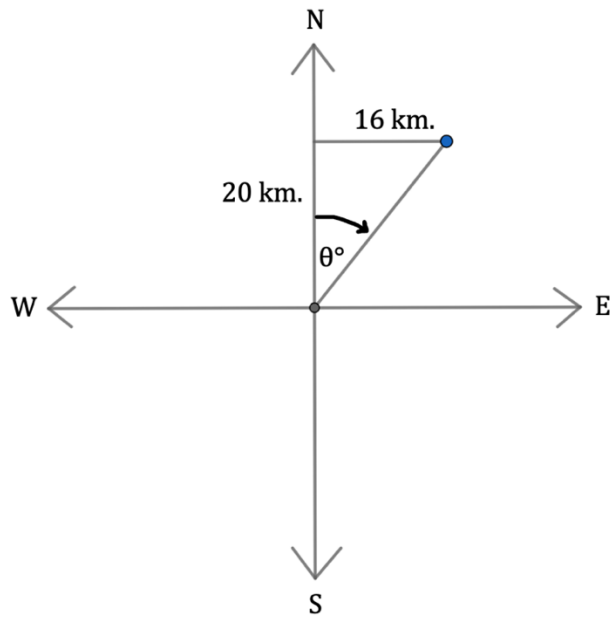
$$10 \left( \frac{1}{2} \right) = w$$

$$w = 5 \text{ km.}$$



Example 2: A bicyclist is doing a weekend ride with some friends. They start at 20 km. north and then slow to stop for lunch at 16 km. east.

What is the bicyclist's bearing from the starting point?



Section 3.7 Vectors and the Law of Sines and CosinesLooking Back 3.7

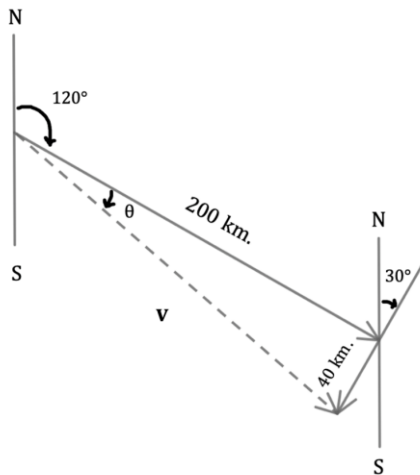
In the previous practice problems section, sine, cosine, and tangent from Geometry and Trigonometry were used to solve some rather direct problems. Here, we will use what you learned in Geometry and Trigonometry to solve some navigation problems that are not so straight forward. The Law of Sines and Cosines can be used to help solve these.

A course is a planned route for travel. What you actually travel is the true course, which may be affected by wind currents (in a plane) or water currents (in a boat). The bearing (as seen in the previous practice problems section) is the degrees measured clockwise from north, and a heading is the direction your mode of travel is pointing.

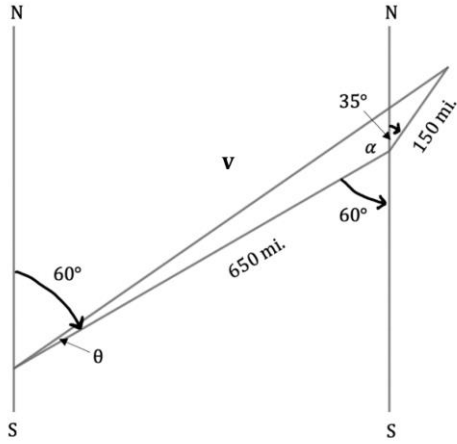
Headings are usually the same as bearings unless you are knocked off course by wind. The velocity a plane travels on a true course is called the ground speed due to these effects, as opposed to the actual air speed. The air speed is the actual speed of the object in the air. The ground speed is the speed of the object relative to the ground.

Looking Ahead 3.7

**Example 1:** A small plane is traveling at 200 km./hr. with a heading of  $120^\circ$ . A wind is blowing east of south at 40 km./hr. from a bearing of  $30^\circ$ . Find the bearing of the true course of the plane and the ground speed of the plane. (In the solution, I write the true bearing of  $56.4^\circ$  but the actual answer is  $55.4^\circ$ .) Watch to see how it is done.



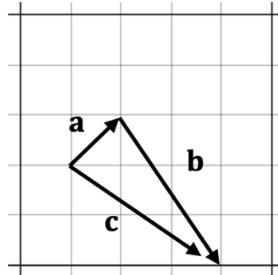
Example 2: A plane flies from the airport to its first layover for 650 miles with a bearing of  $60^\circ$ . It then flies 150 more miles to its destination with a bearing of  $35^\circ$ . What bearing is needed for a direct flight from the plane's takeoff to its destination? What is the distance of this direct flight? Draw a diagram first to help solve the problem.



Section 3.8 Converting Vectors to Component FormLooking Back 3.8

In the last section, you used vectors along with The Law of Sines and Cosines to solve travel and navigation problems. Those problems may be easier to solve another way, which is by using components.

If  $\mathbf{a} = \mathbf{i} + \mathbf{j}$  and  $\mathbf{b} = 2\mathbf{i} - 3\mathbf{j}$ , then  $\mathbf{a} + \mathbf{b}$  can be drawn using vectors and shown geometrically:



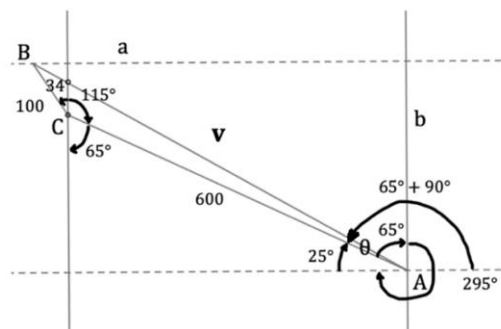
The resultant is  $\mathbf{c} = 3\mathbf{i} - 2\mathbf{j}$ , which is, in component form,  $\mathbf{c} = \langle 3, -2 \rangle$ . This can be solved algebraically using components  $\mathbf{a} = \langle 1, 1 \rangle$  and  $\mathbf{b} = \langle 2, -3 \rangle$ , so  $\mathbf{a} + \mathbf{b} = \langle 1, 1 \rangle + \langle 2, -3 \rangle = \langle 1 + 2, 1 - 3 \rangle$ . Therefore,  $\mathbf{c} = \langle 3, -2 \rangle$  is the same resultant whether computed geometrically or algebraically.

Looking Ahead 3.8

In changing vectors to component form, if  $\mathbf{v}$  is a given vector with a magnitude of  $\|\mathbf{v}\|$  and standard angle  $\theta$ , then it can be written as: “ $a\mathbf{i} + b\mathbf{j}$ ” or “ $\langle a, b \rangle$ ,” or “ $\mathbf{a} = \|\mathbf{v}\| \cos \theta$ ” and “ $\mathbf{b} = \|\mathbf{v}\| \sin \theta$ .” These represent the horizontal and vertical components.

**Example 1:** Let us try to solve Problem 6-10 from the previous practice problems section using components. Earlier, you used the Law of Sines and Cosines. See which you think is easier when you get the same answer.

A plane is flying at 600 mph on a course with a bearing of  $295^\circ$ . It is pushed along by a 100 mph wind blowing to the northwest, and the wind has a direction of  $N34^\circ W$ .



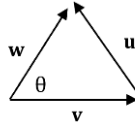
1. First, find the standard angle. Then convert the velocity of the plane to component form.



Section 3.9 The Dot ProductLooking Back 3.9

Both the Law of Sines and Cosines and the component form of vectors have been used to solve applied problems. Most recently, you solved applied problems of force used in physics. Applied problems in physics that are work related can also be solved using the dot product.

In this section, we will investigate how the dot product is used to find the angle between two vectors.



Since  $\mathbf{v} + \mathbf{u} = \mathbf{w} \dots$

$\dots$  then  $\mathbf{u} = \mathbf{w} - \mathbf{v}$ .

If  $\mathbf{v} = \langle v_1, v_2 \rangle$  and  $\mathbf{w} = \langle w_1, w_2 \rangle \dots$

$\dots$  then  $\mathbf{u} = \mathbf{w} - \mathbf{v}$  is  $\mathbf{u} = \langle w_1, w_2 \rangle - \langle v_1, v_2 \rangle$

In component form,  $\mathbf{u} = \langle w_1 - v_1, w_2 - v_2 \rangle$ .

Using the Law of Cosines,  $\|\mathbf{u}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta$ .

The component forms and the Pythagorean Theorem can be used to find  $\|\mathbf{v}\|^2$ ,  $\|\mathbf{w}\|^2$ , and  $\|\mathbf{u}\|^2$ .

$$\|\mathbf{v}\|^2 = v_1^2 + v_2^2$$

$$\|\mathbf{w}\|^2 = w_1^2 + w_2^2$$

$$\|\mathbf{u}\|^2 = (w_1 - v_1)^2 + (w_2 - v_2)^2$$

If we substitute these into the Law of Cosines, we get the following equation:

$$(w_1 - v_1)^2 + (w_2 - v_2)^2 = v_1^2 + v_2^2 + w_1^2 + w_2^2 - 2\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta$$

$$w_1^2 - 2w_1v_1 + v_1^2 + w_2^2 - 2w_2v_2 + v_2^2 - v_1^2 - v_2^2 - w_1^2 - w_2^2 = -2\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta$$

$$-2w_1v_1 - 2w_2v_2 = -2\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta$$

$$\frac{-2(w_1v_1 + w_2v_2)}{-2} = \|\mathbf{v}\|\|\mathbf{w}\|\cos\theta$$

$$w_1v_1 + w_2v_2 = \|\mathbf{v}\|\|\mathbf{w}\|\cos\theta$$

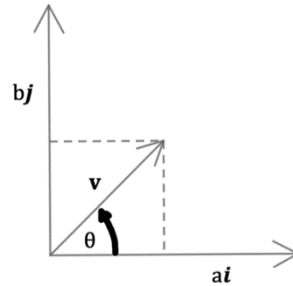
What is on the left side of the last step is called the dot product or scalar product because given two vectors:

$$\mathbf{v} = \langle v_1, v_2 \rangle \text{ and } \mathbf{w} = \langle w_1, w_2 \rangle, \text{ then } \mathbf{v} \cdot \mathbf{w} = v_1 \cdot w_1 + v_2 \cdot w_2.$$

If you solve for  $\cos\theta$  on the right side of the equation, you will see how the dot product can be used to find the angles between two vectors.

$$\cos \theta = \frac{\mathbf{v}_1 \mathbf{w}_1 + \mathbf{v}_2 \mathbf{w}_2}{\|\mathbf{v}\| \|\mathbf{w}\|}$$

The numerator is the dot product, and the denominator is the product of the magnitude of the two vectors.



In the graph above...

$$\sin \theta = \frac{b}{\|\mathbf{v}\|} \therefore \|\mathbf{v}\| \sin \theta = b$$

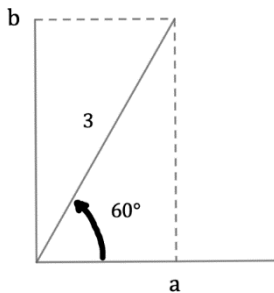
And...

$$\cos \theta = \frac{a}{\|\mathbf{v}\|} \therefore \|\mathbf{v}\| \cos \theta = a$$

**Example 1:** Given the magnitude of a vector and the standard angle, find the components of  $\mathbf{v}$ .

$$\|\mathbf{v}\| = 3 \quad \theta = 60^\circ$$

(Note:  $\theta$  is a standard angle in standard position measured counterclockwise from the positive  $x$ -axis. It is not the bearing angle measured clockwise from the north.)



Looking Ahead 3.9

Example 2: If  $\mathbf{u}$  and  $\mathbf{v}$  are two nonzero vectors, find the dot product where  $\theta$  is the angle between them. Substituting  $\mathbf{u}$  for  $\mathbf{w}$  in the previous formula gives:  $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$ . The angle between them is the following formula:  $\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$ . Find  $\mathbf{u} \cdot \mathbf{v}$  if  $\mathbf{u} = \langle 1, \sqrt{3} \rangle$  and  $\mathbf{v} = \langle \sqrt{3}, 1 \rangle$  and  $\theta = 15^\circ$ .

Example 3: Use the dot product below to find the angle between the vectors in component form.

$$\mathbf{v} = \langle -2, 1 \rangle \quad \mathbf{w} = \langle 4, 3 \rangle$$

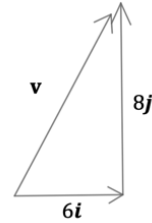
$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \cdot \|\mathbf{w}\| \cos \theta \text{ and } \cos \theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|}$$

Example 4: Given the vectors  $\mathbf{u} = 2\mathbf{i} + 5\mathbf{j}$  and  $\mathbf{v} = \mathbf{i} - 2\mathbf{j}$ , use the dot product to find the angle between them.



Section 3.10 Unit and Orthogonal VectorsLooking Back 3.10

In the diagram to the right, the sum of the first two vectors is  $\mathbf{v}$  ( $6\mathbf{i} + 8\mathbf{j} = \mathbf{v}$ ). The magnitude of  $\mathbf{v}$  is  $\|\mathbf{v}\| = \sqrt{6^2 + 8^2} = \sqrt{36 + 64} = \sqrt{100} = 10$ . The length of  $\mathbf{v}$  is 10.



If we wanted to find the unit vector, we would have to change the components. If the resultant vector is 1 unit, then the resultant vector is 1 unit long. If 1 unit is represented by 10, then the horizontal and vertical components are  $\frac{1}{10}$ th of that. Therefore, the horizontal and vertical vectors must be divided by 10 (the length of the resultant) to get the unit vector.

$$\frac{6}{10}\mathbf{i} + \frac{8}{10}\mathbf{j} = \frac{10}{10}$$

$$\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} = 1$$

Now, if we find the magnitude of the unit vector  $\mathbf{v}$  then  $\sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{25}{25}} = 1$ , which is correct.

Looking Ahead 3.10

Two vectors are orthogonal if they are perpendicular. Because  $\cos 90^\circ = 0$ , the dot product of the two vectors will be 0 when they are orthogonal, and this is because they are perpendicular and meet at  $90^\circ$  angles.

**Example 1:** Find a unit vector orthogonal to  $\mathbf{u} = 3\mathbf{i} - 4\mathbf{j}$ . Let  $\mathbf{v}$  be  $x\mathbf{i} + y\mathbf{j}$ .

Section 3.11 Vector EquationsLooking Back 3.11

We have been using vectors to describe the motion of planes, cars, and boats. Vectors can be written as equations involving speed and velocity. Speed is how fast a plane, car, or boat is traveling, but velocity is how fast and which direction the plane, car, or boat is traveling. Therefore, the speed equals the magnitude of the velocity.

$$\text{speed} = \|\mathbf{v}\|$$

In the vector equation for  $\mathbf{v}(t)$  (read: “velocity at time  $t$ ”), the horizontal direction is decreasing in velocity, while the vertical direction is increasing in velocity for  $\mathbf{v}(t) = \langle 8 - t, 2t \rangle$ .

Looking Ahead 3.11

Example 1: Using  $\mathbf{v}(t) = \langle 8 - t, 2t \rangle$ , find the horizontal and vertical directions for velocity of an object after 4 seconds if it is measured in meters per second.

Example 2: Assuming the object in Example 1 is a remote-controlled car, draw the motion vector for  $t = 4$  and find the speed of the object after 4 seconds.

Example 3: A line can be written as a vector if the two points that connect the vector are known. The line that goes through  $(1, 3)$  and  $(5, 6)$  has the slope  $m = \frac{6-3}{5-1} = \frac{3}{4}$ .

a) Use the point-slope form of the equation to find the equation of the line and convert it to standard form. Use each of the points to derive the same equation.

b) A vector that has initial point  $\langle 1, 3 \rangle$  and terminal point  $\langle 5, 6 \rangle$  is  $\langle 5, 6 \rangle - \langle 1, 3 \rangle = \langle 5 - 1, 6 - 3 \rangle = \langle 4, 3 \rangle$ . How is this vector similar to the slope of the line?

c) One vector equation for the line is  $\langle 1, 3 \rangle + t\langle 4, 3 \rangle$ . What is another vector equation for the same line?

d) When  $t = 3$ , the vector  $\langle 1, 3 \rangle + 3\langle 4, 3 \rangle$  will have the endpoint  $\langle 1, 3 \rangle + \langle 3(4), 3(3) \rangle = \langle 1, 3 \rangle + \langle 12, 9 \rangle = \langle 1 + 12, 3 + 9 \rangle = \langle 13, 12 \rangle$ . Show that the endpoint  $\langle 13, 12 \rangle$  lies on the line.

Section 3.12 Parametric Equations and MotionLooking Back 3.12

We have learned a lot about direction and bearings and true course.

God wants us to stay on the true course for our lives and follow Him. He does not want us to be blown off course like the boats and planes we have studied- by strong currents or winds, or trials that pull against us. If trials in life test us and try to get us off course, we can ask God for wisdom to guide us on our true course. When a boat or plane is off course, we can determine or calculate the true bearing to stay on course by using mathematics.

James 1:5-6 says, "But if any of you lacks wisdom, let him ask of God, who gives to all generously and without reproach, and it will be given to him. But let him ask in faith without any doubting, for the one who doubts is like the surf of the sea driven and tossed by the wind."

There are many ways God has given us to solve problems in both life and mathematics, but they require wisdom and knowledge.

Looking Ahead 3.12

All that being said, there is another way to solve problems of motion in both calculus and physics when more than one variable is involved.

In parametric equations, both  $x$  and  $y$  are dependent on a third variable. Often that variable is time, represented by  $t$ . When you have horizontal motion and vertical motion, both are changing over time.

Example 1: The parametric equations below represent a small boat and a large boat as they pass a lighthouse located at the origin of a graph. The  $x$ -coordinate represents the east location from the lighthouse after  $t$  seconds. The  $y$ -coordinate represents the north location from the lighthouse after  $t$  seconds. The distance is measured in meters.

$$\text{Small Boat: } x = 0 + 15t; y = 200 + 0t$$

$$\text{Large Boat: } x = 0 + 33t; y = 600 + 0t$$

- What will be the coordinates for the small boat after 7 seconds?
- What will be the coordinates for the large boat after 7 seconds?
- How far east of the lighthouse will the small boat be after 5 seconds?
- How far north of the lighthouse will the large boat be after 5 seconds?

Notice that both equations had two variables:  $x$  and  $t$  and  $y$  and  $t$ .

Example 2: The following parametric equations describe a function:

$$x(t) = 3 + 2t$$

$$y(t) = 3 + 4t$$

- a) After 10 seconds, what will the  $(x, y)$  coordinates be?
- b) What will the  $(x, y)$  coordinates be at  $t = 20$ ?
- c) Solve the equation  $(x(t) = 3 + 2t)$  for  $t$  in terms of  $x$ .
- d) Substitute the answer for  $t$  in the second equation  $(y(t) = 3 + 4t)$  in terms of  $y$ .
- e) Are the  $x$ - $y$  coordinates from b) a solution to this equation?
- f) Do the  $x$ - $y$  coordinates satisfy the equation?

Example 3: An object that has projectile motion has a parabolic path as you learned when you studied quadratic equations in Algebra 2.

If an object has an initial position of  $(x_0, y_0)$ , an initial velocity of  $v_0$ , and an angle of  $\theta$  when it is launched, it will travel along a path that can be defined by the following parametric equations:

$$\begin{aligned}x(t) &= (v_0 \cos \theta) \cdot t + x_0 \\y(t) &= -16t^2 + (v_0 \sin \theta) \cdot t + y_0\end{aligned}$$

In these equations,  $x$  and  $y$  are measured in feet and  $t$  is measured in seconds.

If a water bottle rocket is launched from a PVC launcher 3 feet from the ground with an initial velocity of 52 feet per second at an angle of  $60^\circ$ , answer the questions below.

a) What is the initial position  $((x_0, y_0))$  of the water bottle rocket?

b) What is the initial velocity  $(v_0)$  of the water bottle rocket?

c) What is the launch angle  $(\theta)$  of the water bottle rocket?

d) Substitute these values into the parametric equations for the projectile motion to find a model for the flight of the water bottle rocket.

e) What will the height of the water bottle rocket be after 2 seconds?

Section 3.13 Graphing Parametric EquationsLooking Back 3.13

Henri Poincaré (1854-1912) was a French mathematician, physicist, and philosopher. He was considered one of the greatest geniuses of the 19<sup>th</sup> century. Henri was an ill child and struggled to draw or play the piano, but he was gifted in his studies, spending his time taking on the sciences and using parametric equations to solve physics problems.

In 1885, King Oscar II of Sweden had a contest for his 60<sup>th</sup> birthday and offered a prize for anyone who could establish the stability of the solar system. Although the committee to give out the award struggled to fully grasp Poincaré's work, he won the contest, providing ground-breaking techniques and advances in celestial mechanics in his writing. However, he later found a mistake in his submission and had to use almost half his prize money to print a new, updated report. His conclusion: small bodies in orbit could be chaotic (we now call this the phenomenon).

Poincaré said that he believed in religion early in his life but slowly started to doubt it. By the age of eighteen, he stopped believing in religion all together and began believing in the right to search for the truth and tell the truth. He loved the harmony and beauty of mathematics, but made no connection to its Creator, who made all things beautiful. He had a belief that mathematics was intuitive or innate and could not be proven by logic. He was always uncomfortable that he had trouble not believing in the Creator.

Looking Ahead 3.13

We have learned that parametric equations can be used with mathematics and physics to solve problems of motion. Graphing parametric equations allows us to analyze the information graphically.

We will first revisit the problems from the last section and graph them by hand, and then check them using the graphing calculator.

Example 1: The parametric equations for the small boat and large boat are shown as follows:

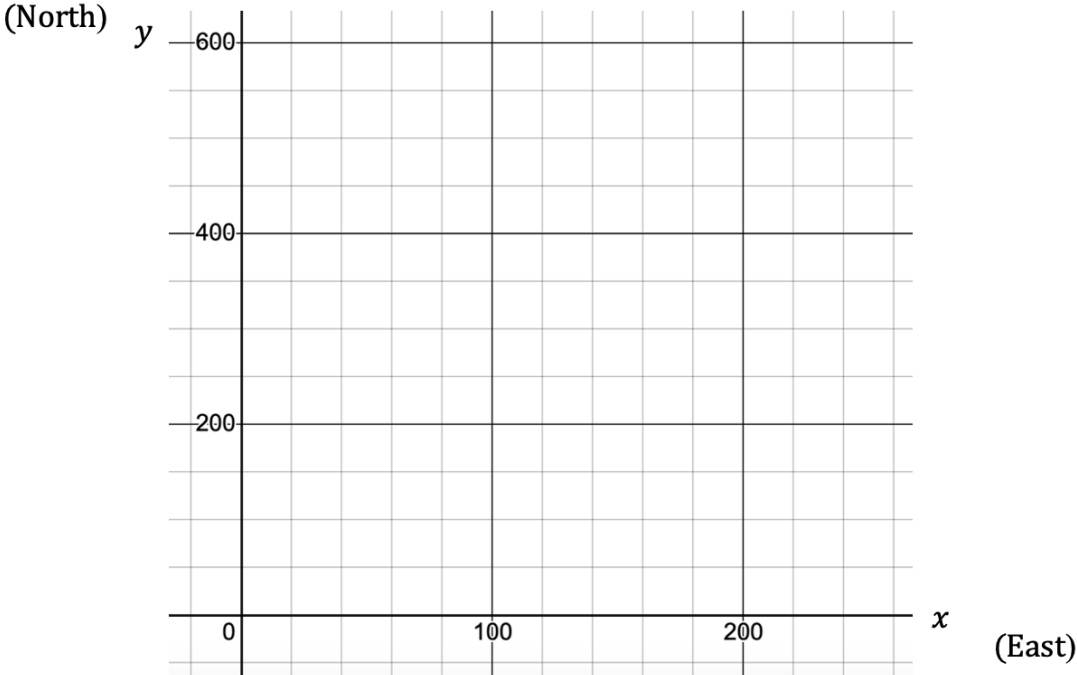
Small Boat:	Large Boat:
$x(t) = 0 + 15t$	$x(t) = 0 + 33t$
$y(t) = 200 + 0t$	$y(t) = 600 + 0t$

Complete the table and graph for the travel of the boats after they pass the lighthouse located at (0, 0).

Small Boat		
$t$	$x$	$y$
0		
1		
2		
3		
4		
5		
6		
7		

Large Boat		
$t$	$x$	$y$
0		
1		
2		
3		
4		
5		
6		
7		

Why will the motion of the small and large boat be horizontal lines?





Example 2: The following parametric equations describe the function  $y = 2x - 3$ :

$$x(t) = 3 + 2t$$

$$y(t) = 3 + 4t$$

To graph the parametric functions on the graphing calculator, go to the “Graphs” page.

- “Menu”
- “#3 Graph Type”
- “#2 Parametric”

At “ $x_1(t) =$ ” type in “ $3 + 2t$ ”

At “ $y_1(t) =$ ” type in “ $3 + 4t$ ”

The default step is “ $0 \leq t \leq 6.28$   $t$  step = 0.13.” Change this to “ $-6.28 \leq t \leq 6.28$   $t$  step = 0.13.” This is the same as  $-2\pi \leq t \leq 2\pi$ . Click “ctrl g” to get a new link line, then click the given titles:

- “Menu”
- “#3 Graph Type”
- “#1 Function”

This will switch the graphing calculator from parametric mode to function mode. Now, type in the function “ $f_1(x) = 2x - 3$ .” What is similar about the graphs? What is different?

Example 3: The takeoff for a model drone in Practice Problem 5 of the last section had the table of values shown below.

Graph the coordinates for  $t = 0$  to  $t = 3$  seconds to see what the takeoff curve looks like. What is the relationship between  $x$  and  $y$ ?

$t$	$x$	$y$
0	1	3
1	3	9
2	9	27
3	27	81
4	81	243
5	243	729

