Pre-Calculus and Calculus Module 5 Limits and Continuity

Section 5.1 Piecewise Functions and Limits

Looking Back 5.1

In working with limits, piecewise functions often come up in the discussion. We have studied piecewise functions in Algebra 2, but it is important to review them here.

A piecewise function is made up of pieces and can be investigated piece by piece. A piecewise function can be continuous, which means you can trace it without lifting your pencil. It can start as a quadratic function, then continue as a linear function over a given domain.

On the other hand, a piecewise function can be discontinuous, which means you must lift your pencil to continue over another interval.
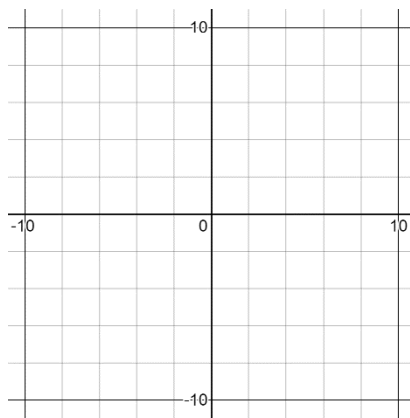
Looking Ahead 5.1

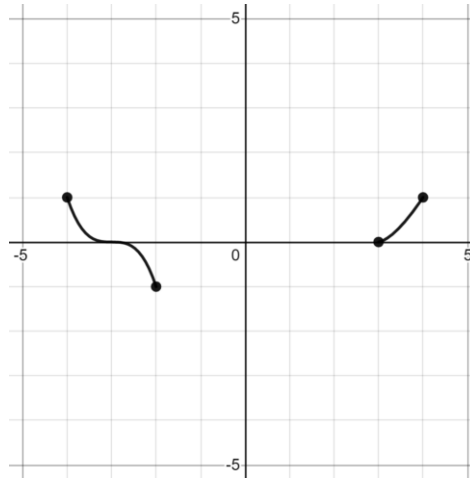| Example 1:	Write the piecewise function, including intervals, for the graph.
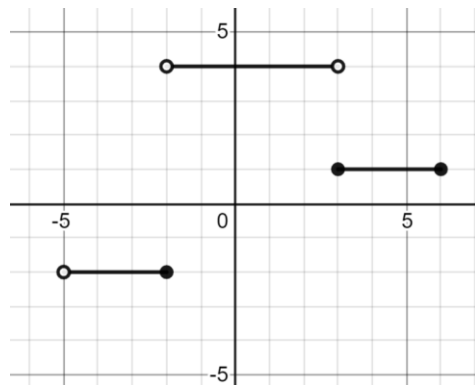
$$f(x) = \left\{ \right\}$$

Example 2:	Graph the given piecewise function. Remember to indicate the interval endpoints with appropriate open or closed circle where necessary.

$$g(x) = \begin{cases} 3x - 4, -2 \leq x < 0 \\ -0.1x + 2, 0 \leq x < 4 \\ -2, 4 \leq x \leq 6 \end{cases}$$

| Example 3: | Answer the given questions for the piecewise function below. |

a) A butterfly starts at point $(-2, -1)$ and flies along the curve to the left of the y-axis. How high will it fly as it gets closer to $x = -4$?

b) If another butterfly starts at point $(3, 0)$ and follows the curve to the right of the y-axis, where will the butterfly be on the x-axis when it reaches the same height as the butterfly from a)?

The concept of a limit is the foundation of Calculus. Graphs can be used to calculate limits as in Example 3. An important use of limits in Calculus is to test functions for continuity. In science, continuous functions often model behavior in God's natural world.

Calculus allows us to find the numerical limits of functions and to confirm the values of functions analytically. The importance of limits is not what happens when a function is at a specific value, but rather as it approaches a specific value.

Reading a limit off of a graph can be a simple matter of tracing a curve through a single value, as in both parts of Example 3. The task becomes much more interesting when the limit involves something that is not or cannot be plotted, such as asking what happens to a graph when x or y become infinitely large, as we will see in upcoming sections.

Section 5.2 Rational Functions and Limits
Looking Back 5.2

In working with limits, rational functions often come up in the discussion (like piecewise functions). Rational functions were also studied in Algebra 2, but it is important to review them here.

A rational function can be expressed as a ratio of two polynomials. Polynomials are functions such as quadratics, cubics, and quartics that have only non-negative integer powers of x. Because $f(x) = \sqrt{x}$ and $g(x) = x^{-1}$ are not polynomials, then $f(x) = \frac{\sqrt{x}}{2}$ and $g(x) = \frac{3x^2}{\sqrt[3]{x}}$ are not rational functions.

Polynomials are of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} \ldots a_2 x_2 + a_1 x + a_0$. The a's are all real numbers, which means the coefficients of each term are real numbers. The general equation for a rational function is $r(x) = \frac{p(x)}{q(x)}$. The rational algebraic fraction has polynomial expressions for both the numerator, $p(x)$, and the denominator, $q(x)$. If the denominator has a degree of at least 1, such as x or y, and xy or $x^2 y$, then this means there is a variable in the denominator. If the denominator is just a constant, such as 3, then the function $f(x) = \frac{2x^3}{3}$ is really the polynomial function $f(x) = \frac{2}{3} x^3$. Polynomials are a special subset of rational functions with a denominator of degree 0. Irreducible rational functions must have denominators of nonzero degree and not share any factors with the numerator.

Because the numerator and denominator in a rational function are polynomials, there can be no fractions, decimals, or negative numbers in the exponents. Because $\sqrt{x}$ can be written $x^{\frac{1}{2}}$, it is not a polynomial.

Looking Ahead 5.2

Arguably the simplest rational function is $f(x) = \frac{1}{x}$.

x	$f(x)$
-3	$-0.\overline{3}$
-2	-0.5
-1	-1
0	undef.
1	1
2	0.5
3	$0.\overline{3}$
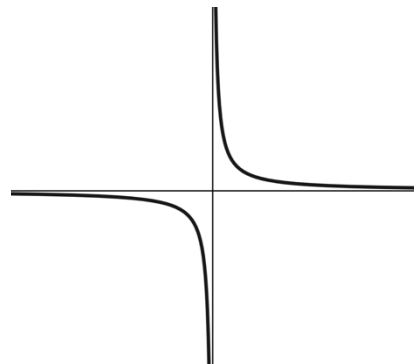
The domain is the set of all real numbers except 0. This is written:

Domain: $\{x \mid x \neq 0\}$
(This is read: "The set of all x where x is not equal to zero.")

It can also be written: $D = \{\mathcal{R}\}$ where $x \neq 0$, where $\{\mathcal{R}\}$ indicates the set of real numbers.

Range: $\{y \mid y \neq 0\}$
(This is read: "The set of all y where y is not equal to zero.")

It can also be written: R $= \{\mathcal{R}\}$ where $y \neq 0$.

There are asymptotes at the horizontal axis ($y = 0$) and the vertical axis ($x = 0$). This will be discussed further in an upcoming section.

We know that division by zero is not allowed and that is why rational functions tend to have values of x which must be excluded from their domains to ensure that the denominator never equals zero.

Example 1:   Use a calculator to generate a graph and table for the parent rational function $f(x) = \frac{1}{x}$ and answer the questions below.

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a) What happens to $f(x)$ as x increases without bounds? In other words, what happens to y when x moves towards positive infinity?

b) What happens to $f(x)$ as x decreases without bounds? In other words, what happens to y when x moves towards negative infinity?
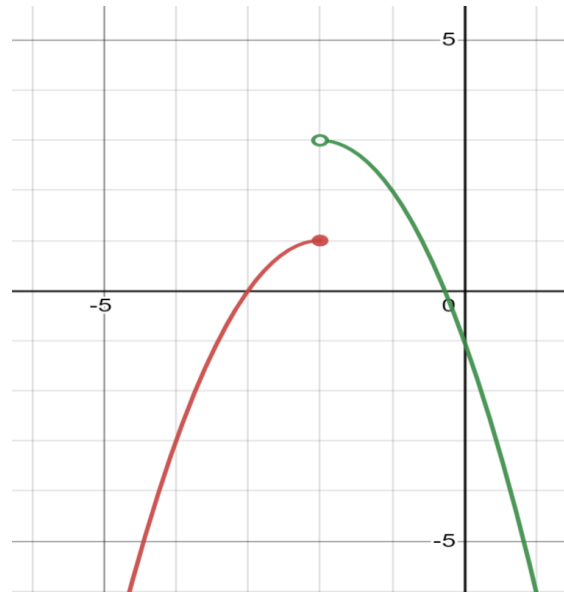
c) What happens to $f(x)$ as x approaches zero from the right?

d) What happens to $f(x)$ as x approaches zero from the left?

Example 2:   Simplify the rational expressions below.

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a) $\dfrac{2x^2+8x-10}{2x+10}$

b) $\dfrac{\frac{-2x}{y}+4}{\frac{2x^2}{y^2}+2}$

<u>Section 5.3 One-sided and Two-sided Limits</u>
<u>Looking Back 5.3</u>

     If a worm starts at the lower end of the function on the left of the graph and begins crawling, it will eventually reach a height of 1 at the point $(-2, 1)$. We can use function notation, $f(-2) = 1$, for the function on the leftmost part of the graph.

     If a worm starts at the lower end of the function on the right of the graph and begins crawling, it will eventually get close to a height of 3, but never reach it. The point $(-2, 3)$ does not exist because it is not included; therefore, $g(-2)$ is equal to D. N. E. for the function on the rightmost part of the graph.

     The function on the left of the graph is $f(x) = -(x + 2)^2 + 1$. We can say $f(-2) = -(-2 + 2)^2 + 1$ and $f(-2) = 1$. The function on the right of the graph is $g(x) = -(x + 2)^2 + 3$. We cannot say $g(-2) = 3$ because -2 is not included in the domain of the function $g(x)$.

Domain $f(x)$ is $(-\infty, -2]$

Domain $g(x)$ is $(-2, \infty)$

The range of each function follows:

Range $f(x)$ is $(-\infty, 1]$
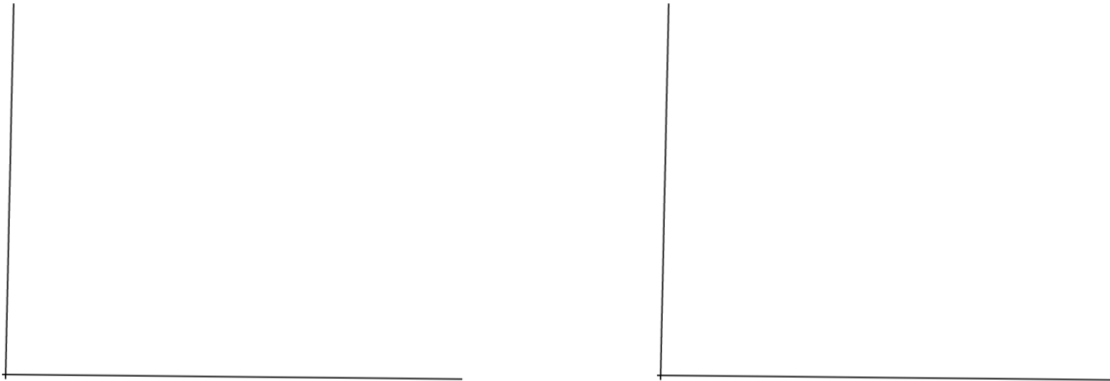
Range $g(x)$ is $(-\infty, 3)$

<u>Looking Ahead 5.3</u>
The formal Calculus definition for limit is:

$$\lim_{x \to c} f(x) = L$$

     Given the real number $\epsilon > 0$, there exists another real number $\delta > 0$ so that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$ for all x.

The value of δ will depend on the value of ϵ. Begin with $\epsilon > 0$ and determine an appropriate corresponding value for $\delta > 0$. Many values for δ will work and any value smaller than that will also work.

The informal Calculus definition (for our purposes here) means that as x gets closer and closer to some constant c, $f(x)$ gets closer and closer to a number L. Our limit is L.

The symbol $x \to c$ means x approaches the constant c from both sides (right and left). In the graph above, as $x \to c$, $f(x) = L$.

If you move from the right along the graph of the function or from the left along the graph of the function to the open circle at c (on the x-axis), you will see that it corresponds to a value of L (on the y-axis).

Therefore, $x \to c^+$ means the values of x get closer and closer to c from the right, $x > c$.

Moreover, $x \to c^-$ means the values of x get closer and closer to c from the left, $x < c$.

These are called one-sided limits if they approach the constant from the right side or the left side.

This is the informal definition of a limit when c and L are real numbers:

$$\lim_{x \to c} f(x) = L$$

If $x \to c^+$ and $x \to c^-$ are the same, then that is the limit, L. If they are different, a limit does not exist. If $x \to c$, this is called a two-sided limit because x approaches c from both the right and left side. Therefore, the function $f(x)$ has a limit, as x approaches c if and only if the right-side and left-side limits exist at c and are equal:

If $\lim_{x \to c^+} f(x) = L$

And $\lim_{x \to c^-} f(x) = L$

Then $\lim_{x \to c} f(x) = L$

Example 1:	Find the following limits of the piecewise function $f(x)$.

$$f(x) = \begin{cases} -2, -5 < x \le -2 \\ 4, -2 < x < 3 \\ 1, 3 \le x \le 6 \end{cases}$$
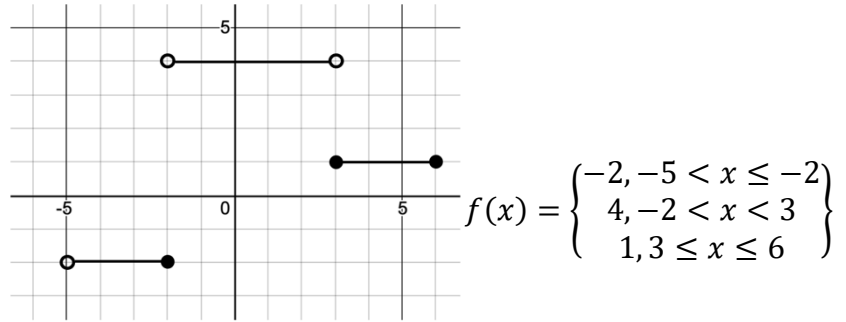
a) $\lim\limits_{x \to 3^+} f(x)$

b) $\lim\limits_{x \to -2^-} f(x)$

c) $\lim\limits_{x \to 3} f(x)$

d) $\lim\limits_{x \to -5^+} f(x)$

Example 2:   Find the limits of the functions below. Make a sketch of the graph and use it to indicate why the limit is sensible.

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a) $\lim\limits_{x\to\infty}\left(\dfrac{2}{x^2}\right)$

b) $\lim\limits_{x\to\infty}\left(-\dfrac{4}{x^3}\right)$

c) $\lim\limits_{x\to\infty}(3x^4)$

d) $\lim\limits_{x\to\infty}\cos x$

<u>Section 5.4 Finding Limits as x Approaches a Constant</u>
<u>Looking Back 5.4</u>

Now that we know what a limit is and we know the official definition of a limit, let us look into finding the limit when x approaches a specific value, such as a constant.

We know that zero divided by any constant is zero: $\frac{0}{c} = 0$. Moreover, any constant divided by zero is undefined: $\frac{c}{0} =$ undefined.

In this section, we will investigate a special case of what happens when zero is divided by zero: $\frac{0}{0} =$ Indeterminate. Indeterminate is not equal to zero. It is also not undefined because zero is in the denominator and the numerator. It is neither $+\infty$ nor $-\infty$. It is called an indeterminate form. This means the answer is "not yet determined" and we need to keep working to find the correct solution.

As a rational function approaches a constant, most limits are fairly straightforward. We will often solve problems algebraically by substituting the value that x approaches in the rational function for x. We will often get a numeric solution. However, if we get $\frac{0}{0}$, we must use other methods to solve the problem, such as factoring and rationalizing the numerator or denominator (to extract roots).

<u>Looking Ahead 5.4</u>

Example 1:   Find $\lim\limits_{x \to 3} \frac{3x^2-x-2}{x^2+x-2}$.

Example 2:   Find $\lim\limits_{x \to 1} \frac{3x^2-x-2}{x^2+x-2}$.

1.   Use synthetic division to find the factors.

2.   Write the rational function in factored form.

3.   Find the limit of the factored form of the rational function.

Example 3:	Find $\lim\limits_{x \to 4} \frac{\sqrt{x}-2}{x-4}$.

Example 4:   If f is the identity function $f(x) = x$, find the limit for any value of c.

Like with a rational function, as a polynomial function approaches a constant, most limits are fairly straightforward. Again (like with a rational function), with a polynomial function substitution may be used to solve a problem algebraically.

Notice in the graph $f(x) = -2$ that for any value of x, $f(x)$ approaches -2 because $f(x)$ is never anything else. Therefore, if f is a constant function, its limit at any input is the same constant. This can be written $\lim_{x \to c} f(x) = \lim_{x \to c}(k) = k$.

Example 5:   a) Find $\lim_{x \to -1} x$.

                          b) Find $\lim_{x \to -1} 5$.

Just as there are rules for operations with polynomials and logarithms, there are rules for operating with limits. Let us investigate these properties.

Let L_1 and L_2 be real numbers. Let $\lim_{x\to c} f(x)$ be equal to L_1 and $\lim_{x\to c} g(x)$ be equal to L_2 for the following properties:

a) $\lim_{x\to c}[f(x) + g(x)] = L_1 + L_2$

The limit of the sum of two functions is the sum of their limits.

b) $\lim_{x\to c}[f(x) - g(x)] = L_1 - L_2$

The limit of the difference of two functions is the difference of their limits.

c) $\lim_{x\to c}[f(x) \cdot g(x)] = L_1 \cdot L_2$

The limit of the product of two functions is the product of their limits.

d) $\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] = \frac{L_1}{L_2}$ and $L_2 \neq 0$.

The limit of the quotient of two functions is the quotient of their limits.

Example 6:   Calculate the limits below using the properties from above.

a) $\lim_{x\to 4} x^2$

b) $\lim_{x\to -1} (x^3 - 4)$

c) $\lim_{x\to 3}(x^2 + 2x)$

d) $\lim_{x\to 2} 5x$

The above examples demonstrate two corollaries of rule c) above.

1.   If k is a real constant, then
$\lim_{x\to c}[kf] = k\lim_{x\to c}[f] = kL_1$

2.   If n is a positive integer, then
$\lim_{x\to c}[f^n] = (\lim_{x\to c}[f])^n = L_1{}^n$

<u>Section 5.5 Finding Limits as x Approaches Infinity</u>
<u>Looking Back 5.5</u>

In the previous section, a limit was defined as a number that $f(x)$ stays arbitrarily close to if x is kept close to c, but not equal to c (where c is a constant).

In the case of the parent rational function $f(x) = \frac{1}{x}$, we saw in Section 5.2 that $\lim\limits_{x \to \infty} \frac{1}{x}$ is equal to 0. This means the limit of $\frac{1}{x}$ as x approaches infinity is 0. According to the definition, $\frac{1}{x}$ stays arbitrarily close to 0 if x gets larger and larger.

We know that $\frac{1}{x} \to 0$ as $x \to \infty$ because $\frac{1}{\text{small}}$ becomes small, $\frac{1}{\text{bigger}}$ becomes smaller, and $\frac{1}{\text{really big}}$ becomes really small. As 1 is divided by larger and larger values of x, $\frac{1}{x}$ gets smaller and smaller. It will never become a negative number or 0 but it will get close to 0.

The function $f(x) = 2x^3$ is a polynomial function. The exponent is a non-negative integer. The limit $\lim\limits_{x \to \infty} 2x^3 = \infty$ is obvious. As x grows larger and larger, $f(x)$ does the same.

<u>Looking Ahead 5.5</u>

Let us look specifically at three rational functions: the first one with the degree in the numerator and the degree in the denominator being the same; the second one with the degree of the numerator being less than the degree of the denominator; and the third one with the degree of the numerator being greater than the degree of the denominator.

Let us investigate what the limits become as x approaches infinity given these rational functions.

| Example 1: | Find the limit of the rational function $f(x) = \frac{3x}{x+1}$ as x approaches infinity. |

When the degree of the numerator equals the degree of the denominator, the limit as x tends to infinity is the ratio of the lead coefficients of the numerator to that of the denominator.

Let us investigate the second case in which the degree of the numerator is less than the degree of the denominator.

Example 2:   Find the limit of the rational function $f(x) = \frac{2x+1}{3x^2+1}$ as x tends to infinity.

Another rule we can use to find the limit of a rational function is: If the degree of the numerator is less than the degree of the denominator, then the limit is 0.

Let us explore the third case in which the degree of the numerator is greater than the degree of the denominator.

Example 3:   Find the limit of the rational function $f(x) = \frac{-2x^4+3x}{x-1}$ as x approaches infinity.

Multiplying by $\frac{1}{x}$ does not always yield an equation that is easier to evaluate. A rule to follow here is: If the degree of the numerator is greater than the degree of the denominator, then the numerator is getting bigger than the denominator and the whole fraction is getting bigger and bigger approaching infinity.

If the ratio of the lead coefficients is positive, the limit is positive infinity. If the ratio of the lead coefficients is negative, the limit is negative infinity.

$$\text{If k is a constant, then } \frac{\infty}{k} = \infty \text{ and } \frac{k}{\infty} = 0.$$

In summary:

a) $\lim\limits_{x \to \infty} \frac{6x^4 + 2x^2}{3x^2 - 2}$

The dominant term is $\frac{6x^4}{3x^2} = \frac{2x^2}{1} = 2x^2$. The higher degree is in the numerator and the quotient is positive, so the limit is positive infinity.

b) $\lim\limits_{x \to \infty} \frac{3x^2 - 2}{6x^4 + 2x^2}$

The dominant term is $\frac{3x^2}{6x^4} = \frac{1}{2x^2}$. The higher degree is in the denominator, so the limit is 0.

c) $\lim\limits_{x \to \infty} \frac{-4x^3 + 1}{2x^3 - 3x}$

The dominant term is $\frac{-4x^3}{2x^3} = -\frac{2}{1} = -2$. The degree of the numerator and the degree of the denominator are equal, so the limit is the ratio of the lead coefficients.

Section 5.6 Asymptotes and Limits
Looking Back 5.6

We looked at a problem in Algebra 1 that showed a can of water with a given amount of liquid. As the diameter of the can increased, and the volume of the water stayed the same, the height of the water decreased.
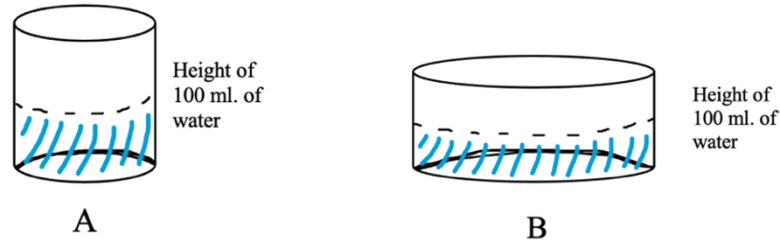
As the diameter of the base of the can increases, the same volume of water spreads out over a larger surface area, and the height of the water in the can decreases. The inverse is also true: if can B is filled with 100 ml. of water and the height is measured, it will be less than the height of the water if it is poured into can A and measured. As the diameter of a can decreases, the height of the water in the can increases.

The graph of this relationship shows that as the diameter of the can increases, the water just barely covers the base and approaches 0. If the diameter of the can approaches infinity, the water in the can approaches 0 but never reaches 0. It never becomes 0 ml. because there is still 100 ml. of water in the can. (Excluding the effects of surface tension, when the water spontaneously bunches into droplets rather than remaining a flat very thin layer covering a large surface area.)
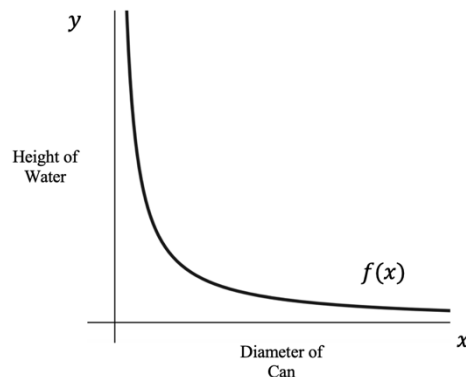
As x approaches ∞, y approaches 0. As the diameter of the can increases, the height of the water decreases.

As x approaches 0, y approaches ∞. As the diameter of the can decreases, the height of the water increases.

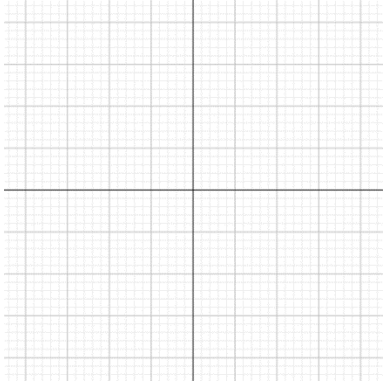The relationship can be written in terms of limits:

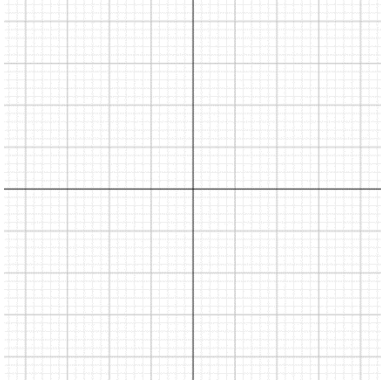$$\lim_{x \to \infty} f(x) = 0$$

$$\lim_{x \to 0} f(x) = \infty$$

The asymptotes are $x = 0$ and $y = 0$. The Greek word "asymptote" means "not due to coincide." The graph gets arbitrarily close to each asymptote but does not touch it.
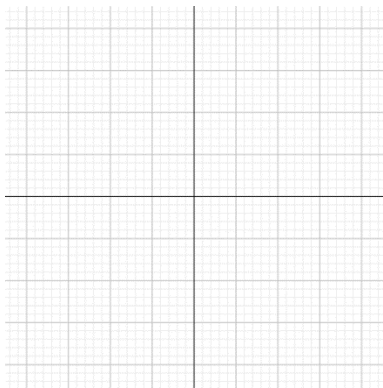
Looking Ahead 5.6

Example 1:   Find the asymptotes for $f(x) = \frac{3x}{x+1}$.

Example 2:   Find the vertical and horizontal asymptotes for $f(x) = \frac{x+1}{x+1}$.

Example 3:   Find the vertical and horizontal asymptotes for $f(x) = \frac{x^2+2x+1}{4x^2-9}$.

To summarize, $f(x) = \frac{x^2+2x+1}{4x^2-9}$. The domain is all real numbers except $x = -\frac{3}{2}$ and $x = \frac{3}{2}$. There are two vertical asymptotes at $x = -\frac{3}{2}$ and $x = \frac{3}{2}$. The horizontal asymptote is at $x = \frac{1}{4}$.

Example 4:   Find all asymptotes for $f(x) = \frac{x^2-2x-3}{x-3}$.

1) Factor if possible

2) Cancel if possible

3) Set denominator equal to 0

Note: The cancelled terms are still indeterminate. They are not horizontal asymptotes but form a hole in the graph at that point.

To summarize, $\frac{x^2-2x-3}{x-3} = x + 1$ except at $x = 3$ so $x + 1$ is identical to $f)x)$ rather than being an asymptote except where there is a hole at $x = 3$ which is an example of a point discontinuity.

<u>Section 5.7 Continuity and Discontinuity</u>
<u>Looking Back 5.7</u>

A continuous function is a function in which the limit of $f(x)$ as x approaches c is equal to the value $f(c)$ for all $x = c$ in the domain. Previously, a function was said to be continuous if you could trace it without lifting your pencil off the paper. That means there are no gaps or discontinuities in the graph of the function. We know that if there is a hole in the graph of the function it is considered a discontinuity. Vertical asymptotes cannot be crossed or touched; however, horizontal asymptotes can be crossed or touched.

<u>Looking Ahead 5.7</u>

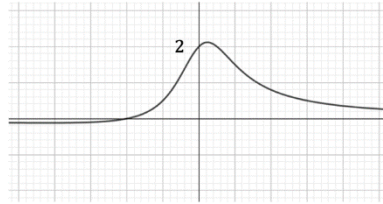Example 1:   Examine the graph of $f(x) = \frac{x+2}{x^2+1}$.

a) Is it continuous or discontinuous?

b) Find $\lim\limits_{x \to \infty} \frac{x+2}{x^2+1}$.

c) Find $\lim\limits_{x \to 0} \frac{x+2}{x^2+1}$.

Example 2:   Given below is the table and graph of $f(x) = \frac{x+2}{x^2+1}$. Explain how they verify the conclusions found in part a), b), and c) from Example 1.

x	$f(x)$
−10,000	−0.0001
−5,000	−0.0002
0	2
5,000	0.0002
10,000	0.0001

A function can cross an asymptote at finite values, but not in infinity. The graph can cross many times, but not in infinity for that is what makes it an asymptote as the function approaches a limit, but does not reach it.

Example 3:   Factor the numerator and denominator of $f(x) = \frac{3x+6}{x^2+5x+6} + 1$ and find all asymptotes and discontinuities.
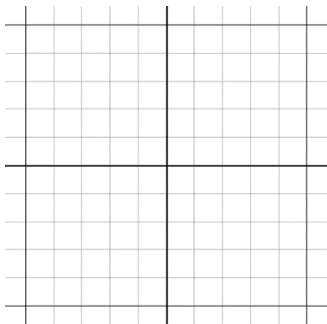
Summing Up Discontinuities:

1.   If the denominator of $f(x)$ contains a factor $(x - c)$ and $f(c)$ is undefined (because division by zero is not allowable), then the graph has a discontinuity at $x = c$.

2.   When the equation for $f(x)$ is simplified and $(x - c)$ is cancelled out, then this is called a removable discontinuity. There is a gap at one point in the graph.

3.   If the factor $(x - c)$ stays in the denominator after all factoring and simplifying is complete and does not cancel out, then the graph of $f(x)$ gets infinitely large positively, negatively, or at both ends. There is a vertical asymptote at $x = c$ as x gets closer and closer to c.

Example 4:   At what point(s), if any, is the piecewise function discontinuous:

$$f(x) = \begin{cases} x - 2, x \leq -1 \\ x^2 + 2, -1 < x \leq 3 \\ x^3 + 1, x > 3 \end{cases}$$

<u>Section 5.8 Definition of Continuity</u>
<u>Looking Back 5.8</u>

The previous section included a "Summing Up Discontinuities" segment, which consisted of three summary statements. If there are no discontinuities, then a function is continuous. Below are three statements that summarize continuities…

Summing Up Continuities:
A function $f(x)$ is continuous at c if…

1. $f(c)$ exists; c is in the domain of the function

2. $\lim\limits_{x \to c} f(x)$ exists; the function has a limit as $x \to c$

3. $\lim\limits_{x \to c} f(x) = f(c)$; the value of the limit is equal to the value of the function

These make up the formal definition of continuity; informally, we could write the following statements:

1.   For every input there is an output such that if x is equal to a, then there is a y equals b that corresponds with it.

2.   Both sides of a function approach the same limit. The right-side and left-side limits are the same. In other words, the two-sided limit is the same as each one-sided limit as it approaches c.

3.   The limit approaches the same value as the function does at c.

There is an Intermediate Value Theorem for functions that are continuous, which states: "$y = f(x)$ is continuous on the closed interval and takes on every value between $f(a)$ and $f(b)$. If c is in the interval $[a, b]$, then some value of y_n is equal to $f(c)$ given y_n is between $f(a)$ and $f(b)$.
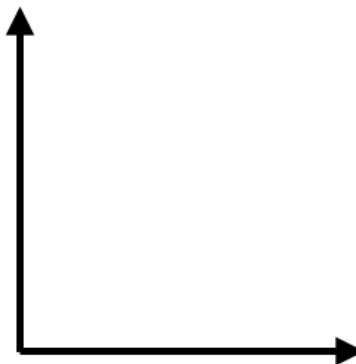
This graphically demonstrates the previous points made concerning continuous functions.

The Intermediate Value Theorem states that if a function f is continuous over an interval $x = [a, b]$, then there must be at least one x value in the interval at which the function takes on every value between $f(a)$ and $f(b)$, its values at the endpoints of the interval. Intuitively, this means a line with no breaks cannot pass from one point to another without smoothly crossing the space in between.

No matter how you draw the function, every x between a and b and every y between $f(a)$ and $f(b)$ correspond to at least one point on the curve.

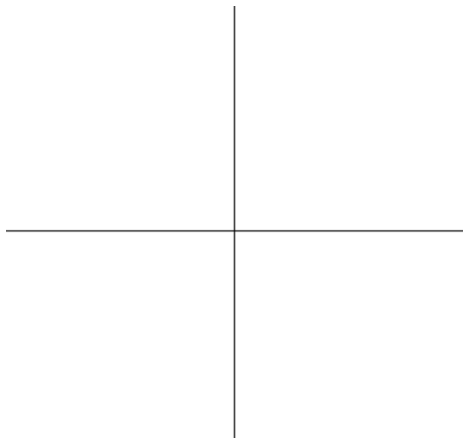This is another way of defining continuity.
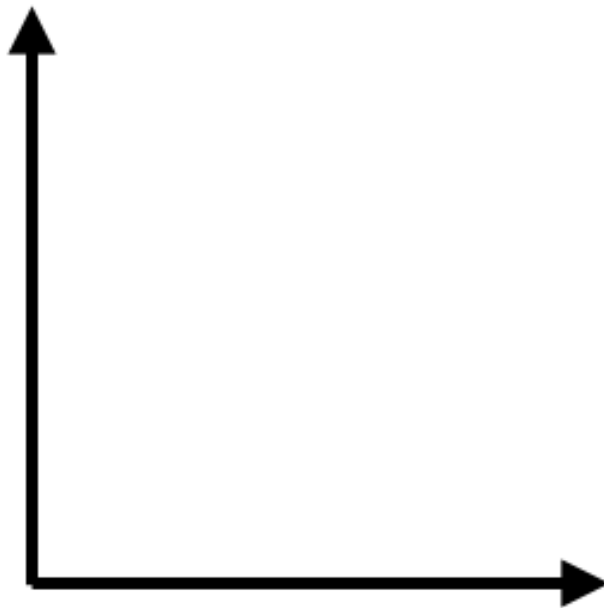
Looking Ahead 5.8

Example 1:   State whether or not the function below is continuous. If it is discontinuous at the point specified, tell which of the three conditions of continuity are not met.

$$f(x) = 3^x \text{ at } x = 0$$

Example 2:   State whether or not the function below is continuous. If it is discontinuous at the point specified, tell which of the three conditions of continuity are not met.

$$g(x) = \begin{cases} 2x + 1 \text{ for } x < 2 \\ x^2 + 3 \text{ for } x \geq 2 \end{cases} \text{ at } x = 2$$
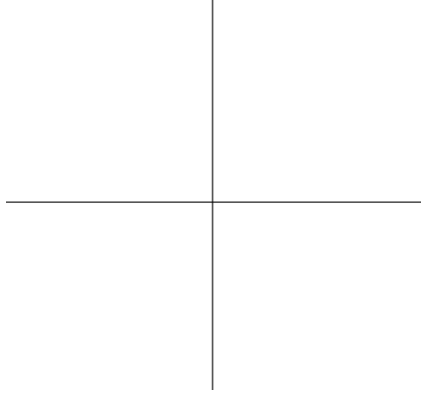
Example 3:   State whether or not the function below is continuous. If it is discontinuous at the point specified, tell which of the three conditions of continuity are not met.
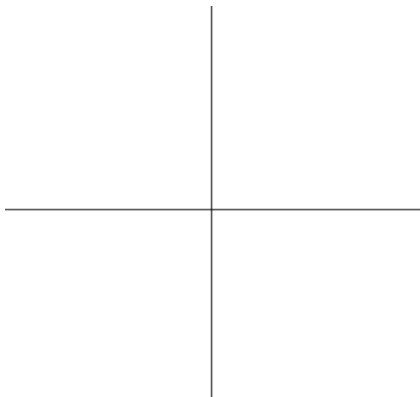
$$h(x) = \frac{1}{x-4} \text{ at } x = 4$$

Example 4:   State whether the function below is continuous. If it is discontinuous at the point specified, tell which of the three conditions of continuity are not met.

$$f(x) = \frac{\sin x}{x} \text{ at } x = 0$$

x	-0.0000001	-0.00001	-0.001	-0.1	0	0.1	0.001	0.00001	0.0000001
$f(x)$									

<u>Section 5.9 Logarithms and Limits</u>
<u>Looking Back 5.9</u>

We have investigated limits as functions approach 0. We have also investigated limits as functions approach infinity.

In the function $g(x) = \frac{100}{x}$, as x gets larger and larger $g(x)$ gets smaller and smaller and closer to 0 (but never 0 or less than 0). At 0, the function is undefined.

In the function $h(x) = 100x$, as x gets larger and larger $h(x)$ gets larger and larger and closer to infinity. We say this function is unbounded and when writing intervals, we use symbols to show it is not included.

When something is said to be infinite, it is boundless. When we say God is infinite, He is infinite in all things. His wisdom and knowledge are infinite; they have no limit. His patience is inexhaustive. His forgiveness is as far as the east is from the west, which is endless- one becomes the other. His ways are above our thoughts, and for any power in the universe and beyond, His is greater still.

How can we measure infinity? Which is greater in infinity: $h(x) = 100x$, $k(x) = 1{,}000x$, or $m(x) = 10x^3$? Which reaches infinity faster?

They are all 0 when $x = 0$. At $x = 1$, $m(x) < h(x) < k(x)$. At $x = 2$, $m(x) < h(x) < k(x)$ again. However, at $x = 10$, $h(x) < m(x)$ and $k(x) = m(x)$.

When x gets infinitely large $m(x)$ overtakes $k(x)$ because the infinitely larger number in $m(x)$ is being multiplied by itself three times, and then being multiplied by 10. In $k(x)$ the infinitely larger number will have three zeroes added to the end; $h(x)$ is always smaller than $k(x)$.

Which will be larger in infinity: $m(x) = 10x^3$ or $n(x) = 3^x$?

Comparing table value shows us that $n(x)$ overtakes $m(x)$ as early as $x = 10$.

x	$m(x)$	$n(x)$
1	10	3
10	10,000	59,049
1,000	$1 \cdot 10^{10}$	132,207,081,948,080,636 …

The power function $m(x) = 10x^3$ has power, but the exponential function $n(x) = 3^x$ has explosive power. It may seem pointless to ask this since both become infinitely large, but if we evaluate $\lim_{x \to \infty} \frac{10x^3}{3^x}$ we see that this is indeterminate just as we encountered in Section 5.4 when dealing with rational functions.

However, $3^x > 10x^3$ for all x beyond $x \approx 7.65$. This ratio will tend toward 0; therefore, $\lim_{x \to \infty} \frac{10x^3}{3^x} = 0$.

<u>Looking Ahead 5.9</u>

You might recall from Algebra 2 that logarithmic functions are the inverses of exponential functions; logarithmic functions can be written as exponential functions. Exponential and logarithmic functions have a base greater than 0 ($b > 0$).

When the base of an exponential function is between 0 and 1 ($0 < b < 1$), the exponential function has a base that is a proper fraction or decimal of a proper fraction and is a decreasing function. Therefore, the base of a logarithm is undefined when $b = 0$ as well. Since it is the inverse of an exponential function or can be written as a logarithmic function, it holds the same properties ($b > 0$).

Example 1:   What is the limit of $\log_b x$ as x approaches 0 if $0 < b < 1$? Let $b = 0.5$ and find $\lim_{x \to 0^+} \log_{0.5} x$.
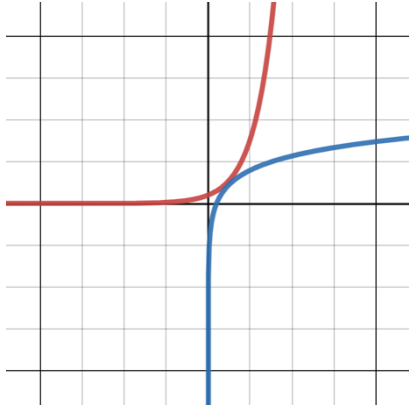
*(The inverse of $y = a^x$ is $x = a^y$ and $y = \log_a x$ is equal to $x = a^y$.)

   The exponential function $f(x) = b^x$ when $0 < b < 1$ is shown in red. Let $b = \frac{1}{2}$. The logarithmic function $f(x) = \log_b x$ is the inverse and is shown in blue. In the example, b is equal to $\frac{1}{2}$.

Example 2:   What is the limit of $\log_b x$ as x approaches ∞. Let $0 < b < 1$.

Example 3:   Draw the graph of $\log_b x$ if $b > 1$. Find $\lim\limits_{x \to 0^+} \log_b x$. Let $b = 1.5$.

The graph of $f(x) = 1.5^x$ is shown in red. The graph of $f(x) = \log_{1.5} x$ is its inverse and is shown in blue as a reflection over the line $y = x$.

Example 4:   Draw the graph of $y = \log_{10}(x - 2)$. Find $\lim\limits_{x \to 2^+} \log_{10}(x - 2)$.

<u>Section 5.10 Limits and the Natural Logarithm</u>
<u>Looking Back 5.10</u>

Let us talk more about logarithms before discussing the natural logarithm. The function $y = \log_3 x$ asks: "What value do we raise 3 to in order to get x?"

$$3^y = x$$

The values of x change, so the values of y will change as well. We do know $\log_3 1$ is equal to 0 because 3^0 is equal to 1. We could make an easy table of values if we just find powers of 3 for the perfect cubes.

x	y
1	0
3	1
9	2
27	3

When we graph these values, we get the shape of the log function.
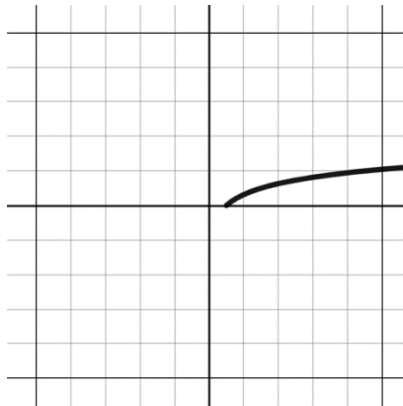
If you invest money at a fixed annual percentage rate of r compounded n times a year, the amount after t years is $P = I(1 + \frac{r}{n})^{tn}$ in which P is the principal, I is the initial amount invested, r is the annual percentage rate (APR), t is time in years, and n is the number of periods of compounding annually.

If you invest $100 at a fixed annual percentage rate of 5% compounded quarterly, the amount after one year is as follows:

$$P = \$100(1 + \frac{0.05}{4})^{1 \cdot 4} \approx \$105.10$$

If you compound it monthly, the earnings after one year are as follows:

$$P = \$100(1 + \frac{0.05}{12})^{1 \cdot 12} \approx \$105.12$$

No matter how many times a year the initial amount is compounded, the investment grows only by a few more cents and never gets more than approximately $105.13. It reaches a limit at this amount because there is a limit to how much money can grow due to compound interest alone.

The limit is actually $P = e^{rt}$ or, in this case, $P = 100^{0.05} \approx \$105.13$. We have explored this previously in Module 2 when we studied logarithms and natural logarithms, but we will investigate this further in Section 5.12 when we look at applications of logarithms.

Remember, the logarithmic function is only defined for positive values of x so the domain is $x > 0$, where the results are real numbers. It is not defined for x equal to 0 or less than 0.

<u>Looking Ahead 5.10</u>

We learned about logarithms and the natural logarithm in Algebra 2 and the Pre-Calculus section of this text. The symbol e is the symbol for the base of the natural logarithm. It is natural because it appears naturally in finance and shows up often in God's created nature. It is a number. Just as pi (π) is 3.14 ... and phi (φ) is 1.618 ..., e is 2.718 ... These numbers are irrational and π and e are also called transcendental numbers. Though irrational; φ is not transcendental. It is an algebraic number because $\frac{1+\sqrt{5}}{2}$ is algebraic as it is the root of a polynomial equation.

The base 10 is the most common base for logarithms ($\log x = \log_{10} x$). Decibels (the measurement for intensity of sound), Richter scales (tools used to measure the intensity of earthquakes), and pH (the measurement of the acidity or alkalinity of substances), all use base 10 scales. The base e is another common base ($\ln x = \log_e x$) because it arises naturally in mathematics and science. The symbol ln (pronounced "ell en") represents the natural logarithm.

Though we rarely see the base ten logarithm in a Calculus course, it is the one most commonly applied in mathematics; hence, it is often written simply as $\log x$ without a base. If we do not put in a base when using a calculator, it is assumed to be 10; base 10 is the common default. Moreover, we will not see $\log_e x$. It will appear as $\ln x$ (pronounced "ell-enn x").

Euler used the most typical names of terms used in this section, and because he was Swiss and spoke French, he probably called le Logarithme Naturel or "ln" (L – N) for short. You can just remember the name Ellen (or L-N) for "logarithm natural."

The question "What power do we raise e to in order to get the absolute value of x (positive values of x)?" comes from $\log_e|x|$.

Example 1: Complete a table of values and graph the function $f(x) = \ln x$; $y = \ln x$ can be written "$e^y = x$."
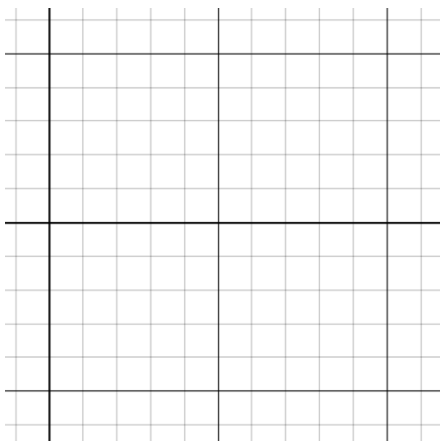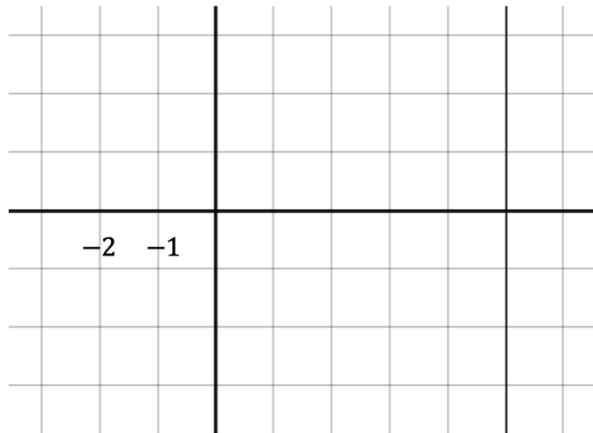

In the previous section, we learned $\lim\limits_{x\to\infty} \log x = \infty$. It increases slower and slower as it goes on but does continue to increase forever. We can say that it increases at a decreasing rate. It is an inverse function of the exponential of any base b in which b is greater than 1 and increases at an increasing rate.

Example 2:	Find $\lim\limits_{x\to\infty} \ln|x|$.

Example 3:	Find $\lim\limits_{x\to-2^+} \ln(x+2)$.

x	$\ln(x+2)$
-1	
0.71	
-1.9	
-1.99	
-1.999	
-1.9999	

Section 5.11 Using Technology to Analyze Limits
Looking Back 5.11

Gottfried Wilhelm Leibniz and Sir Isaac Newton are both credited with inventing Calculus at about the same time. Augustin-Louis Cauchy and Leonhard Euler are other famous mathematicians who used the concepts of limits to further the study of mathematics. However, the precise definition of a limit comes to us from Bernhard Bolzano.

Bernhard Bolzano was born in the late 1700s in Prague. He attended a type of secondary school there called the *Gymnasium*. He then went on to the University of Prague where he received a doctorate of philosophy and became chair for religious doctrine in the Philosophical Faculty. He was also ordained as a Roman Catholic priest.

Bernhard Bolzano studied logic and led his students in worship. He was politically progressive and a great philanthropist. With his pupils, he not only wrote about politics and philosophy, but worked to benefit poor, discriminated against, and disadvantaged people. Bolzano and his students established poor houses and loan banks for the working class, as well as country pride elementary schools and libraries, and homes for the blind.

He was initially appointed and finally dismissed from his professorship by Emperor Franz. He was then forbidden to teach, preach, or publish his works because of his progressive theological and political ideas. Therefore, Bolzano lived out his life on a meager pension and used his time to study logic and write what would go on to become the four volumes of the *Theory of Science*, which were published in 1837.

With his dedication to God and God's work, and his love for the study of mathematics, it is logical that he would have such interest in limits and the unlimited Infinite.

Looking Ahead 5.11

Moving ahead to the present day, studies of limits have been aided by technology. Making graphs and tables by hand and investigating them in terms of what happens when x approaches ∞ can pose problems: one being time constraints.

Let us use a graphing utility or graphing calculator to investigate problems involving limits.

Example 1:   Draw a graph of the function $f(x) = \frac{x^3 - 4x^2 + 6x - 5}{x - 2}$ to investigate the end behavior of the function.

Example 2:   Use a table to find $\lim\limits_{x \to 2} \frac{x^3 - 5x^2 + 7x - 2}{x - 2}$.

Example 3:   Use a graphing calculator to find the limit of the functions below from Example 1 and Example 2.

a) $\lim\limits_{x \to 2} \frac{x^3 - 4x^2 + 6x - 5}{x - 2}$

b) $\lim\limits_{x \to 2} \frac{x^3 - 5x^2 + 7x - 2}{x - 2}$

Section 5.12 Applications of Limits
Looking Back 5.12

In Algebra 2, we learned about the natural logarithm (e). We realized that if you put \$1 in a bank and let it compound continuously, you would not get infinitely wealthy, but approach a limit of about \$2.71.

When we investigate $\lim_{n\to\infty}(1+\frac{1}{n})^n$, we know that $\frac{1}{n}$ gets closer and closer to 0 as n approaches infinity.

That means $1+\frac{1}{n}$ gets close to 1 and 1^∞ is indeterminate just as $0^0, \infty^0, \frac{\infty}{\infty}$, and ∞ and $-\infty$ are indeterminate. This is because they have no direct numerical meaning.

Let us review using substitution to find this limit in terms of compounding interest on \$1 at 100% interest:

$$(1+\tfrac{1}{12})^{12} \approx 2.61304 \text{ in 12 months}$$

$$(1+\tfrac{1}{52})^{52} \approx 2.6926 \text{ in 52 weeks}$$

$$(1+\tfrac{1}{365})^{365} \approx 2.71457 \text{ in 365 days}$$

That looks familiar:

$$\lim_{n\to\infty}(1+\frac{1}{n})^n = e$$

This limit seems to be approaching the base of the natural logarithm. In Section 5.10, we investigated logarithms, the natural constant (e), and the natural logarithm ($\ln x$).

Looking Ahead 5.12

Let us suppose you are trying to decide whether to put your \$100 savings bond money in Gold Bank, which offers an annual interest rate of 5.75% compounded quarterly, or Silver Bank, which offers an annual interest rate of 5.75% compounded daily.

Example 1:   Which bank will you earn more money with after one year, Gold Bank or Silver Bank?

Most banks allow for continuous compounding so that your money is always growing, even if it is growing ever so slightly.

The only thing changing in the expressions is the value of n so it could be written as follows:

a) Gold Bank: $100(1 + \frac{0.0575}{n})^n$ when $n = 4$

b) Silver Bank: $100(1 + \frac{0.0575}{n})^n$ when $n = 365$

We know that the limit of $(1 + \frac{1}{n})^n$ as n approaches infinity is the irrational number e. Therefore, the expression written in terms of e could be $100(e)^{0.0575} \approx \$105.92$. You would have almost $\$106$ in the bank after a year with an annual interest rate of 5.75% compounded continuously.

Example 2:   Let us suppose you deposit your $100 at Bronze Bank, which offers an annual interest rate of 6.75% compounded continuously.

---

a) How much would you have in the bank after one year?

b) How much would you have in the bank after two years?

c) How much would you have in the bank after t years?

<u>Section 5.13 Archimedes and Approximating Pi</u>
<u>Looking Back 5.13</u>

        Archimedes is a famous mathematician of antiquity. He made many great contributions to science as well, but one of his greatest contributions to any field was his approximation for pi. In this section, we will be investigating his method for approximating pi.

        There is evidence that pi was known to the ancient Egyptians, Babylonians, and Chinese. Still, it dates back even farther to the Hebrew people who made use of it when building the temple for God; Solomon was instructed by God on how to build His house, His center of worship. In 2 Chronicles 4:2, we read: "Also he made the cast metal sea, ten cubits brim to brim, circular in form, and its height was five cubits, and its circumference thirty cubits."

        The process Archimedes used to find pi was an iterative process. He started with a circle that had a diameter of 1. Because $C = \pi d$ was widely known, he knew when d was equal to 1, c was equal to π. Archimedes tried to find pi by finding the circumference of the circle. He employed his knowledge of limits by finding the values closest to π. To find those values, he would begin by finding the perimeter of a polygon circumscribed outside of the circle and another with the same number of sides inscribed inside the circle. Archimedes made the polygon with more and more sides, and repeated the process to get more accurate approximations. His idea was that the perimeter of a polygon with infinitely many sides (n) would become closer to the circumference of the circle; he speculated it would approach a limiting value.
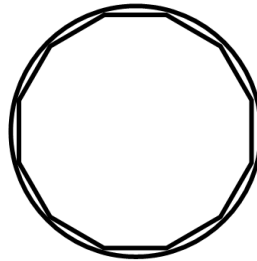
        Archimedes used geometry and inscribed and circumscribed 12-sided polygons, 24-sided polygons, 48-sided polygons, and 96-sided polygons until he found two values that pi was between.

$$3\frac{10}{71} < \pi < 3\frac{1}{7}$$ or $$3.1408 < \pi < 3.1429$$

| Inscribed Polygon Perimeter | Circumscribed Polygon Perimeter | | Inside Polygon Perimeter | Outside Polygon Perimeter |

        We will use Archimedes' method to find the areas of a 12-sided polygon inscribed and circumscribed about a unit circle, and then develop an algorithm to use with the graphing calculator to find the areas of many-sided polygons. These polygons are all regular.

        Many early Greek mathematicians used the area of polygons to approximate pi. Archimedes was the first to use the perimeter of polygons to approximate pi.

<u>Looking Ahead 5.13</u>

The area of a circle is $A = \pi r^2$. If we let the radius be 1, then A is equal to $\pi(1)^2$ and A is equal to π. Therefore, our area will be approximately π.

1.   Draw a circle with a radius of 1 unit (you choose the unit). Draw a regular dodecagon inside it. Divide the dodecagon into triangles.
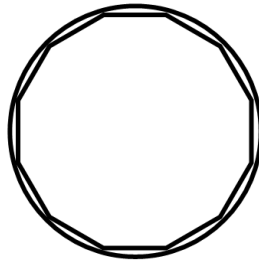

2.   What is the central angle of each triangle? We will try to find the area of each triangle and multiply that by 12 to get the area of the dodecagon.
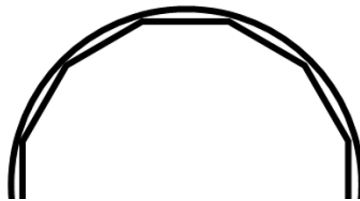
3.   Let us find the area of the triangle with the dashed line as the unknown height. The isosceles triangle is now split into two right triangles. Each central angle is 15° ($\frac{30°}{2} = 15°$). The sides of the isosceles triangles are 1 unit each.

4.   Use trigonometry to find the height of the triangle.
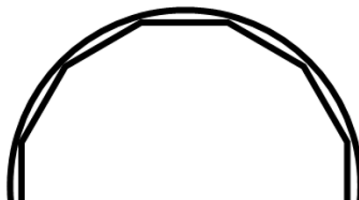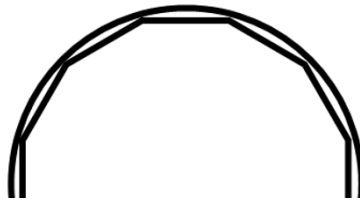
$$\cos 15° = \frac{h}{1}$$

This is because the height of the triangle is the adjacent side to the 15° angle of the right triangle and the side of the isosceles triangle is the hypotenuse.

$$h = \cos 15°$$

5.   Use trigonometry to find the base of the triangle.

$$\sin 15° = \frac{\frac{1}{2}b}{1}$$

This is because the opposite side of the right triangle with the 15° angle is only half the base, and the side of the isosceles triangle is the hypotenuse.

$$\sin 15° = \frac{1}{2}b$$

$$b = 2\sin 15°$$

6.   Use the answers from Step 4 and Step 5 to find the area of one isosceles triangle.

$$A = \frac{1}{2}b \cdot h$$

$$A = \frac{1}{2}(2\sin 15°)(\cos 15°)$$

$$A = \sin 15° \cos 15°$$

7.   Find the area for the dodecagon.

$$A = 12 \sin 15° \cos 15° = 3$$

Make sure the calculator is in degree mode as radians are more often used in Calculus and radian mode is the default on the calculator page.

Now we can get an algorithm.

$$A = 12 \sin 15° \cos 15° \text{ (Final step)}$$

$$A \approx 3$$

The number of sides of the polygon is used to find the final area. Call the number of sides of the polygon used to find the final area n.

$$A = n \sin 15° \cos 15°$$

The 15° comes from the degree in the circle divided by the number of sides and then that was divided in two for each bisected angle of the central angle of each triangle.

$$\frac{360°}{n}\left(\frac{1}{2}\right) = \frac{180°}{n}$$

$$A = n\left(\sin\frac{180°}{n}\right)\left(\cos\frac{180°}{n}\right)$$

Now we have a formula to approximate the area of an n-gon inscribed in a unit circle.

The practice problems for this section will require us to find the area of a dodecagon circumscribed about a unit circle and from that derive an algorithm for an n-gon.

The last part will be using those formulas in the graphing calculator to approximate pi for many-sided polygons and writing inequalities for these approximations.