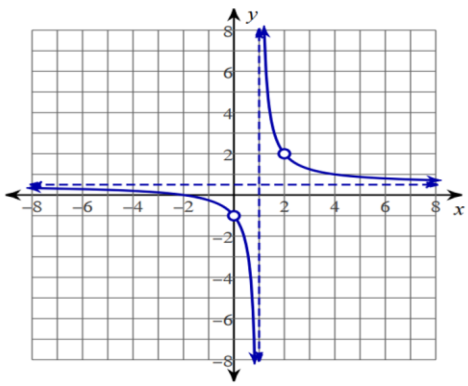
Pre-Calculus and Calculus Module 2 Natural Logarithms and the Logistic Functions

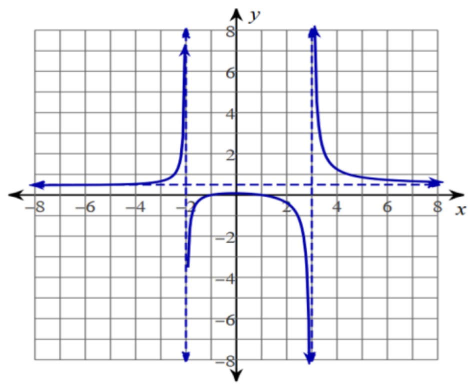
Section 2.1 Exponential Functions
Practice Problems 2.1

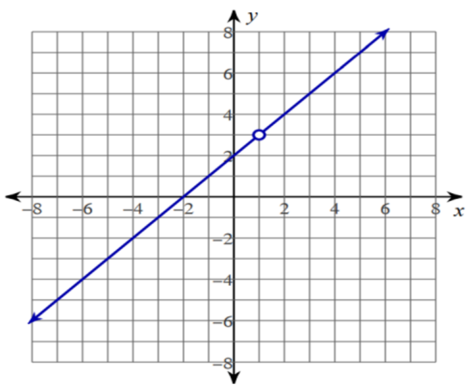
1. Which of the following are rational functions?

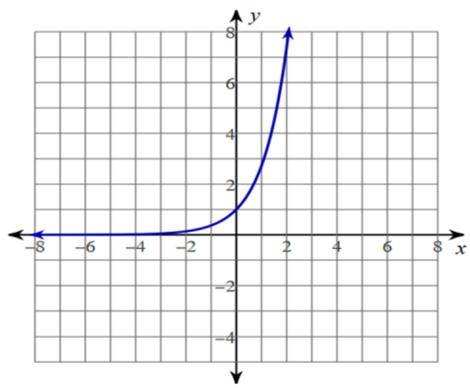
a) $\frac{x}{-2}$ b) $\frac{3x+1}{x-4}$ c) $\frac{\sqrt{x}}{9}$ d) $\frac{5}{x^{\frac{1}{3}}}$

For Problem 2 -5, tell if each graph has a removable or non-removable discontinuity, neither of both.







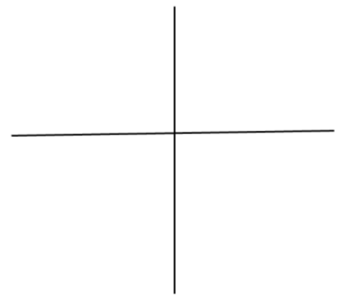


Use the following function to answer Problem 6-9.

$$j(x) = \frac{2x-4}{x^2-2x}$$

6. Simplify the rational expression.

7. Name any vertical or horizontal asymptotes?



8. Are there any removable discontinuities?

9. Sketch the graph of the function.

Use the following function to answer Problem 10 - 13.

$$g(x) = \frac{5x^4}{x^4 - 4x^2}$$

10. Name any vertical or horizontal asymptotes?

11. Are there any removable discontinuities?

12. Are there any non-removable discontinuities?

13. What is the end behavior of the function?

Use the following function to answer Problem 14-15.

$$j(x) = \frac{x^3}{x^2 + 7}$$

14. Write the rational function $\frac{p(x)}{q(x)}$ as $p(x) + \frac{r(x)}{q(x).}$ What is the slant asymptote?

15. What is the end behavior of the function?

For Problem 16-18…

a) Name any asymptotes.

b) Name any removable discontinuities.

c) Tell the end behavior of the function. (Hint: Factor the polynomials first if possible.)

16. $f(x) = \frac{3x-3}{x(x-1)}$

17. $g(x) = \frac{5x^2-10x}{20x^2}$

18. $h(x) = \frac{x^2}{x^4+4x^2}$

19. For the function $j(x) = \frac{5x^4}{x^4-2x^2}$, divide the numerator by the denominator to demonstrate that $y = 5$ is a horizontal asymptote. What are the vertical asymptotes? Are there any removable discontinuities?

20. Find the slant asymptote for the function $k(x) = \frac{x^2-2x-2}{x-3}$. Use long division or synthetic division; are there any non-removable discontinuities?

Section 2.2 Review of Exponential Functions

Practice Problems 2.2

For Problem 1-3, tell which number of the pair given is greater.

1. $4^{\sqrt{2}}$ or $4^{1.2}$

2. 7^{π} or $7^{3.14}$

3. $5^{-\sqrt{2}}$ or $5^{-\sqrt{3}}$

For Problem 4-6, solve for x in each equation and check your answer.

4. $\frac{1}{8} = 2^{x-4}$

5. $625 = 5^x$

6. $\sqrt{7} = 49^{x+2}$

For Problem 7 and 8, use the information given to solve the problem.

7. What is incorrect in the series of numbers below?

$$\frac{7^{1.5}}{49^{1.8}} = \frac{1}{7^{0.3}} = 7^{-0.3}$$

8. Find the error in the steps below and then find the correct solution.

Step 1: $4^{4-x} = 4^{2(x-1)}$

Step 2: $4^{4-x} = 4^{2x-1}$

Step 3: $4 - x = 2x - 1$

Step 4: $-3x = -5$

Step 5: $x = \frac{5}{3}$

For Problem 9 and 10, tell whether the statement is true or false and explain why.

9. A function $f(x)$ is increasing if $f(x_2) < f(x_1)$, where $x_2 < x_1 < 0$.

10. For $f(x) = b^x$ with $b > 1$, b^x grows larger than any polynomial function in the long run (Hint: Let $f(x) = 3^x$ and $g(x) = x^{100}$. Take the ratio of $\frac{f(x)}{g(x)}$ and see if it is greater than any positive integer as x gets larger).

For Problem 11-16, follow the instructions given to solve the problem.

11. Simplify $16^{\frac{1}{2}}$

12. Fill in the blanks to explain the solution to Problem 11:

$$16^{\frac{1}{2}} = (16^1)^{\underline{\qquad}}$$

$$16^{\frac{1}{2}} = (4 \cdot \underline{\qquad\qquad})^{\frac{1}{2}}$$

$$16^{\frac{1}{2}} = 4^{\underline{\qquad}} \cdot 4^{\underline{\qquad}}$$

$$16^{\frac{1}{2}} = 4^{\underline{\qquad} + \underline{\qquad}}$$

$$16^{\frac{1}{2}} = \underline{\qquad\qquad}^{1}$$

$$\therefore 16^{\frac{1}{2}} = 4$$

13. Simplify $(-27)^{\frac{1}{3}}$

14. Simplify $4^{\frac{3}{2}}$

15. Fill in the blanks to explain the solution to Problem 14:

$$4^{\frac{3}{2}} = (4^{\_\_\_\_})^3$$

$$4^{\frac{3}{2}} = (\sqrt[2]{\_\_\_\_\_\_})^3$$

$$4^{\frac{3}{2}} = 2^{\_\_\_\_}$$

$$\therefore 4^{\frac{3}{2}} = \_\_\_\_\_\_$$

16. Let $f(x) = 2^{-x}$ and $g(x) = 2^x$. Show that $2^{-x} = 0.5^x$ and answer the questions.

a) As $x \to \infty$, what do $f(x)$ and $g(x)$ approach?

b) As $x \to -\infty$, what do $f(x)$ and $g(x)$ approach?

c) Are there any vertical or horizontal asymptotes for $f(x)$ and $g(x)$?

d) How do $f(x)$ and $g(x)$ compare?

e) What are the domain and range of $f(x)$ and $g(x)$?

For Problem 17-20, let the parent function be $h(x) = 3^x$ and list the transformations to obtain the graph of the function given.

17. $f(x) = 4 \cdot 3^x$

18. $g(x) = 3^{x-2}$

19. $j(x) = 3^{x+3}$

20. $k(x) = 3^x - 1$

Section 2.3 Logarithmic Functions

Practice Problems 2.3

For Problem 1, answer the word problem.

1. If $\log_b M = \log_b N$, what can you say about M and N?

For Problem 2 and 3, fill in the blank(s).

2. Since $\log_b N = k$ is equal to $b^k = N$, then $\log_b b^k =$ _____.

3. Because $b^1 = b$ and $b^0 = 1$, $\log_b b =$ ________ and $\log_b 1 =$ ________.

For Problem 4-7, simplify the logarithm and check your answer.

4. $5^{\log_5 24}$

5. $\log_7 49$

6. $\log_2 \frac{1}{8}$

7. $\log_4(4^3)$

8. Solve for x if $\log_2 4\sqrt{2} = x$.

9. Logan wrote the steps to solve the equation (shown below to the left). Where did he make a mistake? After you find his mistake, solve the equation.

$2^x = 8\sqrt{2}$

$2^x = 8 \cdot 2^{\frac{1}{2}}$

$2^x = 2^3 \cdot 2^{\frac{1}{2}}$

$2^x = 2^{\frac{3}{2}}$

$x = \dfrac{3}{2}$

10. If $\log_5 y = 4 - x$ is equal to $\log_{25} y = x - 1$, solve for x.

For Problem 11-13, solve for x and demonstrate that the solution is viable.

11. $\log_5 0.2 = x$

12. $\log_5 1 = x$

13. $\log_4 256 = x$

For Problem 14-17, solve for x using the properties of logarithms. Use the calculator and round to the nearest hundredths.

14. $10^x = 8$

15. $\log_{10} x = 3.121$

16. $\log_{10} x = -1.4$

17. $\log_5 125 = x$

18. $\log_2 \frac{1}{2} = x$

For Problem 19 and 20, fill in the blanks.

19. The function $\log_2 10$ lies between the consecutive integers __________ and __________.

20. The function $\log_{10} 101$ lies between the consecutive integers __________ and __________.

Section 2.4 Review of Logarithmic Functions

Practice Problems 2.4

For Problem 1-2, tell which two integers the value of x is between. Then use the calculator to find a decimal approximation for x that is closer to the solution.

1. $2^x = 9$

2. $8^{2x} = 3$

3. Use logarithms to solve Problem 1 and 2. Round to the nearest thousandths place.

For Problem 4-6, use the equation $x^{\frac{2}{3}} = 14$ to solve the problem.

4. Complete the steps to solve $x^{\frac{2}{3}} = 14$ using logarithms.

$$\log_{10} x^{\frac{2}{3}} = \log\_\_\_ \_\_\_\_\_\_\_\_$$

$$\_\_\_\_\_\_\_\_ \log_{10} x = \log_{10} 14$$

$$\log_{10} x = \_\_\_\_\_\_\_\_ \cdot \frac{3}{2}$$

$$\log_{10} x = \frac{3}{2} \log_{10} 14$$

$$\log_{10} x = \log_{10} 14^{\_\_}$$

$$x = \_\_\_^{\frac{3}{2}}$$

$$x = \sqrt[2]{\_\_\_\_\_\_\_\_}$$

$$x \approx \_\_\_\_\_\_\_\_$$

5. Check your answer to Problem 4 and show your steps.

6. Show how to check Problem 4 using logarithms.

For Problem 7-20, use the given instructions to solve the problem.

7. Which is larger: $\log_6 36$ or $\log_{36} 6$?

8. What are $\log_3 27$ and $\log_{27} 3$? Guess first and then solve. How are $\log_a \mathrm{b}$ and $\log_{\mathrm{b}} a$ related?

9. Most scientific calculators will find the logarithm of any base. The common logarithm is base 10 for the Hindu-Arabic system.

Complete the steps below to find the change of base formula for calculators that only use base 10.

$$\log_a x = y$$

$$x = \underline{\hspace{2cm}}^{y}$$

And…

$$\log_{\mathrm{b}} x = \log_{\mathrm{b}} \underline{\hspace{2cm}}^{y}$$

$$\log_{\mathrm{b}} x = \underline{\hspace{2cm}} \log_{\mathrm{b}} a$$

$$\frac{\log_{\mathrm{b}} x}{\underline{\hspace{2cm}}} = y$$

$$\text{Therefore, } \log_a x = \frac{\underline{\hspace{2cm}}}{\log_{\mathrm{b}} a}.$$

10. Use the change of base formula to find $\log_2 9$.

11. Evaluate $\log_{10} 8$ and $3 \cdot \log_{10} 2$ and explain why they yield the same answer.

12. Use powers of base 3 to show that if $\log_3 9\sqrt{3} = x$, then $x = \frac{5}{2}$.

13. Use powers of base 3 and simplify to show that if $\log_9 3 = x$, then $x = \frac{1}{2}$.

14. Check the answer to Problem 13 using exponents.

15. Show that $\log_9 3 = \log_9 9^{\frac{1}{2}}$.

16. Demonstrate that $\log_9 9 + \log_9 27 = \log_9 243$. Simplify each logarithm first.

17. Show that $x = 2$ is a solution for the logarithmic equation $\frac{1}{2}\log_3(x+2) - \log_3 x = 0$.

18. Show that $x = -1$ is not a solution for the logarithmic equation $\frac{1}{2}\log_3(x+2) - \log_3 x = 0$.

19. Use the properties of logarithms to solve for x in $2\log_3 x - 5 = -5$. Check by graphing to see if there are any extraneous solutions.

20. Use two different methods to solve for x in $1.12^x = 4$ (round to the hundredths place). Check by graphing to see if there are any extraneous solutions and then check by finding the intersection of the graph of $y = 4$ with the graph of $y = 1.12^x$.

Section 2.5 The Natural Exponential Function

Practice Problems 2.5

For Problem 1-12, use the information given to solve the problem.

1. You want to buy a car and pay cash for it so that you will not have a loan to pay off and can use your money to give to God’s work. The car is $1,200.00. If you put $1,000.00 in the bank, will you have enough money to pay for the car after 2 years at an interest rate of 1.5% compounded continuously?

2. What amount needs to be deposited so that after 2 years at a rate of 1.75% compounded continuously you will have $1,200.00?

3. An account was opened 25 years ago with an initial amount of $500 paying 3.5% interest compounded continuously. How much is in the account now?

4. Is 5 years at 100% growth the same as 1 year at 500% growth? Pick an initial amount to begin.

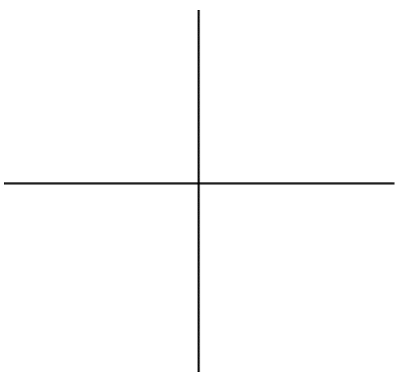
5. Is 50% growth for 20 years the same as 100% growth for 10 years? Pick an initial amount to begin.

6. How can you describe the occurrence in Problem 4 and Problem 5 in terms of the natural exponent?

7. Using the natural exponential function, complete the table. (Approximate to the ten-thousandths place.) What is the y-intercept? Why?

$\boldsymbol{x}$	$\boldsymbol{e^x}$
-2	
-1	
0	
1	
2	
3	

8. Graph the natural exponential function on the graph below? What is the y-intercept? Are there any asymptotes? Label the y-intercept and the point $(1, e)$.



9. Macel has $452 to deposit. One bank offers continuous interest at a rate of 2.5% for 5 years. Another bank has an interest rate of 3% compounded bi-annually for 6 years. Which is a better investment?

10. The periodic compound interest function is $\mathrm{A} = \mathrm{P}(1 + \frac{r}{n})^{nt}$, where P is the initial principal, r is the interest rate, n is the number of periods in which it is compounded annually, and t is the time in years.

The continuous compounding formula for this becomes the limit of the function.

$$\lim_{n \to \infty} = \lim_{n \to \infty} \mathrm{P}(1 + \frac{r}{n})^{nt}$$

To derive the formula known as Pert, complete the following steps (fill in the blanks):

Step 1: Factor out P, which is not involved in the limit.

$$\lim_{n \to \infty} A = \_\_\_\_\_\_\_ \lim_{n \to \infty} \left[(1 + \frac{1}{\frac{n}{r}})^{nt} \right]$$

Step 2: Rewrite the equation as a power to a power function.

$$\lim_{n \to \infty} A = P \lim_{n \to \infty} \left[(1 + \frac{1}{\frac{n}{r}})^{n} \right]^{\_\_\_\_\_\_}$$

Step 3: Multiply the power by $\frac{r}{r} = 1$.

$$\lim_{n \to \infty} A = P \lim_{n \to \infty} \left[(1 + \frac{1}{\frac{n}{r}})^{\frac{n}{r}} \right]^{\_\_\_\_\_\_ t}$$

Step 4:

$$\lim_{n \to \infty} = P \lim_{n \to \infty} \left[(1 + \frac{1}{\frac{n}{r}})^{\frac{n}{r}} \right]^{rt}$$

Substitute e for $\lim_{n \to \infty} (1 + \frac{1}{\frac{n}{r}})^{\frac{n}{r}}$ and A with $A(t)$ (because A depends on t).

$$A(t) = P\_\_\_\_\_\_\_^{rt}$$

11. How long will it take an investment of 2,000 to triple if it is invested in a money market with an annual rate of 13% compounded quarterly?

12. The value of a $25,000 car decreases exponentially at a rate of 8% annually. What will be the value of the car after 3 years?

For Problem 13 and 14, use the given formula to solve the problem.

The double-time growth formula states that if a population size P_0 doubles every d (any unit of time), then the number P in the population at time t is given by the equation $P = P_0 \cdot 2^{\frac{t}{d}}$.

13. A certain bacteria doubles in size every 8 hours. How much will it grow in 3 days?

14. If there are 2 grams of bacteria at the start of the experiment, how many grams of bacteria will there be at the end of the 3-day experiment? Assume the weight of the mass is directly proportional to the same factor.

For Problem 15, use the information given to solve the problem.

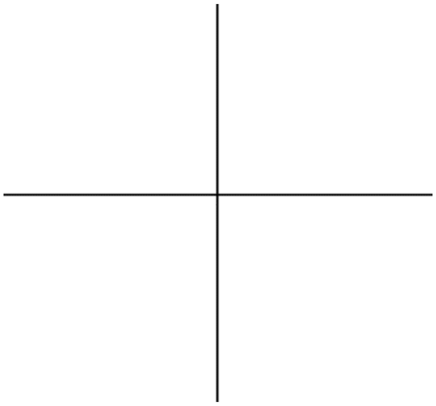
15. If an investment of $1,000 doubles every 2 years, how much money will be earned at the end of 10 years?

For Problem 16-21, complete the table/graph and answer the questions given.

16. Complete the table.

$\boldsymbol{x}$	$\boldsymbol{e^{x}}$	$\boldsymbol{e^{2x}}$	$\boldsymbol{e^{-x}}$	$\boldsymbol{e^{\frac{x}{2}}}$
-2				
-1				
0				
1				
2				
3				

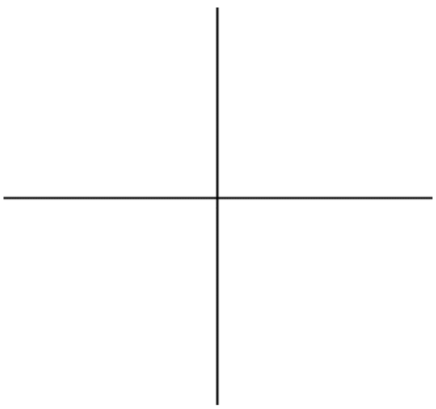
17. Draw the graphs for $f(x) = e^x$ and $g(x) = e^{2x}$ on the same coordinate grid below.



a) What is similar about the two graphs?

b) What is different about the two graphs?

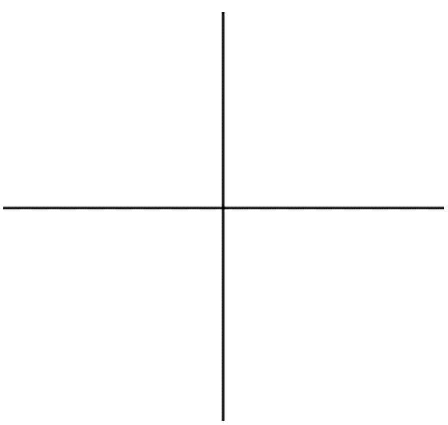
18. Draw the graphs for $f(x) = e^x$ and $h(x) = e^{-x}$ on the same coordinate grid below.



a) How does the graph of $f(x) = e^x$ compare to the graph of $h(x) = e^{-x}$?

b) What is the y-intercept of both graphs?

19. Draw the graphs for $f(x) = e^x$ and $k(x) = e^{\frac{x}{2}}$ on the same coordinate grid below.



a) How does the graph of $f(x) = e^x$ compare to the graph of $k(x) = e^{\frac{x}{2}}$?

b) Are there any common points on both graphs?

20. What are the domain and range of all the graphs in the table in Problem 16?

Section 2.6 The Natural Logarithmic Function

Practice Problems 2.6

For Problem 1, use the given information to solve the problem.

There is a Rule called the Rule of 72 that is a math shortcut to calculate how much time is needed to double your money after it is invested.

The equation $\ln(2) = 0.693$ means it takes 0.693 units of time (years) to double your money with continuous compounding at a rate of 100%.

This means $\text{rate } \cdot \text{time} = 0.693$ or $\text{time} = \frac{0.693}{\text{rate (decimal)}}$.

1. How long would it take for your money to double when invested at a rate of 10% compounded continuously?

For Problem 2 and 3, let $0.693 = 69.3\%$ so that $\text{time} = \frac{69.3\%}{\text{rate (\%)}}$.

2. How long would it take an investment to double when invested at a rate of 20% compounded continuously?

3. How long would it take for your money to double when invested at a rate of 5% compounded continuously?

There is no rule of 69.3 since it is not a very divisible number, but a number near it is divisible; the number 72 is divisible by 2, 3, 4, 6, 8, 12, 18, 24, and 36. This is where the Rule of 72 comes from.

For Problem 4, use the Rule of 72 to solve the problem.

4. Use the Rule of 72 ($\text{time} = \frac{72}{\text{rate}}$) to find the time to double your money at the given rates.

   a) 2%

   b) 3%

   c) 4%

For Problem 5-6, use the formula below to solve the problem.

The previous practice problems section introduced the Time Formula for Growth ($\mathrm{P} = \mathrm{P}_0 \cdot 2^{\frac{t}{d}}$), where P_0 is the initial population that doubles in d periods of time (hours, days, years, etc.) then P is the population at time t.

5. A population doubles every thirty minutes. What is the factor of growth for 6 hours?

6. Bacteria doubles every 6 hours, what is the factor of growth for 3 days?

For Problem 7-10, use the information given to solve the problem.

7. A car loses 20% of its value each year. Its value at the end of a year is $\mathrm{N} = 0.80 \cdot$ value at the beginning of the year. Write the exponential formula for the value of a car initially valued at N_0 after t years.

8. What will be the value of the car after 5 years if it originally costs \$1,500.00.

9. The natural logarithm of x is $\log_e x$ or $\ln x$.

The definition of e is the limiting value of $(1 + \frac{1}{n})^n$ as n gets larger. Complete the table below rounded to five decimal places.

$\boldsymbol{n}$	100	1,000	10,000	100,000
$\boldsymbol{(1 + \frac{1}{n})^n}$				

10. What limiting value does n approach?

For Problem 11-15, use the example below to help solve the problem.

In working with natural logarithms, $\ln x$ works as $\log_e x$. If $\ln x = 3$, then $x = e^3$. If $e^x = 9$, then $x = \ln 9$.

For Problem 11 and 12, convert the natural logarithm to exponents.

11. $\ln x = 5$

12. $\ln x = 7$

For Problem 13-14, convert the exponents to natural logarithms.

13. $e^x = 2$

14. $e^x = 3$

15. What is the value of $\ln e$ when the base is a real number not equal to 1?

For Problem 16 and 17, use the power function $y = 600 \cdot x^{-1.1}$ to solve the problem.

16. Linearize the power function $y = 600 \cdot x^{-1.1}$ by taking the natural logarithm of both sides.

17. a) What is the y-intercept for the linearized power function?

b) What is the slope of the linearized power function?

For Problem 18 and 19, fill in the blanks.

18. In an exponential function, $y = ab^x$, $\log y = \log$ ________ $+ \log$_____ x. If y is a(n) ____________________ function of x, then $\log y$ is a ____________________ function of x.

19. In a power function, $y = ax^b$, $\log y = \log$ _______ $+\, b\ \log$ _______. If y is a ____________________ function of x, then $\log y$ is a linear function of ____________________.

For Problem 20, use the information given to solve the problem.

20. The calculator gives a correlation coefficient of $r = 0.9346$ for a linear regression, $r = 0.9499$ for a power regression, and $r = 0.95$ for an exponential regression; which models the line of best fit?

Section 2.7 Theorems of the Natural Exponential Function

Practice Problems 2.7

For Problem 1-11, use the given information to solve the problem.

The exponential e^x can be defined by the following power series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

The summation notation $\sum_{n=0}^{\infty}$ is new to you. You will learn more about this when you study integrals in Module 8.

The notation 4! means $4 \cdot 3 \cdot 2 \cdot 1$. You first learned about factorials when you studied probability and statistics.

The Taylor Series is written as follows:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

We can look at just these first five terms to get a sound approximation for e^x.

1. Find e^1 using the first four terms of the Taylor Series.

2. Find e^1 using the first seven terms of the Taylor Series. The more terms you use the better the approximation. (Since this goes on for infinity, only God knows the exact answer found by calculating all the terms.)

3. Find the approximate value of e^2 using the first five terms of the Taylor Series.

4. Find the approximate value of e^3 using the first five terms of the Taylor Series.

There are also Taylor Series' for $\sin(\theta)$ and $\cos(\theta)$.

5. $\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$ What are the next two terms of the series (θ is in radians)?

6. $\cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$ What are the next two terms of the series (θ is in radians)?

7. Substitute $i\theta$ for x in the Taylor Series for e^x to get the complex series.

8. Simplify and group the real parts and the imaginary parts of Problem 7 together to get the following equation:

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots)$$

Substitute $\sin\theta$ and $\cos\theta$ into the equation and you have Euler's Identity!

9. Simplify $\ln\frac{1}{e^3}$.

10. Solve $\ln x = 3$.

11. Solve $\ln\frac{1}{x} = 2$.

For Problem 12-14, write the constants as natural logarithms using the given information.
A constant can be written as a natural logarithm: 3 can be written $\ln e^3$ or $3 = \ln e^3$ because $e^3 = e^3$.

12. 4

13. 6

14. 1.06

For Problem 15-17, rewrite the natural logarithm using the power function and show that the numbers are equivalent.

15. $2\ \ln 5$

16. $\ln 2^3$

17. $1.02\ \ln 3^3$

For Problem 18 and 19, use properties of logarithms to write the given expression as a sum or difference without exponents.

18. $\ln(x^4 y^3)$

19. $\ln(\frac{x^2}{3x})$

For Problem 20, follow the instructions to solve the problem.

20. What is ln 1? Demonstrate how you know your answer is correct.

Section 2.8 Properties of the Natural Logarithm

Practice Problems 2.8

For Problem 1, fill in the blank.

1. $\ln(3) = \ln(2) +$ ________ (Double check your answer to see if it is correct)

For Problem 2-5, rewrite the logarithm as an addition or subtraction problem and calculate if possible.

2. $\ln(3 \cdot 6)$

3. $\ln(4y)$

4. $\ln(\frac{10}{3})$

5. $\ln(\frac{7.4}{x})$

For Problem 6 and 7, solve for x using the natural logarithmic function.

6. $e^x = 25$

7. $e^x = 9.5$

For Problem 8-10, follow the instructions to solve the problem.

8. If it takes 3 years to get 100% return on your investment, what is the growth factor per units of time?

9. You want your investment to triple at 100% growth. How many years will it take?

10. What does $e^{2.5} \approx 12.1825$ mean?

For Problem 11-14, use the properties of logarithms to simplify the expression.

11. $2 \ln 5 + \ln 3$

12. $2 \ln 10 - 4$

13. $3 \ln 4 + 2$

14. $\ln 7 + \ln 3 - 2$

For Problem 15, solve Problem 11-14 using the expanded form and using the simplified form. If you do not get the same answer as before, correct any errors and try again.

15. a) $2 \ln 5 + \ln 3$

b) $2 \ln 10 - 4$

c) $3 \ln 4 + 2$

d) $\ln 7 + \ln 3 - 2$

For Problem 16-20, use the information given to solve the problem.

16. Which of the following expressions is the same as $6 \ln 2 - 3 \ln 4$?

a) $\ln(1)$

b) $6 \ln(2) + 2$

c) $\ln 4^3 + e^2$

17. What is $\ln(0)$? Explain why.

18. What is $\ln(e)$? Explain why.

19. What is $\ln(1)$? Explain why.

20. What is another way of writing $-\ln(4)$? What does it mean?

Section 2.9 Applications of the Natural Logarithm

Practice Problems 2.9

For Problem 1-5, solve for the variable k. Leave in exact form.

1. $e^{2k} = 21$

2. $6e^{2k-1} = 13$

3. $6 + 11e^{1-2k} = 9$

4. $10^{k^2-k} = 100$

5. $k - ke^{3k+3} = 0$ (Factor out the k first)

For Problem 6-10, follow the instructions given to solve the problem.

6. Fill in the missing steps:

$$2(x^2 - 1) = (x^2 - 1)e^{2-x}$$

$$2(x^2 - 1) - \underline{\hspace{3cm}} = 0$$

$$(x^2 - 1)(2 - \underline{\hspace{3cm}}) = 0$$

$$x^2 - 1 = 0 \quad \text{or} \quad 2 - e^{2-x} = 0$$

7. Solve for x to get three solutions for Problem 6.

8. Using the function from Example 3 of the Looking Ahead section, if there were only $50{,}000$ bacteria after 3 hours, how does that change k?

9. Using the new constant in Problem 8, how many bacteria would there be after 5 hours?

10. Simplify $\ln(e^{\log_4(x)^3})$.

For Problem 11-15, find the solution of x in terms of e, natural logarithms, or constants.

11. $\sqrt[3]{e^x} = 3$

12. $e^{2x} = 16$

13. $e^{\ln x} = 11$

14. $|\ln x| = 1$

15. $\ln(\ln x) = 0$

For Problem 16-20, use the formula $A = Pe^{rt}$ for P dollars invested at an APR (annual percentage rate) r (in decimal form), compounded continuously for t years.

16. Niles invests $\$550$ at a 2.75 APR. How much will his investment be worth after 4 years?

17. How much will Niles' investment be worth after 10 years?

18. If Monica invested $\$900$ at an 11% APR, how much will the investment be worth after 5 years?

19. An investment is worth $\$1{,}648.72$. What was the initial amount invested at a 10% APR 5 years earlier?

20. An investment is worth $\$1{,}200$. What was the initial amount invested at a 1.5% APR 5 years earlier?

Section 2.10 The Logistic Function
Practice Problems 2.10

Use the function below to answer questions 1-6.

$$\mathrm{F}(x) = \frac{10}{1 + 10 \cdot e^{-x}}$$

1. What is the value of a?

2. What is the value of b

3. What is the value of c?

4. What is the y-intercept?

5. Use the formula $x = \log_c b$ to find the x value of the inflection point of the logistic function:

6. Substitute the value of x in the equation and find the y value of the inflection point.

7. One horizontal asymptote in the logistic function $f(x) = \frac{3}{1+0.1^x}$ is $y = 0$. Find the other horizontal asymptote.

8. What is the point of inflection for the graph of Problem 7?

9. Write $f(x) = \frac{3^x}{1+3^x}$ as a logistic function?

10. Populations can be constrained in their growth by limited food resources. Give another example of a type of growth called logistic growth.

For Problem 11-16, use the given information to solve the problem.

Review of Power Functions

A cardboard box, with no lid, is made from templates. The box is made by folding up the corners. The cut-outs of the corners are x units long by x units wide. The cardboard is 20 x 24 inches.

11. Draw a diagram of the template.

12. What is the height, length, and width of the box in terms of x?

13. Write the standard form equation for the volume, $V(x)$, of the box.

14. Graph the equation on a graphing calculator over the interval $[0, 20]$ for the domain and $[-25, 800]$ for the range. What is the local maximum over the interval $[0, 10]$ and what does it represent?

15. What intervals of the domain and range are important to this problem?

16. What are the x-intercepts and what do they represent?

For Problem 17-20, use the given information to solve the problem.

The volume of a cardboard box is 360 cubic inches. Squares that are 5 inches long by 5 inches wide are cut from the corners of the template used to build the box. The width of the box template is half the length. Find the dimensions of the box.

17. Draw the template of the box and the 3-dimensional box folded up at the corners.

18. Write the formula for volume of the box, $V(l)$, in terms of length.

19. Write the formula for the volume of the box, $V(w)$, in terms of width.

20. Find the width and length of the box that has a volume of 360 cubic inches.

<u>Section 2.11 The Natural Exponential Logistic Function</u>

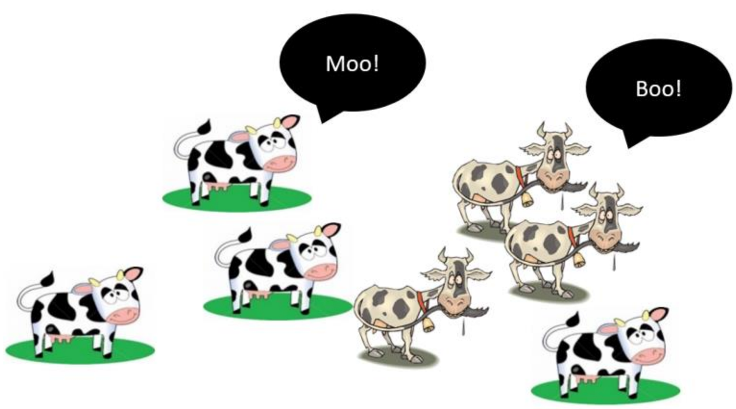
<u>Practice Problems 2.11</u>

For Problem 1-3, name the limiting factor for the logistic function.

1. Seedling Growth

2. Island Populations

3. Mad Cow Disease Growth



For Problem 4-5, evaluate the logistic function at $x = -2$, $x = 0$, and $x = 3$.

4. $g(x) = \frac{200}{1+629e^{-4x}}$

5. $h(x) = \frac{100}{1+7e^{-x}}$

6. Find where the maximum growth rate occurs for the logistic functions in Problem 4 and 5.

For Problem 7-12, use the given information to solve the problem.

A strain of mad cow disease is spreading through a herd according to the logistic function…

$$\mathrm{P}(t) = \frac{96}{1 + 9e^{-0.06t}}$$

… where $\mathrm{P}(t)$ is the population of cows infected and t is the number of days.

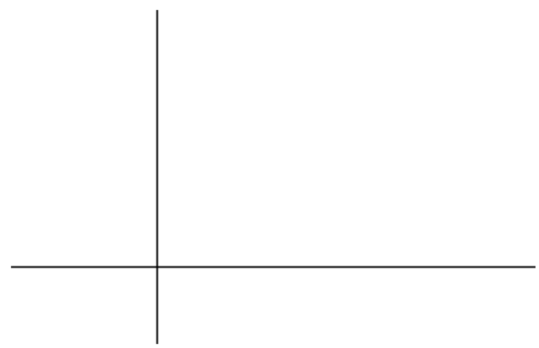
7. How many cows are in the herd? Explain your conclusion.

8. What is the point of the maximum growth rate?

9. How many cows were initially infected?

10. How many cows will have mad cow disease after 7 days?

11. Sketch a graph of the logistic function using the solutions for Problem 7-10.



12. When will 93 cows in the herd be infected?

For Problem 13-20, use the given information to solve the problem.

A subdivision has 1,200 lots left for sale. The table below shows the number of houses sold in the subdivision over a given period of months.

x (number of months)	**y (number of houses)**
3	130
6	154
9	183
12	215

Follow the given steps to solve for a system of equations and find the logistic function that models the growth.

13. What is the value of a in the logistic function $y = \frac{a}{1+be^{-cx}}$?

14. Substitute the point $(3, 130)$ and a into the equation.

15. Substitute the point $(12, 215)$ and a into the equation.

16. Use the logistic equation from Problem 14 and solve for b in terms of c.

17. Substitute the expression for b in the equation from Problem 15 and solve for c.

18. Now that we know the value of c, substitute the value of c in the equation for b from Problem 16 and find the value of b.

19. Substitute the constant values a, b, and c in the general logistic function from Problem 13 to find the specific logistic function for the number of houses sold in a given number of months for the subdivision with $1{,}200$ lots left.

20. Check the point $(6, 154)$ in the equation from Problem 19 to see if it is a good fit for the data.

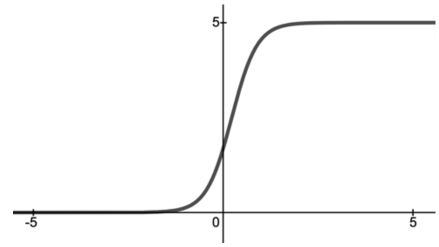
Section 2.12 Graphing and Solving Logistic Functions

Practice Problems 2.12

For Problem 1-3, match the graph with the logistic function.

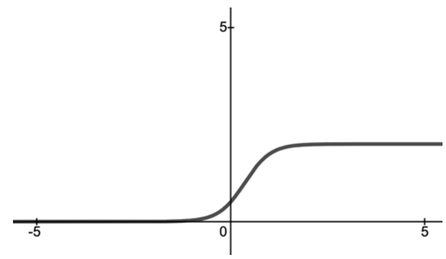
1. $f(x) = \frac{4}{1+2e^{-4x}}$

a)



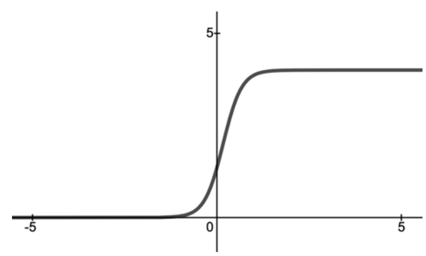
2. $g(x) = \frac{5}{1+2e^{-3x}}$

b)



3. $h(x) = \frac{2}{1+3e^{-3x}}$

c)



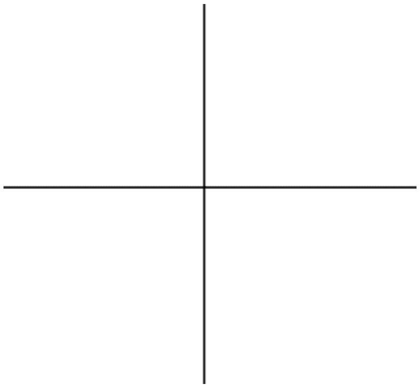
For Problem 4 and 5, solve for x in the logistic function.

4. $\frac{10}{1+3e^{-x}} = 7$

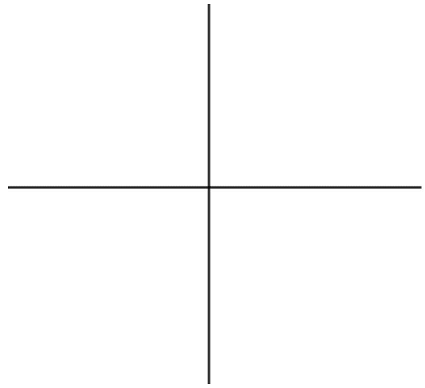
5. $\frac{9}{1+16e^{-2x}} = 5$

For Problem 6 and 7, graph the function given. Name the y-intercept, asymptotes, and point of inflection (this is also the point of maximum rate of growth).

6. $\frac{2}{1+3e^{-x}}$



7. $\frac{4}{1+0.06e^{-1.2x}}$



For Problem 8, use the information given to solve the problem.

8. The logistic function has an inflection point $(\frac{\ln b}{c}, \frac{a}{2})$ for the growth function $y = \frac{a}{1+be^{-cx}}$. Use the halfway point $y = \frac{a}{2}$ to show that $x = \frac{\ln b}{c}$.

For Problem 9 and 10, use the given information to solve the problem.

Plant growth is given by the table below. The time (t) represents weeks and the height (h) is measured in centimeters (cm).

t	0	1	2	3	4	5	6	7	8	9
h	3	10	24	37	49	86	92	101	110	117

Graph the table on the calculator and write a logistic growth equation that models the growth.

9. Which week does the rate of maximum growth occur?

10. Use the equation to calculate how tall the plant will be by the 3rd week?

For Problem 11-14, use the information below to simplify the natural exponential function and specify the domain, range, and any asymptotes.

We have investigated four mathematical models for exponential or logarithmic functions.

1. Exponential Growth:

$y = b^x$ $b > 1$ and $y = e^{cx}$ $c > 0$

2. Exponential Decay:

$y = b^x$ $0 < b < 1$ and $y = e^{-cx}$ $c > 0$

3. Logarithmic:

$y = \log_{10} bx$ and $y = \ln bx$

4. Logistics Growth:

$$y = \frac{1}{1 + be^{-cx}}$$

11. $y = e^{3x} \cdot e^{4x}$

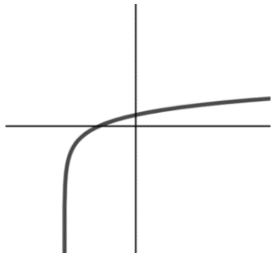
12. $y = (3e^{-x})^2$

13. $y = 3(e^x + 1)$

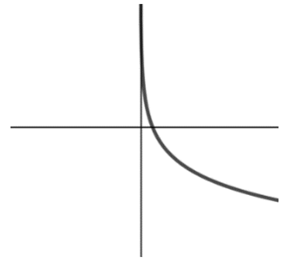
14. $y = (\frac{2}{3}e^{-0.5x})^3$

For Problem 15-18, match the graph with its function.

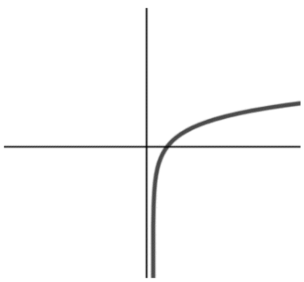
15.



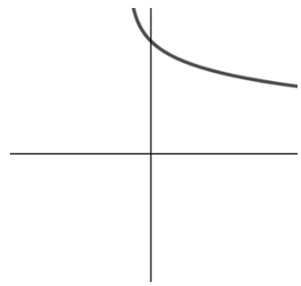
16.



17.



18.



a) $f(x) = \log_{10}(x+2)$

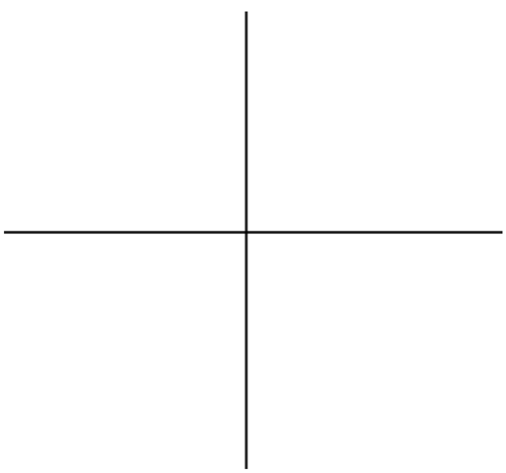
b) $f(x) = 5 - \ln(x+1)$

c) $f(x) = -\frac{5}{2}\ln x$

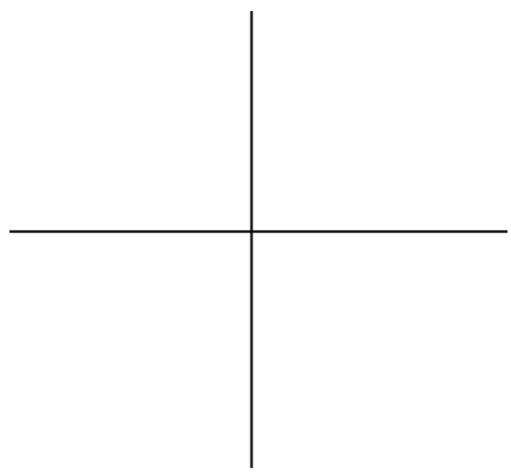
d) $f(x) = 3\log_{10}(x - \frac{1}{2})$

For Problem 19 and 20, sketch the graph of each logistic function. Specify the horizontal asymptote.

19. $f(x) = \frac{4}{1+2e^{-x}}$



20. $f(x) = \frac{3}{1+0.6e^{-x}}$



Section 2.13 Applications of Logistic Functions

Practice Problems 2.13

For Problem 1-10, use the information given to solve the problem.

1. Dane planted a seedling and kept track of its height for 9 weeks. Below is the data where w represents week and h represents height in centimeters.

$\boldsymbol{w}$	0	1	2	3	4	5	6	7	8	9
$\boldsymbol{h}$ **(cm)**	9	16.5	28	45	65	85	101.5	113	119.5	174

Find the logistic growth function that models the growth of the seedling.

2. Predict the height of the seedling in Problem 1 by the 10^{th} week.

3. In Example 2 of this section, a bacterial growth in a petri dish with an area of 46 square centimeters was modeled by the following function:

$$A = \frac{46}{1 + 129e^{-1.4t}}$$

If t represents days, when will the area of the bacterial growth be approximately 35 square centimeters?

4. When will the bacterial growth in Problem 3 reach its limiting value since the exact maximum capacity cannot be reached?

5. The logistic function for the spread of a flu epidemic is $\mathrm{P}(d) = \frac{3{,}000}{1+1{,}250e^{-kd}}$. If the epidemic has spread to 45 people by the end of 1 week, how many people will it spread to by the end of 2 weeks? (Let d be time in days and $\mathrm{P}(d)$ be the total population infected). Solve for k first, which is the constant determined by the strain of the flu.

6. The gross domestic product (GDP) of a small third world country in the last decade can be modeled by the following equation:

$$G(t) = \frac{800}{1 + 6.02e^{-0.2t}}$$

GDP in millions of dollars is represented by $G(t)$. Time in years is represented by t. In what year was the GDP $\$300$ million?

7. The population (in millions) of the same country from Problem 6 over the last decade is represented by the following logistic function:

$$P(t) = \frac{500}{1 + 499e^{-0.6t}}$$

How many people (in millions) lived in the country when the GDP was $\$300{,}000{,}000.00$?

8. The number of people owning computers from Problem 7 was modeled by the following equation:

$$P(t) = \frac{22.8}{1 + 10.04e^{-0.5t}}$$

In this equation, t represents time in years and P represents thousands.

How many thousands of people owned computers at the $6\frac{1}{2}$ year mark of the last decade in this small third world country?

9. The growth of bacteria under ideal laboratory conditions is represented by the equation $\mathrm{B}(t) = \mathrm{B}_0 e^{kt}$, where $\mathrm{B}(t)$ is the total bacteria after t hours and k is the constant determined by the bacteria strain.

If a culture of bacteria starts with 1,000 grams and grows to 2,000 grams in 3 hours, what is the constant of determination (k) for the particular bacteria?

10. How many grams of bacteria from Problem 9 will be in the culture after 7 hours?

For Problem 11-12, use the given information to solve the problem.

Four models of exponential or logarithmic functions have been investigated:

1. Exponential Growth:

$$y = b^x \quad b > 1 \quad \text{and} \quad y = e^{cx} \quad c > 0$$

2. Exponential Decay:

$$y = b^x \quad 0 < b < 1 \quad \text{and} \quad y = e^{-cx} \quad c > 0$$

3. Logarithmic:

$$y = \log_{10} bx \quad \text{and} \quad y = \ln bx$$

4. Logistics Growth:

$$y = \frac{1}{1 + be^{-cx}}$$

11. The level of sound, D (decibels) is related to the intensity of a sound, I (watts per square centimeter) and is modeled by the equation $D = 10\log_{10}\frac{I}{I_0}$, where I_0 is $\frac{1}{10^{16}}$ watts per square centimeter. Anything above 100 decibels can be damaging to hearing. What type of function is this?

12. If you buy a new car for $\$36{,}000.00$ in 2025 and it decreases by 15% each year, the model for what it will be in years, t, after 2025 is the following equation:

$$V = 36{,}000(0.85)^t$$

What type of model is this?

13. What is the intensity of sound for 10 decibels of music? Use the formula from Problem 11 to find it.

14. What will be the value of the car from Problem 12 in 2030?

15. To estimate a person's time of death, a coroner can take the person's temperature twice at 30-minute intervals, or twice at 1-hour intervals. As long as the room temperature stays constant, Newton's Law of Cooling can be used to find the person's time of death; the equation for Newton's Law of Cooling is $Kt = \ln\frac{T-S}{T_0-S}$, where K is the constant of cooling, S is the surrounding air temperature, and t is the time the body cools from T to T_0, where T is the temperature taken and T_0 is the normal body temperature. A coroner takes the temperature of a person who died during the night at 8:00 a.m. in the morning and it is 85.3°F. At 9:00 a.m., the temperature of the body is 78°F. The room temperature is a constant 70°F. Estimate the person's time of death using Newton's Law of Cooling.

16. To estimate the temperature of an object that has been cooling over time, Newton's Law of Cooling may again be used. The temperature (in Fahrenheit degrees) for a cup of hot chocolate that has been cooling over time is modeled by the equation $T(t) = S + (T_0 - S)e^{-0.024t}$ where t is the time (in minutes) that the hot chocolate cools from T_0, the temperature when it was removed from the microwave, and S is the surrounding ambient room temperature. What is the temperature (in degrees Fahrenheit) of the cup of hot chocolate that has been cooling for 20 minutes after being removed from the microwave with a temperature of 180° in a room that is 70°?

17. The shape of a seashell fossil forms a spiral. A point $S(1,0)$ on the positive $x-$axis moves counterclockwise around the origin. The distance, r, between the point S and the origin is modeled by the equation $r = e^{0.002132\theta}$. The Greek letter Θ represents the degrees through which the point spirals. What is the distance between the origin and point S when point S has spiraled through 270°?

18. After how many degrees will the r value in Problem 17 be 2?

For Problem 19 and 20, use the information given to solve the problem.

A flu epidemic in a high school of 1,491 students is spread by one student who was ill over Christmas break. The equation that models the spread of the virus is $f(t) = \frac{1{,}491}{1+1{,}300e^{-0.6t}}$, where $f(t)$ is the number of students infected over t days.

19. How many students will be infected in 3 days, 5 days, and 10 days?

20. When will 1,000 students have the flu?

Section 2.14 Module Review

For Problem 1-3, solve the exponential equation and check your answer.

1. $4^x = \frac{1}{16}$

2. $\sqrt{5} = 25^{x+1}$

3. $49^x = 7^{x+3}$

For Problem 4-6, solve the logarithm and check your answer.

4. $\log_3(\frac{1}{81}) = x$

5. $\log_7 \sqrt{7} = x$

6. $\log_3(-8) = x$

For Problem 7-9, complete the table using the Properties of Logarithms.

			Product Property		**Quotient Property**
7.	$\log_3(90)$	=	$\log_3(10) +$ ________	=	$\log_3(180) -$ ________
8.		=	$\log_6(4) + \log_6(3)$	=	$\log_6(12) -$ ________
9.		=	$\log_4(2) + \log_4(15)$	=	

For Problem 10-12, use the given information to find the pH level for the values given.

The pH level of blood having bicarbonate concentration b and caloric acid c is given by the equation

$$\mathrm{pH} = 6.1 + \log\left(\frac{b}{c}\right).$$

10. $b = 32$ and $c = 3$

11. $b = 20c$

12. $\frac{b}{c} = 15$

For Problem 13, use the information given to solve the problem.

13. The cost of living for a salary is 2% for a 5-year period. If a salary is $\$43{,}000$ now, how much more will the salary be in 5 years? Use the equation $\mathrm{A} = \mathrm{P}(1 + \mathrm{r})^{t}$, where t is years.

For Problem 14-16, convert the natural logarithm to an exponent or the exponent to a natural logarithm and solve for x.

14. $\ln(x) = 4$

15. $e^{x} = 7.2$

16. $10e^{x} = 52.9$

For Problem 17, solve for x in the equation.

17. $3e^{2x-1} = 9$

For Problem 18-20, use the information given to solve the problem.

18. A bacterial growth function $\mathrm{B}(t)$ is given by the exponential growth function $\mathrm{B}(t) = 200e^{0.03t}$, where t is minutes.

a) What is the constant K for this strain of bacteria?

b) What was the initial amount of bacteria (in grams)?

c) How many grams of bacteria will there be after 20 minutes?

19. Solve the logistic function, $\mathrm{F}(x) = \frac{5}{1+9e^{-0.01x}}$, for $\mathrm{F}(0)$ and $\mathrm{F}(3)$.

20. Using the logistic function $\mathrm{F}(x) = \frac{30}{1+40e^{-2x}}$, find the asymptote, the point of inflection, and the y-intercept of its graph.

Section 2.15 Module Test

For Problem 1-3, solve the exponential equation and check your answer.

1. $5^x = \frac{1}{625}$

2. $\frac{1}{3} = 9^{x-1}$

3. $81^x = 3^{x-2}$

For Problem 4-6, solve the logarithm and check your answer.

4. $\log_2(-1) = x$

5. $\log_{16}(\frac{1}{8}) = x$

6. $\log_3(x) = -2$

For Problem 7-9, complete the table using the Properties of Logarithms.

7.	$\log_5(36)$	=	________ $+ \log_5(4)$	=	________ $- \log_5(2)$
8.		=	$\log_4(1) +$ ________	=	$\log_4(20) - \log_4(4)$
9.		=	$\log_9(3) + \log_9(1)$	=	

For Problem 10-12, use the information given to solve the problem.

10. The Richter Scale for measuring the intensity of earthquakes has been used since its invention in 1932 by Charles R. Richter (1901-1985). This scale has the property that its values are proper logarithms.

The Great Japan Earthquake of 1923 is estimated to have measured 8.9 on the Richter Scale. The Great San Francisco Earthquake of 1906 is estimated to have measure 8.3 on the Richter Scale.

Let $\log_{10}(x) = 8.9$ and $\log_{10}(y) = 8.3$. How much greater is x than y? The answer will give you some idea of how much more intense the Great Japan Earthquake of 1923 was than the Great San Francisco Earthquake of 1906.

11. Solve for x without using a calculator (then check you answer using a calculator):

$$\log_3(2x) + \log_3(4) = 5$$

12. An investment decreases by 1.4% over the course of 3 years. How much less will an initial investment of \$1,200.00 be worth after 3 years?

For Problem 13-16, convert the natural logarithm to an exponent or the exponent to a natural logarithm and solve for x.

13. $\ln(x) = 3.6$

14. $e^x = 5.103$

15. $1.2(e^{-3x}) + 2.4 = 3$

16. $2\ \ln(-x) + 5 = 10$

For Problem 17, solve for x.

17. $10^{x^2-x} = 100$

For Problem 18-20, use the information given to solve the problem.

18. A bacterial growth function $\mathrm{B}(t)$ is given by the exponential growth function $\mathrm{B}(t) = 400e^{0.02t}$, where t is minutes.

a) What is the constant k for this strain of bacteria?

b) What is the initial amount of bacteria (in grams)?

c) How many grams of bacteria will there be in half an hour?

19. Solve the logistic function for F(0) and F(3):

$$\mathrm{F}(x) = \frac{2}{1 + 8e^{-0.3x}}$$

20. Find the asymptotes, the point of inflection, and the y-intercept of the equation:

$$\mathrm{F}(x) = \frac{50}{1 + 200e^{-4x}}$$