Pre-Calculus and Calculus Module 2 Natural Logarithms and the Logistic Functions

Section 2.1 Exponential Functions

Looking Back 2.1

Exponential functions are of the form $f(x) = b^x$, where x is the input value and occurs as an exponent. While the base b may generally adopt any real value, it is useful to restrict it to positive real values and exclude $b = 1$. Note that this will guarantee the expression b^x is positive for any value of x. First, $b = 0$ and $b = 1$ present trivial cases because $0^x = 0$ (except for possibly 0^0, which is an ambiguous case because it is equal to 1) and $1^x = 1$.

Negative bases provide their own challenges. The expression b^x is only defined for negative b given x is an integer or rational number with an odd denominator such as $\frac{1}{5}$ or $\frac{2}{3}$. Strictly speaking, b^x when $b < 0$ is defined for any value of x, but the result is non-real for any values that do not meet the stated criteria (look ahead to complex numbers for such expressions). Moreover, the function, where defined, rapidly jumps between positive and negative values for negative values of b, making it useless for modeling anything realistic. It is beautiful, however.

For our purposes, there are cases in which $0 < b < 1$, which are known as exponential decay. There are also cases in which $b > 1$, which are known as exponential growth. Both these cases of exponential functions have the unique property that when points are taken at evenly spaced intervals of x, then the y values are successively related by a common ratio.

For example, take the function $f(x) = 2^x$ and input the values of $x = (1, 2, 3, 4, \ldots)$ or the values of $x = (1.1.\ \ , 1.2, 1.3, 1.4, \ldots)$. Each value f generated in each sequence is one constant multiplied by the previous f value in the sequence (2 for the first sequence and $2^{\frac{1}{10}}$ or 1.072 for the second sequence). Exponential functions have a sense of repeated multiplication as we investigated in Section 8.1 of Algebra 2; they are associated with integer powers and extend to all real values of x.

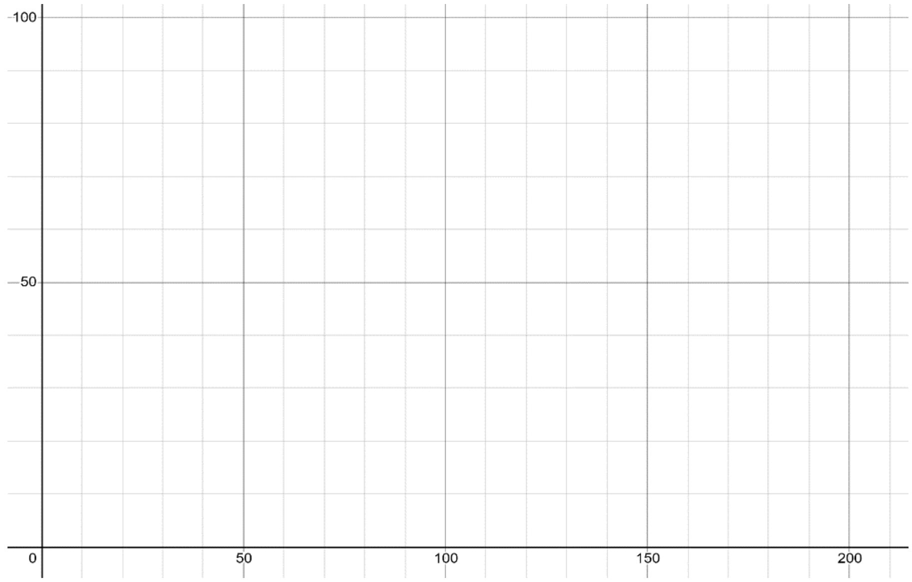
Through the techniques of Calculus, we find exponential functions have another unique property: the rate at which the function is growing or decaying at any value x is directly proportional to the value of the function itself. In other words, for the case of exponential growth in which $b > 1$ given x doubles or triples, then the rate of growth of $f(x) = b^x$ also doubles or triples! The “rate of growth” is expressed as a number and is known as the derivative, one of the key concepts of Calculus that we will explore more fully in Module 7.

Looking Ahead 2.1

In Algebra 2, you performed an experiment in order to find the bounce ratio of various balls.

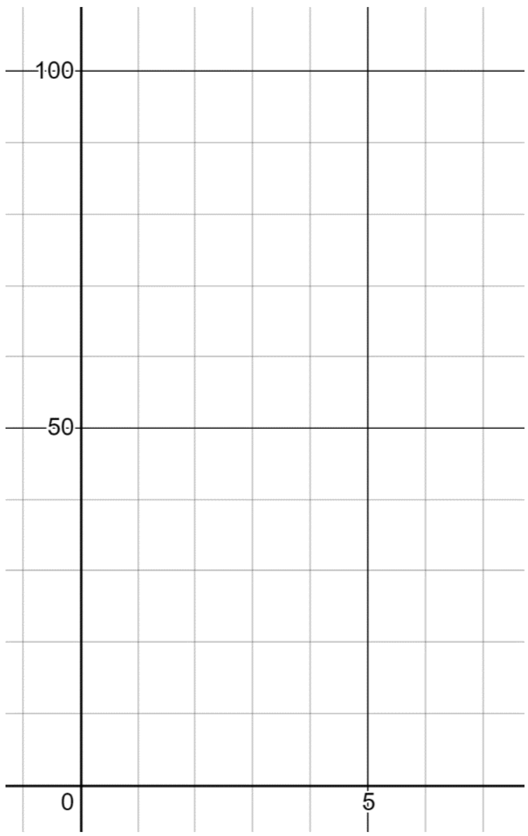
Example 1: In one experiment, the drop height changed. Each drop height and the average bounce height is recorded in the table below. Graph the drop height on the *x*-axis and the average bounce height on the *y*-axis, then analyze the graph.

Bouncer Supreme Ball Drop Changing Heights					
Drop Height	**200 cm.**	**175 cm.**	**150 cm.**	**125 cm.**	**100 cm.**
Average Bounce Height (cm.)	86	75	66	55	44


100
50
0
50
100
150
200

Example 2: In another experiment students dropped the ball from 200 centimeters (2 meters), but this time marked the bounce height after 1 bounce, 2 bounces, 3 bounces, 4 bounces, and 5 bounces. Graph the number of bounces on the *x*-axis and the bounce height on the *y*-axis, then analyze the graph.

Bouncer Supreme Ball Bounce Heights					
Number of Bounces	1	2	3	4	5
Bounce Height (cm)	86	39	17	7.5	3.3



The major focus of this module will be natural logarithms, which occur naturally in the natural world. Mathematics is everywhere around us and these beautiful phenomena and extraordinary displays of patterns are part of the wonder of God's universe.

Like exponential functions, rational functions have asymptotes. Whereas an exponential function has horizontal asymptotes, a rational function has horizontal, vertical, and non-vertical asymptotes (such as curved or slant asymptotes).

A vertical asymptote is never touched or crossed as it is a domain restriction. Horizontal asymptotes may be crossed. Here, we will investigate these type of rational functions.

Example 3: Name any asymptotes and any removable or non-removable discontinuities for the functions $f(x) = \frac{x^3+x^2-4x-4}{x-2}$ and $g(x) = \frac{x^3+x^2-4x-3}{x-2}$.

If $p(x)$ (the polynomial part of the function) is linear, but not a linear constant function, then it is a slant asymptote. The graph of the function (curve) approaches the graph of the asymptote as x approaches positive or negative infinity.

Example 4: Find the slant asymptote for $f(x) = \frac{x^2-3x+3}{x-2}$. Since the degree of the numerator is greater than the degree of the denominator, what happens to the end behavior of the function?

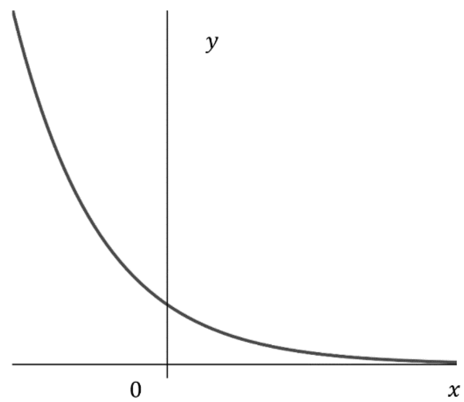
Example 5: Find the asymptotes for the function $g(x) = \frac{x+1}{x^2+2}$. Does the curve cross the asymptote? What can be said about the end behavior of the function since the degree of the denominator is greater than the degree of the numerator?

Section 2.2 Review of Exponential Functions

Looking Back 2.2

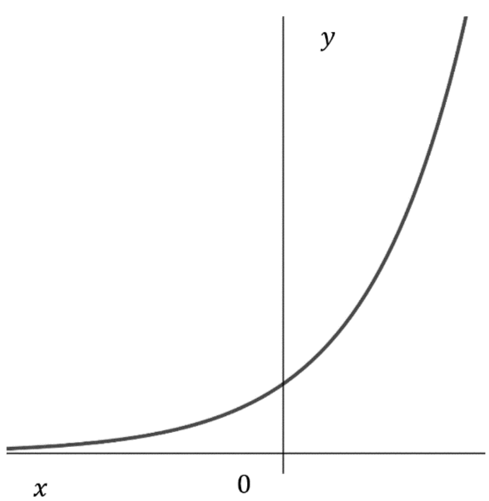
An exponential function is defined by the equation $y = b^x$. This is an exponential function with a base b. The initial value is 1.

We have already stated why $b > 0$ and $b \neq 1$. If $0 < b < 1$, then the graph takes the following shape:



- As x decreases, the function approaches positive infinity.
- As x increases, the function approaches 0.
- The function is decreasing.
- The function has a horizontal asymptote along the x-axis ($y = 0$).
- The point (0,1) is on the graph.
- The point (1, b) is on the graph.

If $b > 1$, then the graph takes the following shape:



- As x increases, the function approaches positive infinity.
- As x decreases, the function approaches 0.
- The function is increasing.
- The function has a horizontal asymptote along the x-axis ($y = 0$).
- The point (0, 1) is on the graph.
- The point (1, b) is on the graph.

In general, for exponential functions:

- The function is on the positive side of the y-axis.
- The function does not cross the x-axis.
- If $y = b^x$, the function passes through the point $(1, b)$.
- If $y = b^x$, the function passes through the point $(0, 1)$.

Looking Ahead 2.2

The law of exponents that you learned in Algebra 1 and reviewed in Algebra 2 states that for real numbers a and b and positive integers m and n:

$$a^m \cdot a^n = a^{m+n}$$

$$(ab)^m = a^m b^m \text{ or } (ab)^n = a^{1 \cdot n} b^{1 \cdot n} = a^n b^n$$

$$(a^m)^n = a^{m \cdot n} = a^{mn}$$

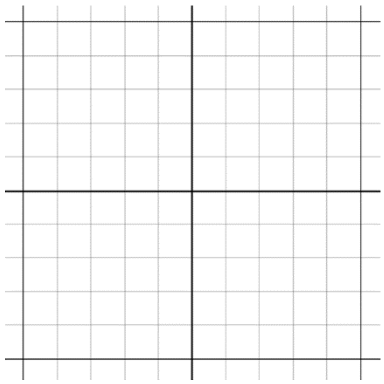
$$\frac{a^m}{a^n} = a^{m-n}, \text{ where } a \neq 0$$

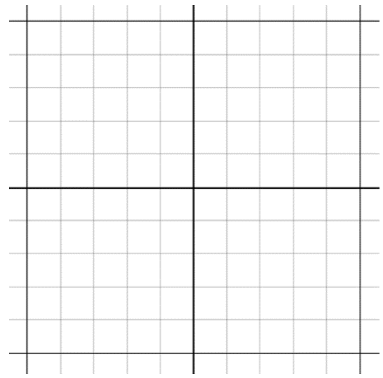
To solve exponential equations, express each side of an equation as a power to the same base, then set the exponents equal and solve.

Example 1:	Solve for x in $9^x = \frac{1}{27}$.

Example 2: Graph the pair of exponential equations and describe the similarities and differences:

$$y = 5^x \qquad y = 5^{x-3}$$





The graphs have the same shape. The graph of $y = 5^{x-3}$ is shifted 3 units to the right of $y = 5^x$.

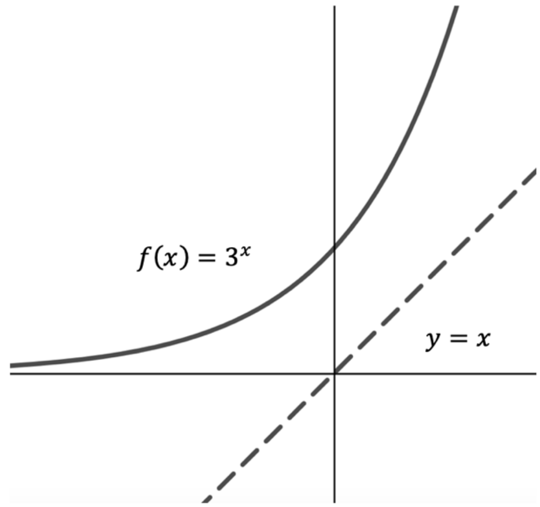
The graph of $f(x) = 5^x$ has a y-intercept of $(0, 1)$. The graph of $f(x) = 5^{x-3}$ has a y-intercept of $(0, 0.008)$.

Section 2.3 Logarithmic Functions

Looking Back 2.3

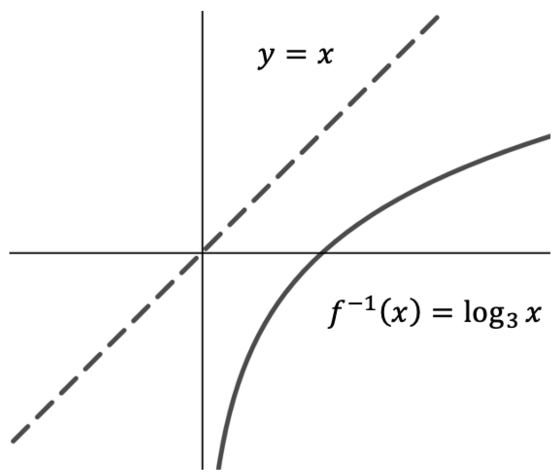
The function $f(x) = 3^x$ passes the Vertical Line Test and therefore, has an inverse. The table and graph for $f(x) = 3^x$ are shown below.

x	$f(x)$
-3	$\frac{1}{27}$
-2	$\frac{1}{9}$
-1	$\frac{1}{3}$
0	1
1	3
2	9
3	27



The table and graph for the inverse of $f(x)$ are shown below.

x	$f(x)$
-1	Undefined
0	Undefined
0.5	-0.63
1	0
1.5	0.36
2	0.63
2.5	0.83
3	1

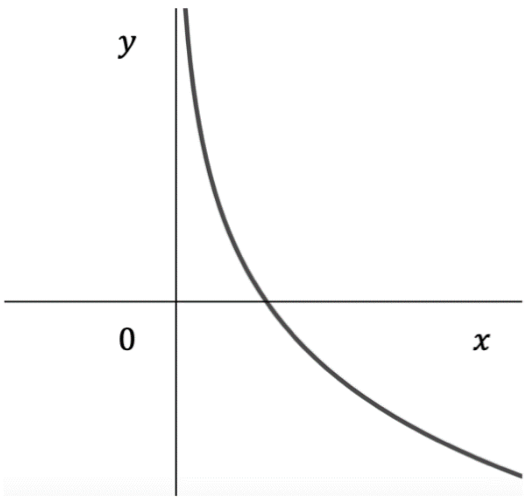


The inverse of the exponential function with base 3 is the logarithmic function with base 3, written: "$f^{-1}(x) = \log_3 x$." (This is read: "log base 3 of x," which you might recall from Algebra 2.)

If no base is shown for a logarithm, it is assumed to be base 10. Logarithms with base 10 are called "common logarithms." The common logarithmic function is $y = \log_{10} x$. Logarithms with base e also have a special name. They are called "natural logarithms." The natural logarithmic function is $y = \log_e x$.

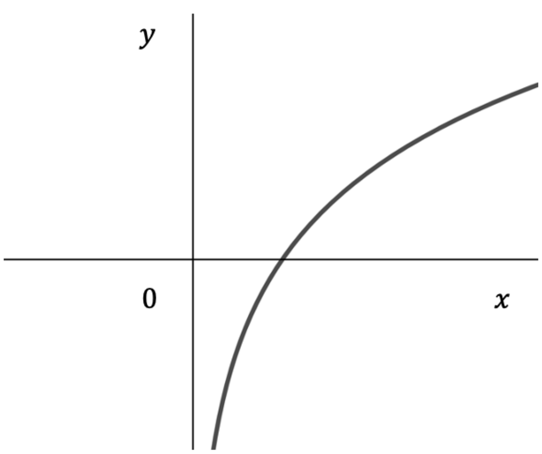
<u>Looking Ahead 2.3</u>

The logarithmic function $f(x) = \log_b(x)$ for $0 < b < 1$ is shown below.



- As x increases, the function approaches negative infinity.
- As x approaches 0, the function approaches positive infinity.
- The function is decreasing.
- The function has a vertical asymptote along the y-axis ($x = 0$).
- The graph crosses the x-axis at the point (1, 0).
- The point (b, 1) is on the graph.

If $b > 1$, the graph takes the following shape:



- As x increases, the function approaches positive infinity.
- As x approaches 0, the function approaches negative infinity.
- The function is increasing.
- The function has a vertical asymptote along the y-axis ($x = 0$).
- The graph crosses the x-axis at the point (1, 0).
- The point (b, 1) is on the graph.

In general, for logarithmic functions:

- The function is on the positive side of the x-axis.
- The function does not cross the y-axis.
- If $y = \log_b x$, the function passes through the point $(b, 1)$.
- If $y = \log_b x$, the function passes through the point $(1, 0)$.

Therefore, the function $y = 2^x$ can be written in logarithmic form as $\log_2 y = x$ and the inverse of that is $\log_2 x = y$. This means $y = \log_b x$ is the inverse of the exponential function $y = b^x$.

Example 1:	Write the exponential functions in logarithmic form to solve them. Check each solution.

a) $2^x = \frac{1}{2}$

b) $3^x = 15$

c) $4^x = 1$

Example 2:	The function $\log_3 10$ lies between which two consecutives integers? Find the decimal approximation without a calculator; then check your solution using a calculator.

Example 3:	Use exponents to solve for x in the logarithmic equations below.

a) $\log_3 x = 4.12$

b) $\log_x 7 = 3$

c) $\log_4(x - 1) = 2$

Section 2.4 Review of Logarithmic Functions

Looking Back 2.4

Because an exponential function can be written as a logarithm, and because logarithmic functions are inverses of exponential functions, the properties that hold true for exponents may be used to derive corresponding properties that also hold true for logarithms.

When you multiply logarithms with common bases, you add the exponents. When you divide logarithms with common bases, you subtract the exponents. The Golden Rule of Algebra, which you might remember, states that whatever you do to one side of an equation you must also do to the other side. This means you can take the logarithm of both sides of an equation to solve for a variable.

One last property that will help in solving logarithmic equations is the Power Rule:

$$\log_b M^N = N \log_b M$$

Let us investigate a proof for the Power Rule:

Let $\log_b M = a$.

Proof:

$$b^a = M$$

$$(b^a)^N = M^N$$

$$b^{aN} = M^N$$

$$\log_b M^N = aN$$

$$\log_b M^N = Na$$

Then, substituting $\log_b M$ for a yields $\log_b M^N = N \log_b M$.

Looking Ahead 2.4

Example 1: Use logarithms to solve for x in the equation below. Approximate the decimal to the thousandths place. Use base 10 as the common logarithm.

$$4^{2x} = 7$$

Remember, you are not getting exact answers, but decimal approximations.

Example 2: Use the power rule to solve for x in the equation below. Approximate the decimal to the thousandths place. Use base 10 as the common logarithm.

$$\log x = \frac{1}{3}\log 42$$

Just as when two exponents on either side of the equal sign have the same base and the exponents can be set equal to one another, when the logarithms have the same base, the remainder of the expressions on either side of the equal sign can be set equal to one another to solve for x.

The properties of logarithms can be used to solve logarithmic equations but may introduce extraneous solutions. This is extremely important! Forgetting this leads may lead to solutions that do not work.

Example 3: Solve for x in the equation using the properties of logarithms. Then graph the logarithmic equation to find the x-intercept. What is the extraneous solution when the equation is solved algebraically?

$$\frac{1}{2}\log_3(x+2) - \log_3 x = 0$$

On the graphing calculator, $(2, 0)$ is the only x-intercept. The only solution is $x = 2$ and $x = -1$ is an extraneous solution.

Section 2.5 The Natural Exponential Function

Looking Back 2.5

An exponential function occurs when a quantity grows or decays at a rate that is proportional to its original value. We can see this in bacterial growth. For example, if bacterial growth triples every minute with a start of 2 bacteria, then the exponential function for the number of bacteria at any given time is $\mathrm{p} = 2 \cdot 3^h$ (from $y = ab^x$), where p is the population and h is the hours of growth. After 5 hours, the number of bacteria is $\mathrm{p} = 2 \cdot 3^5$, $\mathrm{p} = 486$ bacteria.

Exponential growth usually starts slowly and increases rapidly. We can see where the exponential function increases naturally in banking when the amount of interest is compounded. For example, if you deposit $\$300.00$ in a savings account with an interest rate of 1.75%, at the end of one year you will have the following amount (given the rate is compounded annually):

$$\mathrm{A} = 300(1 + 0.0175)^1 \text{ (from } \mathrm{A} = \mathrm{p}(1 + \tfrac{r}{n})^n)$$

$$\mathrm{A} = \$305.25$$

Looking Ahead 2.5

The natural exponential function, $y = e^x$, is a special function that demonstrates continuous growth over a certain amount of time. The base e is an exponential constant that occurs often in economic growth. The amount of growth after x amount of time is e^x, and e^x allows us to substitute values in for time and find growth.

Example 1: Let us say you invest \$1 in a savings account at a bank for 1 year at an interest rate of 100% compounded annually. What would be the amount of money in the account after 1 year?

How much will be in the savings account if the interest is compounded quarterly (4 times a year)?

How much will be in the savings account after 1 year if the interest is compounded monthly (12 times a year)?

How much will be in the savings account after 1 year if the interest is compounded daily (365 times a year)?

If the amount of interest is compounded by the hour and the minute and the second, then the amount of money after 1 year will still be approximately $\$2.71$. The constant e is approximately $2.71828\ldots$ like pi, it does not repeat or end; so, though it is called a constant, it is an irrational number.

The constant e is named after Leonard Euler, who discovered how important e is as a natural occurring number in physical phenomena. Its decimal approximation to thirteen places of the graphing calculator is as follows:

$$e = 2.7182818284590\ldots$$

It is found on scientific calculators because it is widely used in science and mathematics.

Like Euler, you have previously learned about Johann Bernoulli, who was tutored by his older brother Jacob. Both Jacob and Johann studied the Calculus of Leibniz, and both broke family tradition by studying mathematics. Johann earned a degree in theology, however, and was offered a position at a church, but still, he declined it to pursue mathematics. He and his brother Jacob later became rivals over advancing the subject, which was unnecessary because both made great contributions to it.

Jacob researched exponential series, which came about from his study of compound interest. He defined e as follows:

$$e = \lim_{n \to \infty} (1 + \frac{1}{n})^n$$

This is read: "e equals the limit of one plus one over n to the nth as n approaches infinity." We know that as n gets larger and larger…

$$\ldots e \approx 2.71828182459$$

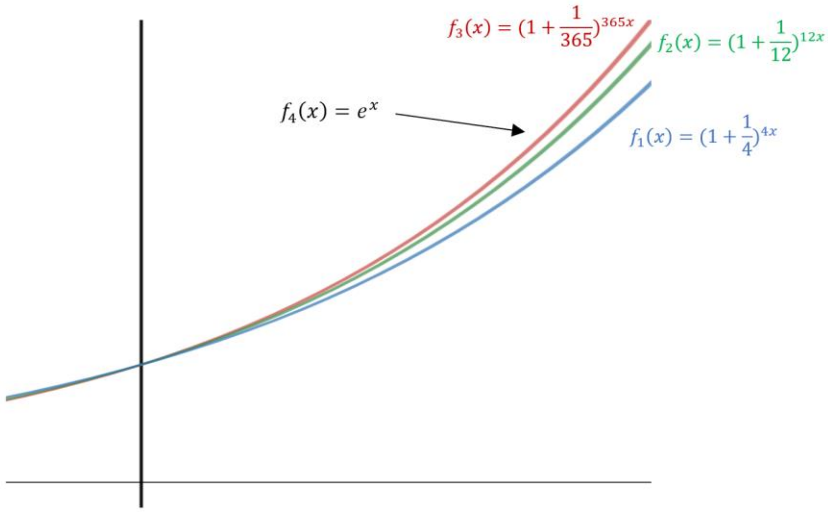
Or

$e \approx 2.72$ to the hundredths place and e^x is how much growth occurs after x units of time with 100% continuous growth. Therefore, $e^2 = 7.38$ means that after 2 periods of time there will be 7.38 times the amount of growth. So, e^x is a scaling factor for the amount of growth after x units of time.

Because x lets us substitute time to get growth, 4 years at a rate of 100% growth is the same as 1 year at a rate of 400% growth. Converting rates to 100% allows only time to be considered:

$$e^x = e^{\text{rate}\cdot\text{time}} = e^{1.0\cdot\text{time}} = e^{\text{time}}$$

Example 2:	What is happening to the graphs as x is getting increasingly larger?



Example 3: The continuous compound interest formula is as follows:

$$A(t) = Pe^{rt}\ldots$$

… where P is the principal amount invested, r is the interest rate, t is the time in years, and e is the natural exponential function.

If $\$400$ is deposited in an account that earns an interest rate of 1.05% compounded continuously, what is the balance after 3.5 years?

Both π and e are considered transcendental numbers because they are not roots or solutions of a non-zero polynomial equation with integer coefficients.

Jacob Bernoulli died in 1705 and had the golden spiral put on his tombstone with the Latin inscription: “Eadem Mutata Resurgo,” which translates to “I shall rise the same though changed.” Even in death, he was a mathematician and theologian.

Section 2.6 The Natural Logarithmic Function

Looking Back 2.6

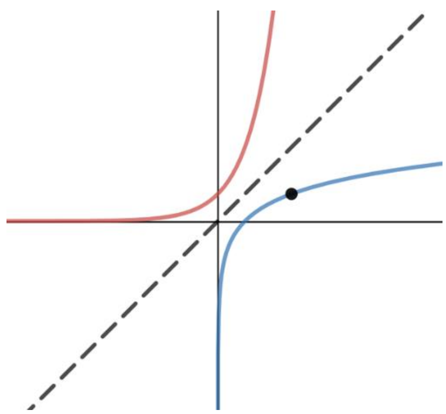
In Algebra 1, you learned about exponential functions. In Algebra 2, you learned about logarithmic functions. In the previous section of this text, you were introduced to the natural exponential function $f(x) = e^x$, where e is the transcendental, irrational Euler number approximately equal to 2.7182818 ... and e^x is the amount of continuous growth over a given amount of time.

In this section, you will be introduced to the natural logarithmic function. It is, by definition, the inverse of e^x; therefore, it is the inverse of the natural exponential function. If $f(x) = e^x$, then $f^{-1}(x) = \ln(x)$. The natural logarithmic function seems almost supernatural- by the design of a supernatural Designer.

The natural logarithmic function is written $f(x) = \log_e x$ or $f(x) = \ln(x)$. The "ln" is the abbreviation for the Latin word "logarithmus naturalis." The natural logarithmic function is just a logarithmic function with a base of e.

It would make sense that because the natural logarithmic function is the inverse of the natural exponential function, it is the amount of time needed to reach a given amount of continuous growth.

Looking Ahead 2.6



The line that is tangent to the curve $f(x) = \ln(x)$ at point $(e, 1)$ has a slope of $\frac{1}{e}$.

The function e^x lets us substitute time to find the growth that takes place.

The function $\ln x$ lets us substitute growth to find the time it would take.

Example 1:	What does $e^2 \approx 7.389$ mean? What is $\ln 7.389$? Why?

Example 2: What does $\ln 1 = 0$ mean? If you type "ln 1" in the calculator and press "=," you get 0. Why does $\ln(-2) = \text{undefined}$? If you type "$\ln(-2)$" in the calculator and press "=," you get "undefined;" what does this mean?

Although exponential decay may be seen as negative growth (before one unit of time), you cannot find the natural logarithm of a negative number.

However, you can ask how long it would take to get one-half of your amount assuming you have continuous growth at a rate of 100%. That would be $\ln(\frac{1}{2}) = \ln(0.5) \approx -0.693147$. To find one-half of the amount is the reverse of doubling (taking the negative time) so that $-\ln(2) \approx -0.693147$. Therefore, $-\ln(2)$ gives the time needed to get one-half of the current value or amount and $-\ln(3)$ is the time needed to get one-third of the current value (1.0) or amount.

Example 3: Given:

$$\ln(\frac{1}{2}) \approx -0.693$$

$$-\ln(2) \approx -0.693$$

$$\ln(\frac{1}{3}) \approx -1.099$$

$$-\ln(3) \approx -1.099$$

$$\ln(\frac{1}{4}) \approx -1.386$$

$$-\ln(4) \approx -1.386$$

$$\ln x = -\ln(\frac{1}{x})$$

What is $\ln(4)$? What is $-\ln\left(\frac{1}{4}\right)$?

Example 4: Match the problems below with the correct solutions from the following answers:
$\frac{1}{2}$, DNE, 1, 0 or 3.

a) $\ln(e)$

b) $\ln(e^3)$

c) $\ln(0)$

d) $\ln(1)$

e) $\ln(\sqrt{e})$

Sometimes, data for an experiment can be difficult to read because the exponential numbers get so large. Other times, only a small amount of data from a power function may be confused as exponential. In both of these cases, it helps to analyze the data, if it is linearized.

This is how a calculator finds a line of best fit: it takes exponential data, transforms it to a linear function of x in terms of $\log y$, does a linear regression, and then transforms it back to an exponential function.

Example 5: Graph the data in the table below and determine a good model or line of best fit for the data. Is it a power function or an exponential function?

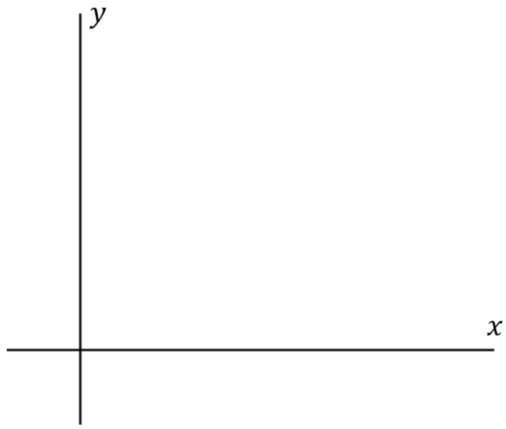
x	y
3	27.59
4	36.43
7	83.79
8	110.6
11	254
12	335

To linearize the data, take the common logarithm of both sides and use the laws of logarithms to simplify and evaluate them.

To graph the log, find the $\log y$ values for $f(x)$.

x	y
3	27.59
4	36.43
7	83.79
8	110.6
11	254
12	335

x	$\log y$
	1.44
	1.56
	1.92
	2.04
	2.40
	2.52



Leonard Euler, who you might remember from Geometry and Trigonometry, was a Swiss mathematician who contributed many symbols to the field of mathematics, which include the following: i for $\sqrt{-1}$, π for pi, $f(x)$ for functions, Σ for summation, and e for the natural logarithm.

Euler derived the following formula to calculate e:

$$e = 1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \ldots}$$

The precise notation for this formula is $\sum_{n=0}^{\infty} \frac{1}{n!}$.

It is told that when the French philosopher Diderot was trying to convert the court of Catherine the Great of Russia to atheism, Euler sent him the following problem and note:

"Sir, $\frac{a+b^n}{n} = x$, hence God exists, reply!"

Diderot did not reply to this power series, which requires pages of work, but he did leave the court alone after receiving Euler's note!

Section 2.7 Theorems of the Natural Exponential Function

Looking Back 2.7

As you learned at the start of these courses (many, many years ago), John Napier, a Scottish mathematician, was dedicated to implementing devices to make the arduous task of computing less complicated. Napier's bones (for multiplication) is one of these devices. Exponents deal with very large numbers, so John Napier wrote tables of logarithms, now embedded in scientific calculators, to work with these. The Greek word "logos" means "proportion" and the Greek word "arithmos" means "number;" therefore, Napier combined these to form the word "logarithm."

Napier knew that long and tedious calculations were prone to error. So, though one could have the process correct, the outcome could still be faulty. Napier's pages upon pages of tables saved mathematicians from repeating these errors.

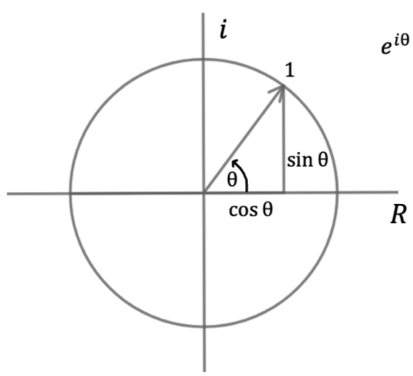
Moreover, Napier applied mathematical principles to spiritual principles in his own life, stating that doctrine does matter; however, one must be reminded that to be right on doctrine does not necessarily mean that one is right with the Lord.

Looking Ahead 2.7

In Module 4 of Pre-Calculus and Calculus, you will investigate Polar Functions and Complex Numbers. The natural exponential function can be graphed on the complex number plane.

Example 1: On the polar coordinate grid, the horizontal component of a complex number is given by $\cos\theta$ and the vertical component is given by $\sin\theta$. This may be familiar to you from your study of the unit circle in Geometry and Trigonometry.

A complex number is part real and part imaginary.



$e^{i\theta} = \cos\theta + i\sin\theta$

This is the natural exponential function on the complex number plane written in Euler's Identity:

$$e^{i\theta} = \cos(\theta) + i\sin\theta$$

a) Which part of the identity is the real part?

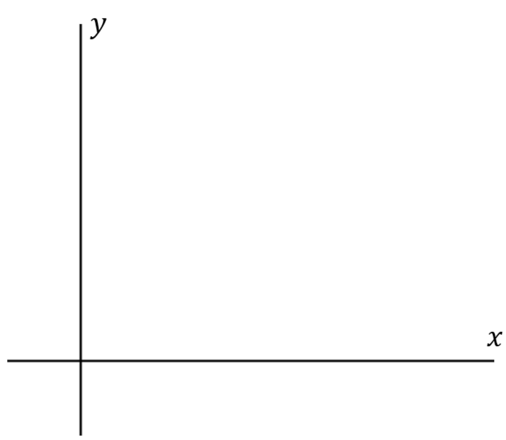
b) Which part of the identity is the imaginary part? ($sin\theta$)

c) What does the 1 represent?

As previously studied, the inverse of e^x is $\ln x$. Another unique characteristic about e^x is that it is its own derivative. You will learn about derivatives in Module 7 of this text. Also, $x = \int_1^y \frac{1}{t}\, dt = e^x + c$ is the indefinite integral of x. You will learn about integrals in Module 8 of this text. You have already learned that the limit of $(1 + \frac{x}{n})^n$ as n approaches infinity is e^x:

$$\lim_{n \to \infty}(1 + \frac{x}{n})^n = e^x$$

Let us talk about limits. You will learn more about limits in Module 5 of this text. A limit is the number $f(x)$ approaches as x approaches a given value c. The one number L that represents the limit is what $f(x)$ gets close to as x gets close to c from either side, but not equal to c. It is written: "$\lim_{x \to c} f(x) = \mathrm{L}$." The graph for this limit looks like this:



Example 2: Find the limit of $(1+\frac{1}{n})^n$ as n approaches infinity for:

a) $n = 1$

b) $n = 10$

c) $n = 100$

Notice that it approaches the limiting value of $e = 2.71828\ldots$

Example 3: Find the $\lim_{n \to \infty} n^{\frac{1}{n}}$ by using natural exponential and logarithmic functions. Let $\lim_{n \to \infty} n^{\frac{1}{n}} = \mathrm{C}$ and solve for C.

This can be found using L' Hospital's Rule, which you will learn about in Module 8 of this text. You will use the Taylor Series to find e^x in the practice problems.

Section 2.8 Properties of the Natural Logarithm

Looking Back 2.8

You learned the rules of exponents in Pre-Algebra and continued to use them in Algebra 1. In Algebra 2, you learned that logarithms are inverses of exponents, so the same rules apply.

For like bases, where $n \neq 0$ and $n \nless 0$:

$$\log_n(a \cdot b) = \log_n a + \log_n b$$

$$\log_n(\frac{a}{b}) = \log_n a - \log_n b$$

Because e^x is the natural exponential function and $\log_e x$ is the natural logarithm function (denoted by $\ln x$) the same rules apply. Let us see if that makes sense.

Looking Ahead 2.8

Example 1: $\ln(2)$ is the time it takes for your current value to double so if we double $\ln(2)$, $\ln(2) + \ln(2)$, that is how long it should take to quadruple your current value:

$$\ln(4) = \ln(2) + \ln(2)$$
$$\ln(2 \cdot 2) = \ln(2) + \ln(2)$$
$$1.38629 \ldots \approx 0.693 \ldots + 0.693 \ldots$$
$$1.38629 = 1.38629$$

Therefore, if we double $\ln(2)$, this should be the same amount of time as $2\ \ln(2)$.

$$2\ \ln(2) \approx 1.38629 \ldots$$

Does this pattern work for any amount of growth as long as growth is strictly natural: $y = ae^x$?

a) Does 10 times the amount of growth require an amount of time equal to that required for 2 times the amount of growth plus that required for 5 times the amount of growth?

$$\ln(10) = \ln(2) + \ln(5)$$

b) Does 20 times the amount of growth require an amount of time equal to that required for double the growth plus that required for 10 times the amount of growth?

$$\ln(20) = \ln(2) + \ln(10)$$

Example 2: Does $\ln(\frac{a}{b}) = \ln a - \ln b$?

a) Solve $\ln(\frac{5}{3})$ both ways.

b) Solve $\ln(\frac{20}{2})$ both ways.

Section 2.9 Applications of the Natural Logarithm

Looking Back 2.9

The fact that e^x is its own derivative makes it quite easy to work with in Calculus. In fact, it is the only function, other than $f(x) = 0$, which is its' own derivative. You will learn what this means and how important it is in Module 7.

By definition, e^x is the amount of continuous growth over a certain amount of time and the natural logarithm is the time needed to reach that amount. Therefore…

$$\ln(e^1) = 1$$
$$\ln(e^2) = 2$$
$$\ln(e^x) = x$$

The natural logarithmic function $\ln(x)$ is the inverse of the natural exponential function e^x. This makes it easy to solve problems in Calculus using the natural logarithm and to find missing exponents when e is involved.

Looking Ahead 2.9

Example 1: Solve for x in the equation:

$$2e^{2x+1} = 9$$

Example 2: Solve for x in the equation:

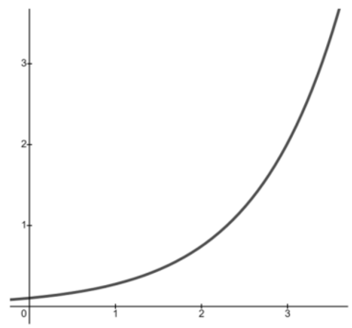
$$10^{x^2-2x} = 1{,}000$$

Example 3: A bacterial culture grows under the general function $P(t) = P_0 e^{kt}$, where P_0 is the initial bacteria, k is the rate given the strain of bacteria and factors affecting it, and t is time. If there are initially $20{,}000$ bacteria and this grows to $80{,}000$ after 3 hours, how many bacteria will be present after 5 hours?

Section 2.10 The Logistic Function

Looking Back 2.10

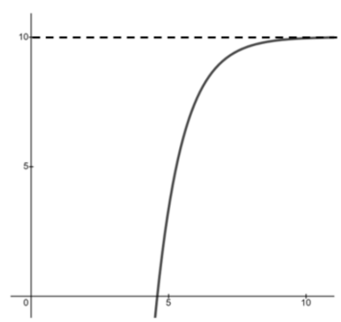
There are two kinds of exponential growth functions. The first kind is the pattern we have seen in savings' deposits with interest rates; it increases at an increasing rate. The growth is exponential, so the growth rate is proportional to the size of the function's actual value.



Increasing Exponential

$$F(x) = \frac{1}{10}e^x$$

The second kind of exponential growth is a bounded exponential growth. A decaying exponential reaches up to a fixed boundary. This kind of growth is limited by some fixed value, which is the maximum that can be obtained.



Bounded Exponential

$$F(x) = 10(1 - \frac{100}{e^x})$$

Looking Ahead 2.10

The logistic function combines the two types of exponential functions and models an exponential function that is limited by the maximum value that can be obtained.

The logistic function combines small outputs for the first kind of exponential growth with outputs near capacity for the second kind of exponential growth. Initially, exponential growth reaches a bound or upper limit. It mixes both exponential powers and proportions involved in rational functions. Note that for the function to behave as it should $a, b,$ and c must all be positive and $c \neq 1$.

$$F(x) = \frac{a}{1 + bc^{-x}}$$

The exponential bc^{-x} determines whether the function rises or falls and has a y-intercept of b. In the short run, when x is near 0, then c^{-x} is near 1 and the value of the function approximates $\frac{a}{1+b}$. In the long run, the function approaches either "a" or 0 depending on c.

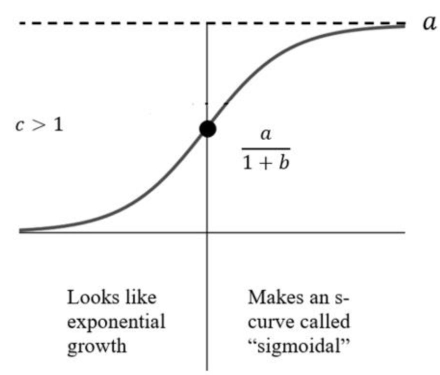
Because c^x grows when $c > 1$, c^{-x} decays for $c > 1$. As c^{-x} decays for $c > 1$, then the denominator approaches 1, and the function approaches "a." Therefore, the function never passes "a," but reaches a horizontal asymptote at "a." For the growth of resources, "a" is called the limiting value, and for the growth of population, "a" is called the carrying capacity. The function first increases at an increasing rate and then decreases at a decreasing rate after the halfway point.

Small Initial Rates

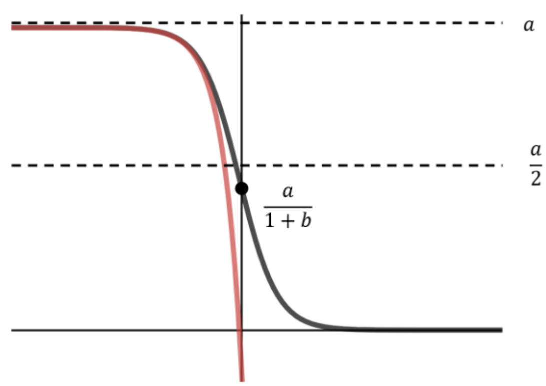
Acceleration to Point of Inflection

Growth Slows

Approaches a Limiting Value



Because c^x decays when $0 < c < 1$, c^{-x} grows for $0 < c < 1$. As c^{-x} grows for $0 < c < 1$, then the denominator grows larger, and the function approaches 0 in inverse proportion. Therefore, the function first decreases from "a" at an increasing rate and then decreases at a decreasing rate after the halfway point.



Example 1: Compare the end behavior of the logistic function $F(x) = \frac{a}{1+bc^{-x}}$ for $0 < c < 1$ in black above and the exponential function $G(x) = a(1 - bc^{-x})$ in red above.

Example 2: The rate at which a logistic function increases, or decreases is dependent on the exponential function in the denominator, particularly b and c. In Example 1, the function, F, decreases at an increasing rate then at the halfway point begins to decrease at a decreasing rate. Describe the concavity of the graph before and after the critical point of inflection.

Example 3: The inflection point of the logistic function always coincides with the halfway point of growth and decay. Complete the calculation below to solve for x. Find the y-intercept and the exact value of the point of inflection for $y = \frac{10}{1+1.1(0.2)^{-x}}$

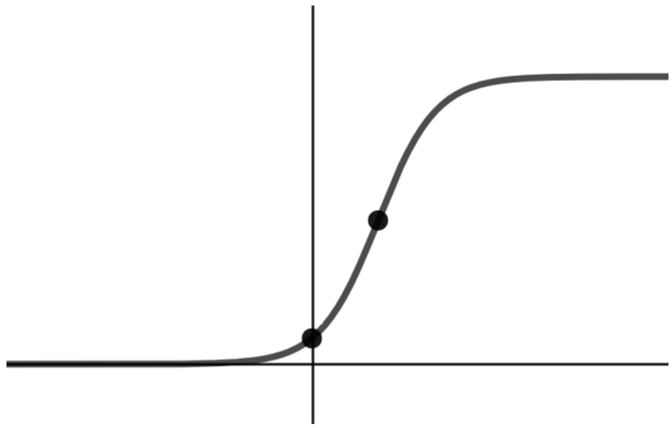
$$\frac{a}{1 + bc^{-x}} = \frac{a}{2}$$

Section 2.11 The Natural Exponential Logistic Function

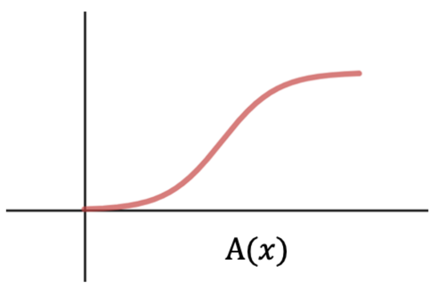
Looking Back 2.11

In Problem 8 of the previous Practice Problems section, you calculated the inflection point to be approximately $(2.31, 5.02)$ for the logistic function given below. The graphing calculator shows the point of inflection to be approximately $(2.3, 5)$; the equation and graph are shown below:

$$\mathrm{F}(x) = \frac{10}{1 + 10e^{-x}}$$



The characteristic S-shape is modeled in an accumulation of a function with a peak value.



The S-shape is referred to as a sigmoidal graph.

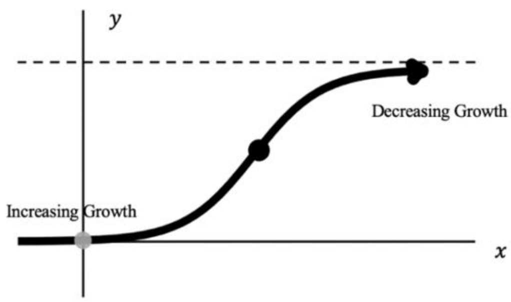
Looking Ahead 2.11

When bacteria are grown in a petri dish, they have a limiting factor, which is the size of the petri dish. They will grow rapidly at first until they near the circumference of the dish and crowd. The bacteria will cease growing when there is no more room in the crowded petri dish.

We have seen e, the natural exponential function, which occurs naturally in problems of growth and decay.

The family of functions of the form $y = \frac{a}{1+be^{-cx}}$ are logistic growth functions where a, b, and c are constants. Exponential functions increase without bound. The logistic growth function $y = \frac{a}{1+be^{-cx}}$ has $y = a$ as an upper bound. It models real-world growth that changes from an increasing growth rate to a decreasing growth rate. The inflection point is the maximum growth rate and occurs at $(\frac{\ln b}{c}, \frac{a}{2})$.

Example 1:	List the characteristics of $y = \frac{a}{1+be^{-cx}}$.



Example 2:	What are the domain and range of the function $y = \frac{a}{1+be^{-cx}}$?

Example 3:	Evaluate the logistic growth function at $f(-1), f(0),$ and $f(5)$: $$f(x) = \frac{100}{2 + 7e^{-3x}}$$

Section 2.12 Graphing and Solving Logistic Functions

Looking Back 2.12

You have already learned how to find the asymptotes (the floor and the ceiling of the logistic function), as well as the inflection point (which is at the halfway point) of a graph. Whether $c > 1$ or $0 < c < 1$ will determine the end behavior of the function (whether the function represents growth or decay).

For the graphs of logistic growth functions, the graph is increasing from left to right for the following equation:

$$y = \frac{a}{1 + be^{-cx}}$$

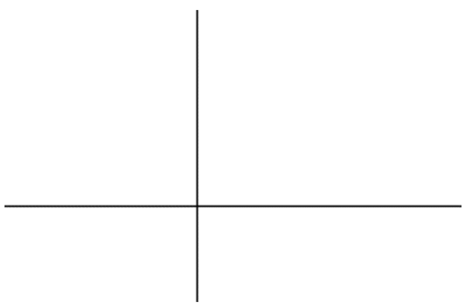
The rate of increase is increasing to the left of the point of maximum rate of growth, $(\frac{\ln b}{c}, \frac{a}{2})$. The rate of increase is decreasing to the right of the point of maximum rate of growth, $(\frac{\ln b}{c}, \frac{a}{2})$.

Looking Ahead 2.12

Example 1: Graph the equation:

$$\frac{9}{1 + 3e^{-0.3x}} = y$$

Check the solution using a graphing calculator.



Example 2: Solve for x in the logistic growth function below (this will be a decimal approximation, not an exact answer).

$$\frac{30}{1 + 9e^{-2x}} = 25$$

Section 2.13 Applications of Logistic Functions

Looking Back 2.13

We have said that logistic functions model growth and decay, much like exponential functions. However, they model these functions at a limited resource capacity. Therefore, logistic functions are not without bound, as exponential functions, but are bounded.

One example of a logistic function is the growth of a seedling (which we looked at in the practice problems of Section 2.11). Seedling growth is limited in its growth because a seedling will not grow on forever as Jack's beanstalk did in the fairy tale *Jack and the Beanstalk*. Plants are limited by their stem size. Trees are limited by their trunk size; a tree too tall could not be supported.

Looking Ahead 2.13

Example 1: A seedling's growth and height are being tracked in centimeters per week. Use the table below and a graphing calculator to find the logistic regression equation that models the growth and predict the height and growth of the seedling by the 8th week.

Let $w =$ weeks and $h =$ height (in centimeters).

$\boldsymbol{w}$	0	1	2	3	4	5	6	7
$\boldsymbol{h}$ $(\mathbf{cm})$	2.5	6.5	13	19.5	25.5	44	47	51.5

Example 2: A bacteria is growing in a petri dish that has an area of 46 square centimeters where t represents days. The growth is modeled by the logistic function shown below:

$$\mathrm{A} = \frac{46}{1 + 129e^{-1.4t}}$$

a) What is the initial area of the bacteria?

b) What will be the area of the bacteria after 3 days?