Module 7 Cubic Equations

Section 7.1 The Parent Cubic Equation

Looking Back 7.1

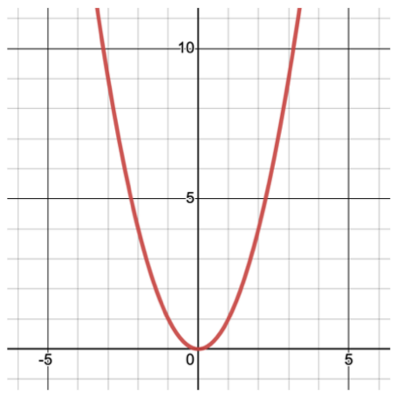
The parent function of a quadratic equation is $y = x^2$. It is called an *even function*. We will learn about even functions in Section 7.11. The parabola of an even function has a vertex at $(0, 0)$ and opens upward. There are no shifts right, left, up, or down from the origin. When the input values for x are positive numbers, the output values for y are also positive numbers. A positive number multiplied by itself (squared) is a positive number. When the input values for x are negative, the output values for y are still positive numbers. This is because a negative number multiplied by itself (squared) is also a positive number.

$$3 \cdot 3 \longrightarrow 3^2 = 9$$
$$(-3)(-3) = 9 \longrightarrow (-3)^2 = 9$$

Therefore, when the inputs are positive or negative, the outputs are always positive, so all values of the graph are in Quadrant I and II.

Quadrant II	
x	y
-3	9
-2	4
-1	1
0	0

(x, y)
$(-, +)$



(x, y)
$(+, +)$

Quadrant I	
x	y
3	9
2	4
1	1
0	0

When 0 is squared, $0 \cdot 0 = 0$. Therefore, $(0, 0)$ is the vertex. It is the lowest point on the graph; so, it is a minimum point. This is why the parabola opens upward. The parabola is symmetric over the y-axis as can be seen by the symmetric values in the y-column of the tables.

Looking Ahead 7.1

The parent function for a cubic equation is $y = x^3$. It is called an *odd function*. We will learn about odd functions in Section 7.11. There are no shifts right, left, up, or down from the origin.

Example 1: Fill in the blanks.

When we cube a positive number, we multiply it by itself three times.

When the input is positive, the output is __________________________.

$(3)(3)(3) =$ ________________ or $3^3 =$ ________________

When we cube a negative number, we multiply it by itself three times.

When the input is negative, the output is __________________________.

$(-3)(-3)(-3) =$ ________________ or $(-3)^3 =$ ______________.

Therefore, when the inputs are positive, the outputs are also positive, and when the inputs are negative, the outputs are also negative, so all of the values of the graph are in Quadrant I and III.

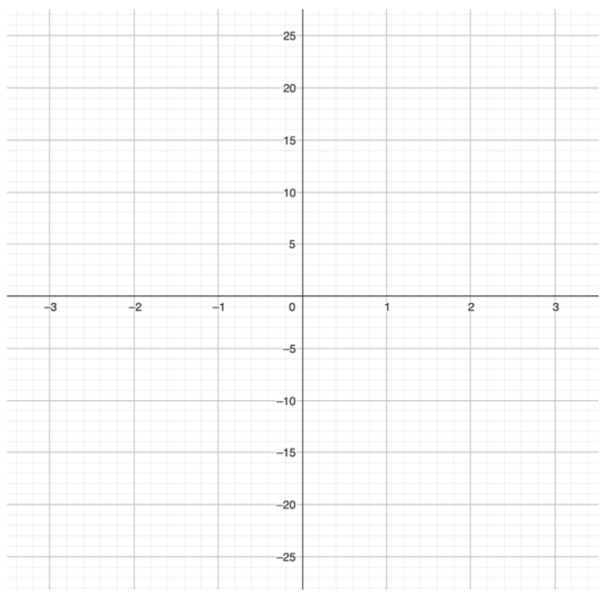
In tabular form:

Input	**Output**	**Quadrant**
+	+	I
−	−	III

The graph of the parent function of a cubic equation (to the third power) is in Quadrant I and III and goes through the origin. The origin is called the *point of inflection*.

Example 2: Complete the table and graph of the cubic equation $y = x^3$.

x	y
-3	
-2	
-1	
0	
1	
2	
3	



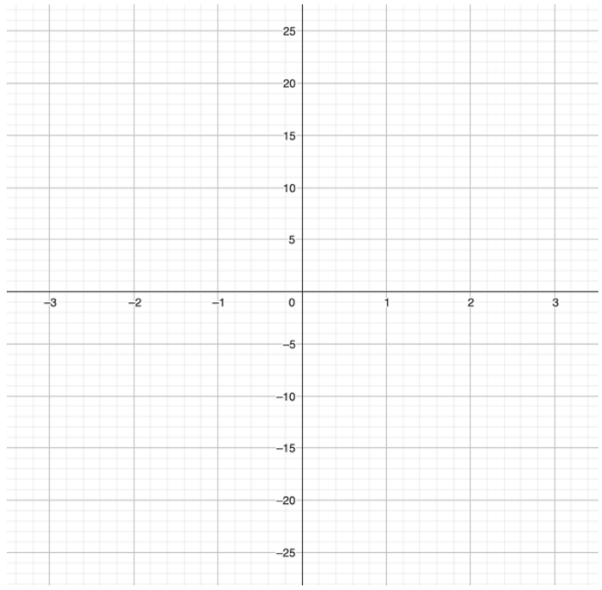
Example 3: Fill in the blanks.

The parent function of a cubic equation is ____________________. The point ____________________ is still on the graph but is not a ____________________. There is no ____________________ or ____________________ point. Instead, the point $(0, 0)$ is called the point of ____________________. In math, this indicates a change in direction.

The parabola starts at the left side ____________________ down at ____________________ infinity and goes up at the right side ____________________ up to ____________________ infinity.

Example 4: Sketch what you think the graph of $y = -x^3$ will look like.

x	y
-3	
-2	
-1	
0	
1	
2	
3	



Quadratics are to the second power. Cubics are to the third power. These are exponents. Throughout the first half of this module, we will review exponents in the Lesson Notes; we will also review several mathematics concepts by playing games in the Practice Problems.

Throughout this module, we will specifically study equations with an exponent of three (to the third power). Equations to the third power are cubic equations. Both quadratic and cubic equations are called *power functions*.

Example 5: What is the multiplication rule for exponents? Do we multiply or add the exponents?

Example 6: Simplify the expression: $x \cdot y^3 \cdot x^2 \cdot x^3 \cdot y$.

Example 7:	Simplify the expression: $(-2x^3)(4y^3)(6x^5)$.

Example 8:	Simplify the expression: $(-3x^3)(4y^4)(5y^5)$.

Section 7.2 Standard Form of the Cubic Equation

Looking Back 7.2

Remember, the standard form of a quadratic equation is $y = ax^2 + bx + c$. In standard form, when $a = 1$, $b = 0$, and $c = 0$, then $y = 1x^2 + 0x + 0$ or $y = x^2$; $y = x^2$ is the parent function of the quadratic equation.

The standard form of a cubic equation is $y = ax^3 + bx^2 + cx + d$. In standard form, when $a = 1$, $b = 0$, $c = 0$, and $d = 0$, then $y = 1x^3 + 0x^2 + 0x + 0$. Using 1 as the identity element of multiplication and 0 as the identity element for addition, this equation simplifies to the parent function of a cubic equation: $y = x^3$.

The parent quadratic equation has only a quadratic term, and the parent cubic equation has only a cubic term.

Looking Ahead 7.2

The factored form of a cubic equation is $(x - 3)(x + 1)(x + 2) = y$. It is cubic because there are three binomial (or linear) factors. They are:

$x - 3$ $\qquad$ $x + 1$ $\qquad$ $x + 2$

The binomials are called linear factors because the factors are all in the form $y = mx + b$ and the degree of x is to the first power. If graphed individually, each factor would give a straight line, but when graphed collectively as a product, they give a wave, which is characteristic of a cubic equation.

To convert the factored form of a cubic equation to standard form, multiply the first two binomials, then multiply that solution by the third binomial and simplify the equation.

Example 1: Use the Distributive Property to multiply $(x - 3)(x + 1)(x + 2) = y$ and convert it to standard form to identify a, b, c, and d.

Example 2: Use a geometric model (array) to multiply $(x - 3)(x + 1)(x + 2) = y$.

Example 3:	Use long multiplication to multiply $(x-3)(x+1)(x+2)=y$.

In the previous section, we reviewed the rule for multiplying terms with common bases and exponents. When we multiply terms with common bases, we add the exponents. Now, we want to look at dividing terms with common bases and exponents.

Example 4:	Simplify: $\frac{2^4}{2^2}$

Example 5:	Simplify: $\frac{x^3}{x^5}$

Example 6:	Simplify: $\frac{2x^7y^5}{8x^5y^8}$

Remember, anything to the zero power is _______________.

Section 7.3 Revisiting the Zero-Product Property

Looking Back 7.3

In the previous module, the standard form of a quadratic equation $y = ax^2 + bx + c$ could be factored into two binomial (or linear) factors: $(ax + c)(bx + d)$ in which $a \neq 0$ and $b \neq 0$. When we have factored a quadratic equation, we have used the Zero-Product Property to find the zeroes of the functions. Letting $y = 0$ and solving for x results in the x-intercepts of the equation. This is where the graph crosses the x-axis.

$$(x - 2)(x + 8) = 0$$
$$x - 2 = 0 \text{ or } x + 8 = 0$$
$$x = 2 \text{ or } x = -8$$

The ordered pairs on the graph that are the x-intercepts are $(2, 0)$ and $(-8, 0)$.

Remember that a, b, and c of the standard form are not the same as a, b, and c of the linear factors, even though they represent the same equation. They are two different forms of the same equation. They are constants for each specific equation.

For example, in the quadratic equation $2x^2 + 7x + 3 = y$ of the form $ax^2 + bx + c = y$, $a = 2$, $b = 7$, and $c = 3$, but when factored into its binomial (linear) factors $(x + 3)(2x + 1)$ of the form $(ax + c)(bx + d)$, we see $a = 1$, $b = 2$, $c = 3$, and $d = 1$.

Looking Ahead 7.3

Example 1: For the factored form of the cubic equation $y = (x + 3)(x - 2)(x - 1)$, find the zeroes of the function.

Example 2: Use the Zero-Product Property to find the x-intercepts of the graph of the cubic function in its factored form $3x(2x + 1)(2x - 2) = 0$.

The quadratic equation from the Looking Back section had two zeroes. These cubic equations in the Looking Ahead section have three zeroes. Sometimes, zeroes are repeated as seen later in this module. These are called double zeroes (remember, zeroes are roots, etc.), which means a quadratic equation may have only one zero and a cubic equation may have only two zeroes.

Example 3: Simplify: $(2^2)^3$

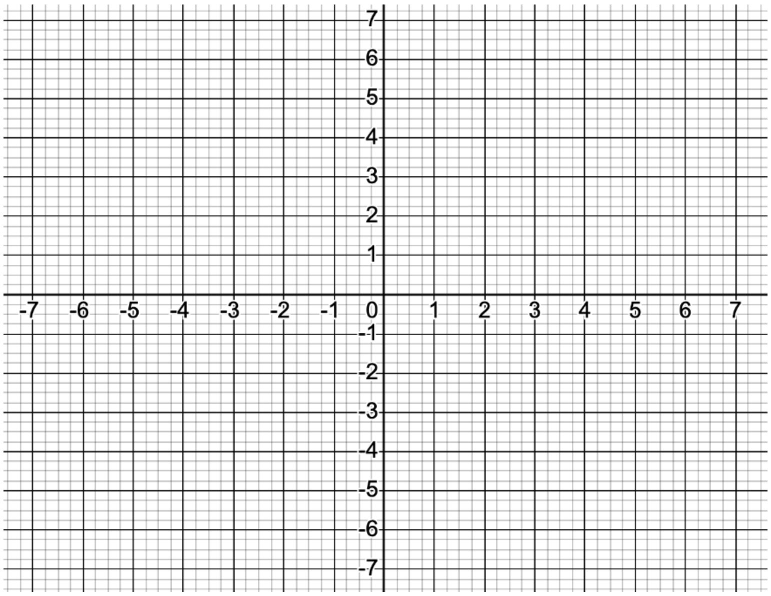
Example 4: Simplify: $(x^3y^2)^4$

Example 5: Simplify: $(\frac{x^2}{y^3})^3$

Section 7.4 Graphing Cubic Equations

Looking Back 7.4

The zeroes of the function are also called the roots of the function. These are also the x-intercepts, which are the points on the graph in which y is 0. Because these are the points that cross the x-axis, they are very helpful when graphing equations. In the equation from the previous Practice Problems section, $x(x+4)(x-3)=0$ has three zeroes: $x=0$, $x=-4$, and $x=3$. The three corresponding ordered pairs are $(0,0)$, $(-4,0)$, and $(3,0)$. These can be located on the graph the same way the x-intercepts of a quadratic equation were found.



Make a sketch on the graph to the left. The cubic equation has a positive a value in front of the lead x, so (like in the parent function), it starts down and ends up, and it crosses through the origin $(0,0)$.

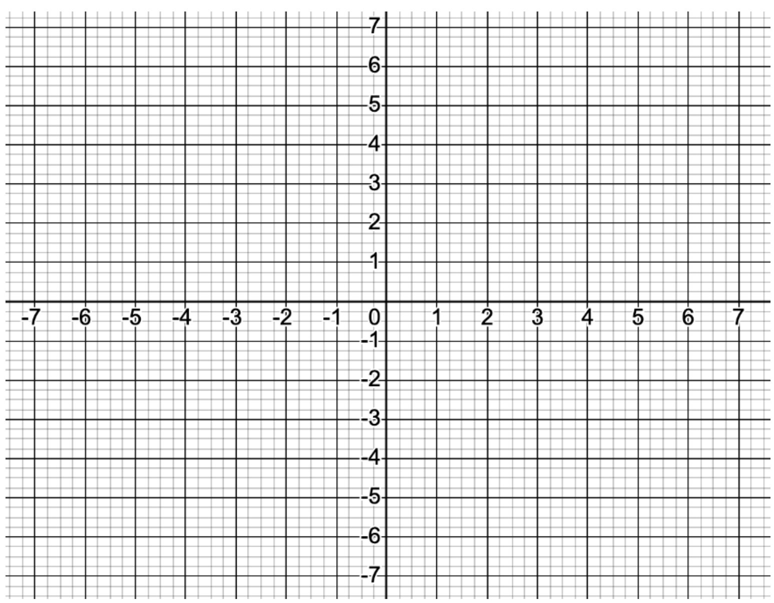
Converting from standard form to vertex (graphing) form for a cubic equation is a concept that is above the level of this *Algebra 1* course; we will begin studying that concept in *Algebra 2*.

If the equation is in factored form, a sketch of the graph can be drawn. As we get into more advanced mathematics courses, the graphs will get more precise. For now, they are sketches.

Looking Ahead 7.4

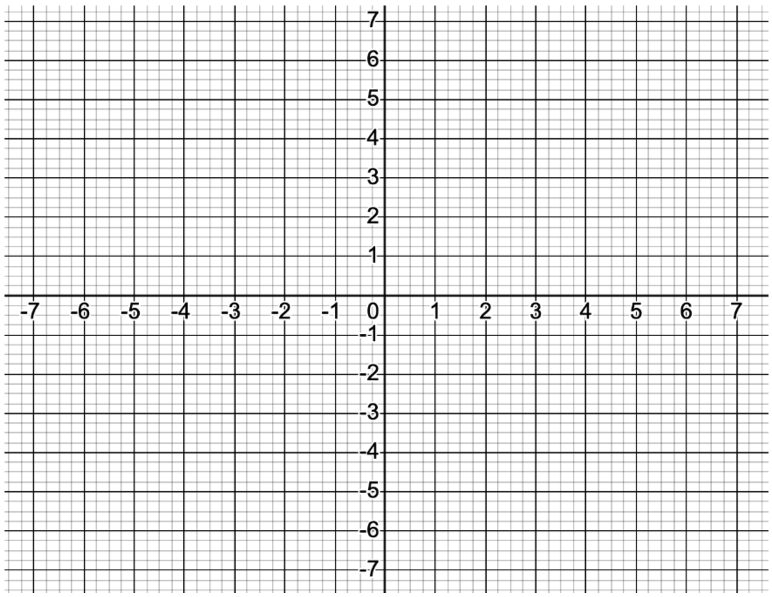
Example 1: Sketch the graph of the cubic equation $(x-1)^2(x+1)=y$.

A double zero is __________________________ that can be written ___________________________.



Example 2: Find the y-intercept of the previous graph as a third point because it has a double zero. This will show how high the hump in the middle of the graph goes.

At the y-intercept, x is ____________________.



Example 3: Simplify: $3^0 \cdot 3^4$

Example 4: Simplify: $x^1 \cdot x^{-1}$

Example 5: Write an equivalent form of each of the expressions below.

$\frac{1}{x}$ x^{-5}

$\frac{1}{x^r}$ x^{-2}

Section 7.5 Finding Cubic Equations from the Graph

Looking Back 7.5

In the previous section, we saw that if the equation given was factored, we could use the Zero-Product Property to find the x-intercepts, and we could substitute 0 for x to find the y-intercept. The point could be plotted and a sketch of the graph could be drawn. So, moreover, if the graph is given, the zeroes of the function could be used to find a reasonable equation.

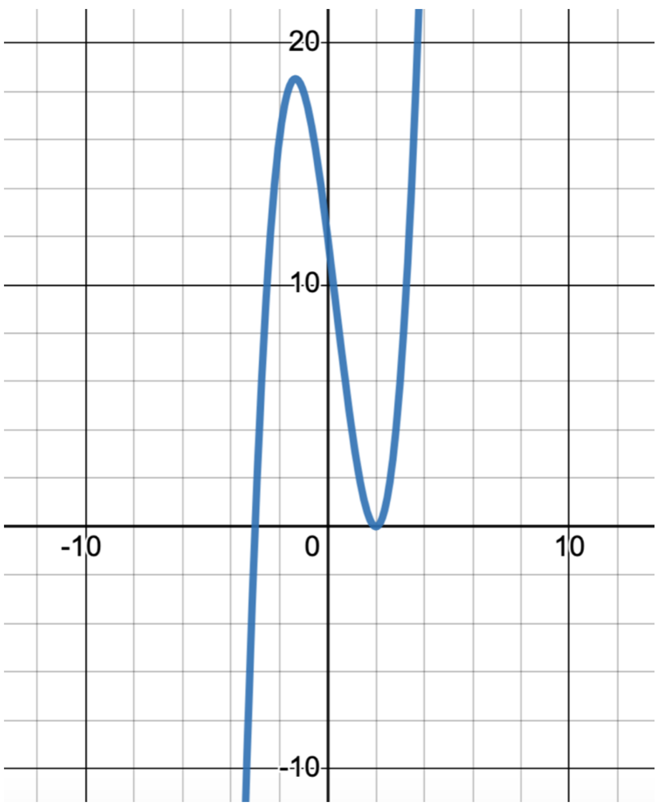
Cubic equations do not have a vertex; there is no minimum or maximum, but rather a point of inflection. The point of inflection is a change in direction. The graph of the parent function of a cubic equation goes from negative infinity to positive infinity. There are some local maximum points and local minimum points which we will learn about in *Algebra 2.*

Our lives sometimes have points of inflection, or changes in direction. Romans 10:9-10 says: "… If you confess with your mouth Jesus as Lord, and believe in your heart that God raised Him from the dead, you will be saved. For with the heart, man believes, resulting in righteousness and with the mouth, he confesses, resulting in righteousness."

This belief becomes an inflection point in our lives. We go from walking away from God to walking towards God. Just as the cubic equation goes from concave down to concave up, or concave up to concave down at the point of inflection.

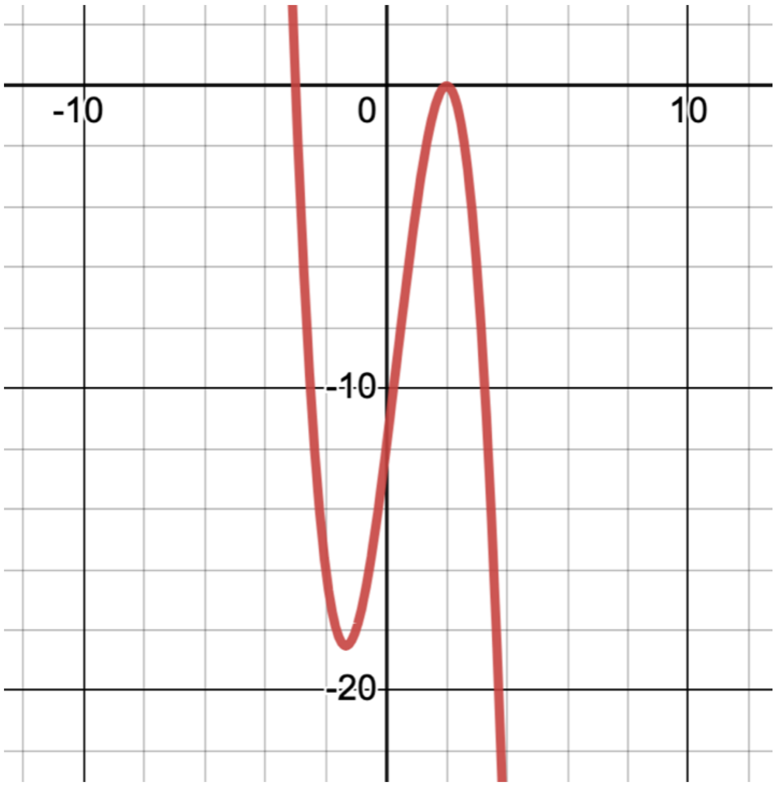
Looking Ahead 7.5

Example 1: Letting $a = 1$, find the cubic equation of the graph below. Find the linear factors and identify the y-intercepts of the equation.



The x-intercepts give the linear factors; using reverse thinking, the signs are opposite the values on the x-axis.

Example 2:	How does the graph in Example 1 change when $a = -1$ and the equation is $y = (x + 3)(x - 2)^2$.



This means the equation is starting ____________________ and ending ____________________. It goes from

__________________________ infinity to ___________________________ infinity. The y-intercept is now

_____________________. The graph is reflected in the y-axis.

Fill in the blanks:

Rules of Exponents

To multiply terms with common bases, ____________________ the exponents

$$x^2 \cdot x^5 = x^7$$

To divide terms with common bases, _____________________ the exponents

$$\frac{x^4}{x^2} = x^2$$

To take the power of a term with an exponent, ____________________ the exponents

$$(x^4)^3 = x^{12}$$

Any exponent to the zero power is ___________

$$x^0 = 1$$

To take the power of a fraction, take the power of the ______________________ and the _____________________

$$\left(\frac{x^5}{x^2}\right)^3 = \frac{x^{5\cdot 3}}{x^{2\cdot 3}} = \frac{x^{15}}{x^6}$$

When an exponent is negative, make the exponent positive and move the monomial term either from the numerator to the denominator, or from the denominator to the numerator.

$x^{-3} = \frac{1}{x^3}$ or $\frac{1}{x^{-4}} = x^4$

Section 7.6 Comparing Quadratic and Cubic Equations

Looking Back 7.6

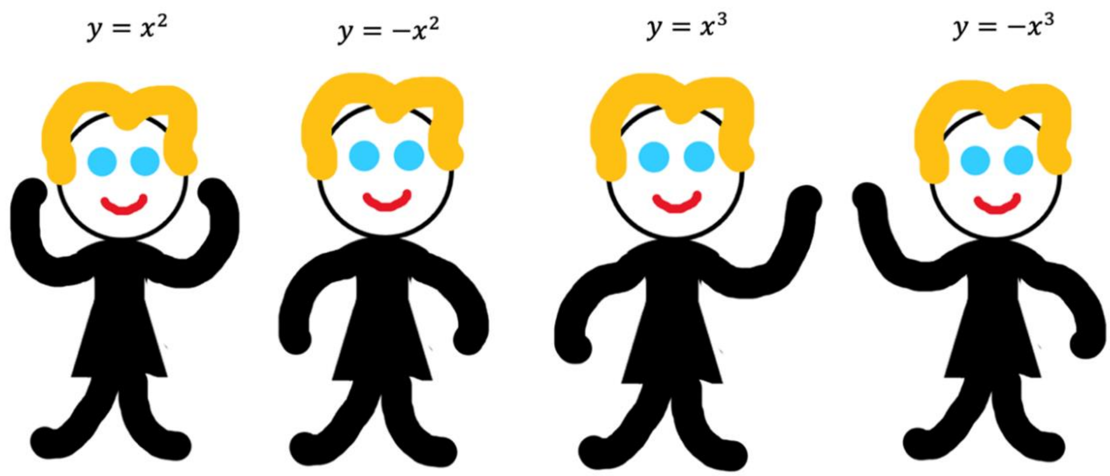
The graph of the parent function of a quadratic equation starts up and ends up. Actually, they do not really start and end because they go from infinity to infinity, but we use start and end to explain what we see on the graph. This is called *end behavior*.

Stand up and make a U with your right and left arm (shown below). If your head is the origin, you are demonstrating the parent function $y = x^2$ for a quadratic equation. It starts the same as it ends. This is also called an *even function*. The end behavior of all even functions is to start the same direction they end. We will learn more about even functions and end behavior in Section 7.11.

Now, use your arms to form the equation $y = -x^2$. What is the difference between $y = x^2$ and $y = -x^2$? Your arms should be open downward (shown below) like an upside-down U. Again, your head represents the origin. The function starts down and ends down because $a = -1$ instead of $a = +1$. The graph opens the opposite direction as it flips (reflects) in the x-axis.

Now, put your left arm up and your right arm down like a wave. This is the graph of $y = x^3$. It starts and ends in opposite directions. Again, your head is the origin. Now, move your arms to make the graph of $y = -x^3$. This is a reflection of $y = x^3$ in the x-axis. Your right arm should be up and your left arm should be down. This is called an *odd function*; we will learn more about odd functions in Section 7.11.

Even degree functions start and end in the same direction. Odd degree functions start and end in the opposite direction. Even degree functions start up when a is positive and down when a is negative. Odd degree functions start down when a is positive and up when a is negative.



It is up to U to figure it out

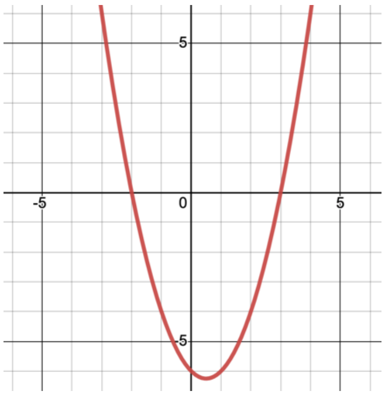
I see U did figure it out

I am starting to feel wavy

The "Cubic Wave" Dance

Looking Ahead 7.6

Example 1: A quadratic equation can have 0 zeroes, 1 zero, or 2 zeroes. Identify which quadratic equation below has 0 zeroes, 1 zero, and 2 zeroes. Write the equation for each graph in factored form.

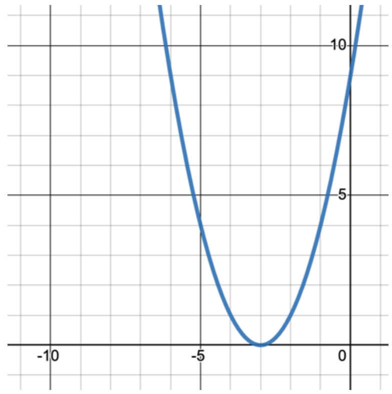


___________ solution(s)

The zeroes are:

____

$y =$

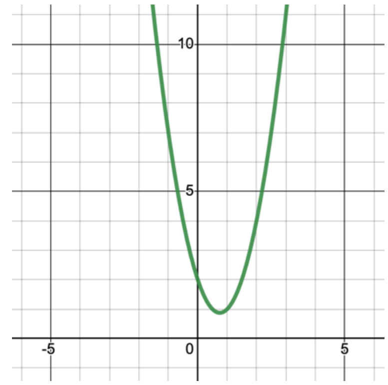


___________ solution(s)

The zeroes are:

____

$y =$



___________ solution(s)

The zeroes are:

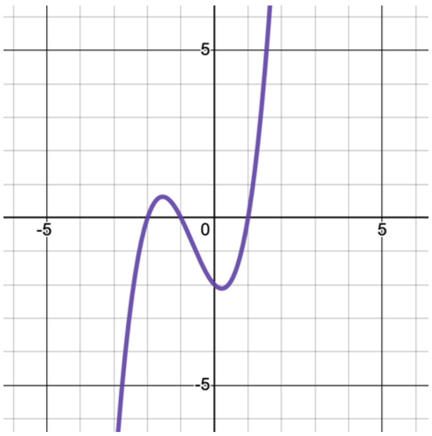
____

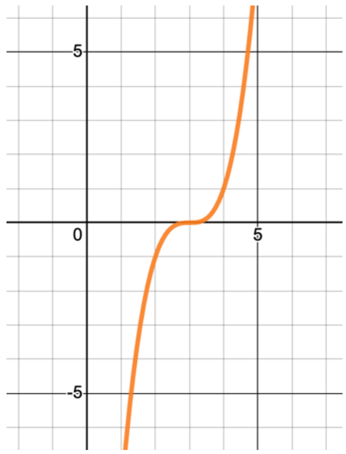
$y =$

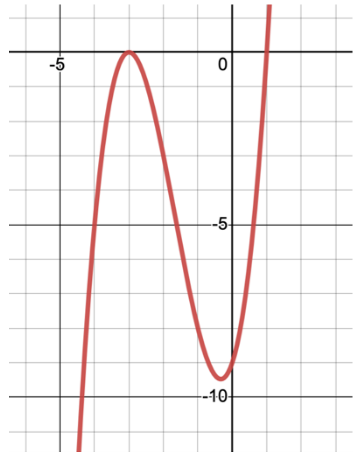
Example 2: Convert the first two equations in Example 1 from the factored form of a quadratic equation to the standard form of a quadratic equation.

Example 3: A cubic equation can have 3 zeroes, 2 zeroes, 1 zero, or 0 zeroes. We cannot graph a cubic equation that has no zeroes with real numbers. Those zeroes are real and imaginary numbers, which are called *complex numbers*. We will learn about complex numbers in *Algebra 2*.

Identify which quadratic equation below has 3 zeroes, 2 zeroes, or 1 zero. Write the equation for each graph in factored form. Let $a = 1$.







__________ solution(s)

The zeroes are:

____

$y =$

__________ solution(s)

The zeroes are:

____

$y =$

__________ solution(s)

The zeroes are:

____

$y =$

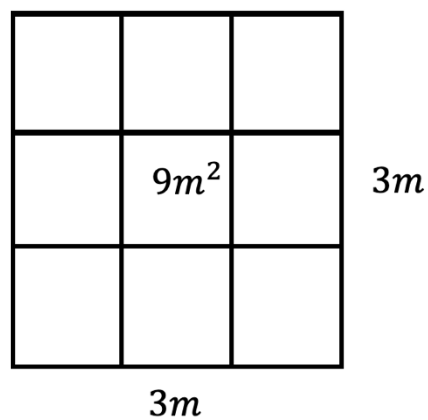
Example 4: Convert each equation in Example 3 from the factored form of the quadratic equation to the standard form of the quadratic equation.

In summary, a quadratic equation can have 0, 1, or 2 real zeroes, but no more than 2 real zeroes. A cubic equation can have 0, 1, 2, or 3 real zeroes, but no more than 3 zeroes. Power functions can have the same number of zeroes as their highest power.

The bounces or double zeroes are called a multiplicity.

The Fundamental Theorem of Algebra states that a polynomial equation with real coefficients and positive degree n has exactly n zeroes.

Quadratic equations are used to model area problems. The sides of a square are equal, so $A = s^2$ in which A is the area and s is the length of the sides. The length of the side is the factor, and the area is the quadratic.



Cubic equations are used to model volume problems. The sides of the cube are equal. So, $V = s^3$ in which V is the volume and s is the length of the side. Again, the length of the side is the factor, and the volume is the cubic.

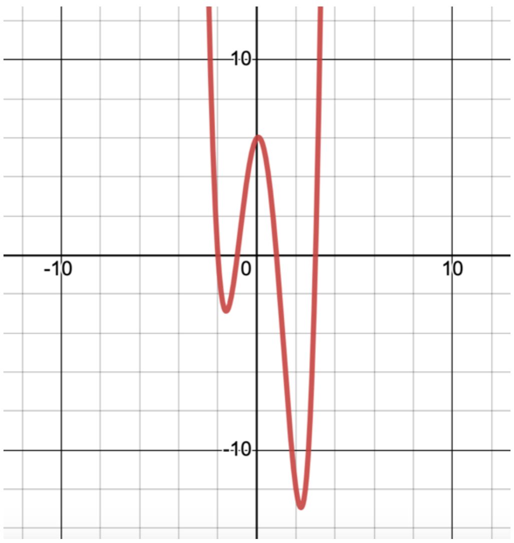
Problems such as these will be explored at the end of this module.

Section 7.7 Patterns in Power Functions

Looking Back 7.7

We know that even powered functions start and end the same way. We know that there can be no more zeroes than the highest degree power of the function. We also know that when $a > 0$ an even powered function starts up and ends up, but when $a < 0$ the graph starts down and ends down. We also know how to find the x-intercepts, which are also called zeroes, roots, or solutions of the function. Moreover, we know how to find the y-intercepts. We can use all this information to find an equation given the graph.

Example 1: Fill in the blanks given the graph below and find the standard form equation using the Distributive Property.



The highest degree is __________. There are __________ real zeroes. Therefore, there are four intercepts:

$x =$ _______ $x =$ _______ $x =$ _______

$x =$ _______

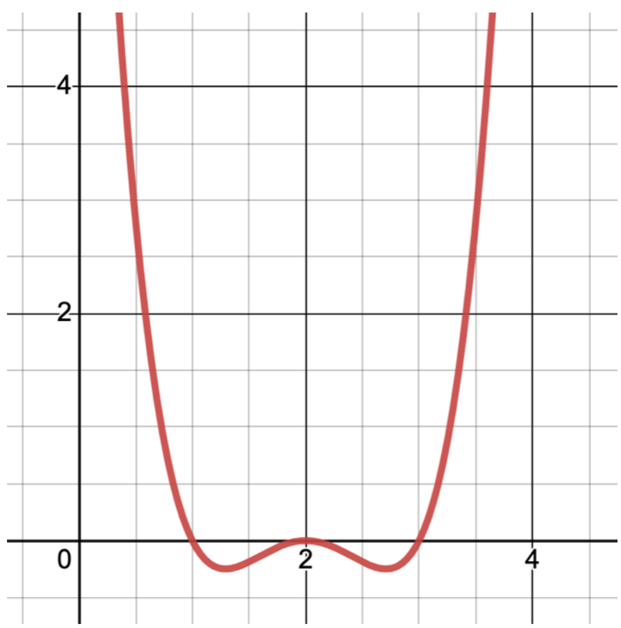
Because a is positive and because the power is of even degree, the graph starts __________ and ends __________.

Example 2: Fill in the blanks given the graph below and use long multiplication to find the standard form equation for the graph.

There are __________ real zeroes. One is a __________ zero, which means there is a bounce at that point on the x-axis rather than a crossing through of the x-axis at that point. The zeroes are below:

$x =$ _______ $x =$ _______ $x =$ _______

The graph is still even because there is a linear factor whose degree is ___________, a quadratic factor whose degrees is __________, and another linear factor whose degree is also ___________. This is

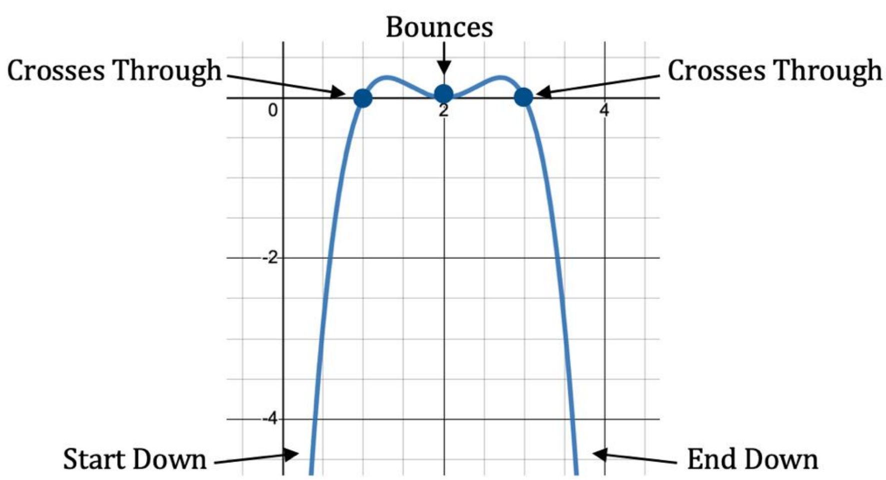


$1 + 2 + 1 = 4$, which is the highest degree of the standard form of the equation. Let $a = 1$.

Example 3:	Find the standard form equation for the graph in Example 2.

Example 4:	Fill in the blanks given the graph below and find the standard form equation using a geometric model (array).

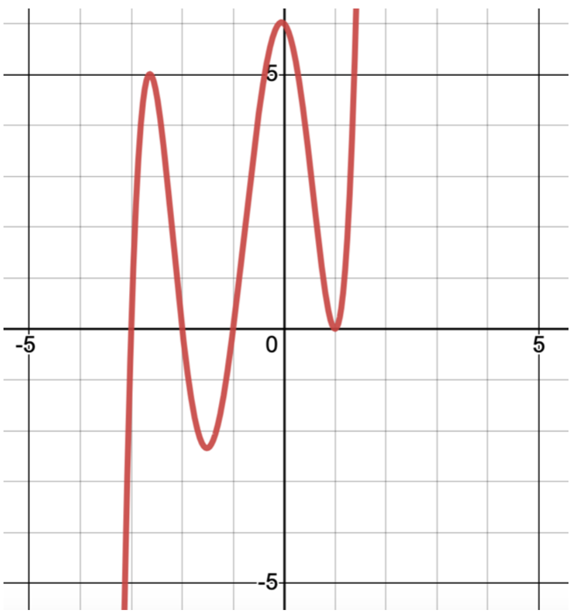
If the equation were $y = -(x-1)(x-2)^2(x-3)$ with a being negative, the graph would start ________________ and end ________________ with the same x-intercepts and a bounce at the point _____________________.



Looking Ahead 7.7

Later in the module, we will explore finding the value of a when it is a number other than 1. When there is an inflection point, the graph changes direction from concave up to concave down or vice versa, but with a bounce, the graph keeps moving the same direction, concave up to concave up, or concave down to concave down. As you can see, there are many identifiable patterns in quadratic and cubic equations. Just as Galileo said: "Mathematics is the language with which God has written the universe." There are many patterns in mathematics, which is God's universal language.

Example 5: If an equation in factored form is $y = (x-1)^2(x+1)(x+2)(x+3)$, what is the highest degree of the equation? What are the x-intercepts? Sketch a graph of the equation. Fill in the blanks before answering the questions.



The degree is __________. Because a_______0, the graph starts _________________ and ends _________________.

The x-intercepts are $x =$ ________, $x =$ ________, $x =$ ________, $x =$ ________. There is a double zero and a bounce at $x =$ ________.

In the last part of this module, we will review working with radicals.

Example 6: The n^{th} zero of x is $\sqrt[n]{x}$. Find $\sqrt[3]{125}$.

Example 7: Simplify $\sqrt{50}$.

Example 8: Simplify $\sqrt{12x^2}$.

Example 9: Simplify $\sqrt[5]{\frac{32}{x^5}}$

Section 7.8 Multiple Representations of Cubic Functions

Looking Back 7.8

We have learned there are several names for different forms of equations or functions. The parent function is centered at the origin, $(0, 0)$. The parent functions for linear, quadratic, and cubic equations are shown below in order from left to right, all parameters being equal to 0 or 1 (the identity elements for addition and multiplication):

$$y = x \qquad y = x^2 \qquad y = x^3$$

These equations are all functions because they all pass the Vertical Line Test. Each value of x maps to only one value of y. Each input has only one output.

These are called power functions because they are to the first, second, third power, etc. In fact, all polynomial equations are power functions by the definition of polynomials.

We have also seen the standard form of linear, quadratic, and cubic equations, which are as follows:

$$y = ax + c$$
$$y = ax^2 + bx + c$$
$$y = ax^3 + bx^2 + cx + d$$

These are called standard (general) form. When there is only one variable in each term, the variables are in alphabetical order with the degree of the monomial term decreasing from left to right. When there are two or more variables in each term, the variable that comes first alphabetically is in descending order. The others may be in ascending order, as in:

$$y = ax^3y + bx^2y^2 + cxy^3 + d$$

This is actually a quartic equation because the highest degree of each monomial term is 4. Generally, these equations are simply called polynomials after the third degree.

These equations can also be in vertex (graphing/factored) form, which we have seen previously, but will investigate further in the next few sections.

Looking Ahead 7.8

Example 1: For linear equations, the slope is $m = \frac{y_2 - y_1}{x_2 - x_1}$. Convert this equation to the point-slope form of the equation.

Example 2: Compare the point-slope form $m(x - x_1) + y_1 = y$ of a linear equation to the vertex form of a power function $y = a(x - h)^1 + k$.

Combining the h and k in the vertex form of a linear equation becomes b in the slope-intercept form of a linear equation, $y = mx + b$.

Example 3: Expand $m(x - h) + k = y$ and solve for its slope.

The point-slope form of a linear equation is $m(x - x_1) + y_1 = y$ in which m is the slope of an equation and (x_1, y_1) is a specific point on the line. It is very similar to $y = a(x - h)^1 + k$, which is the graphing form of a linear equation.

Example 4: Use the real numbers $(2, 3)$ and $(5, 8)$ in the point-slope form, and convert the equation to the slope-intercept form of the linear equation. Find the slope first.

Example 5: Use the slope you found in Example 4 and substitute one of the points $(2, 3)$ or $(5, 8)$ into the slope-intercept form. Solve for b and write the slope-intercept equation. Is it the same solution you got in Example 4?

The vertex (graphing) form can be used to graph a cubic equation, or as we have seen, the factored form can give us the linear factors, which can be graphed through plotting the x-intercepts using the Zero-Product Property.

Example 6: Given the equation $y = x^3 - 2x^2 + x - 2$ and that 2 is an x-intercept, find another factor using an area model. We also call this a geometric model or array. If $x = 2$ is an $x -$ intercept , then $(x - 2)$ is a factor.

x			
-2			

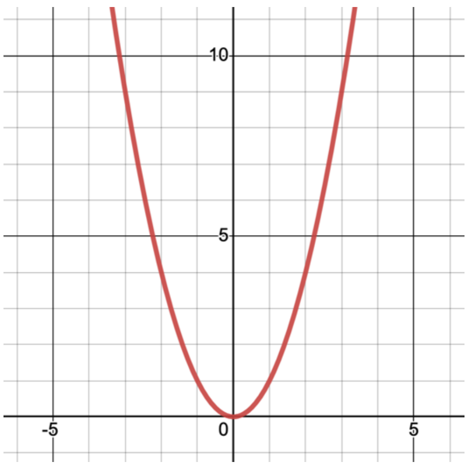
You will not be using area models to solve cubic equations. This section was simply to show you that the same methodologies can be used for the power functions, whether it is a quadratic or cubic function. Though there are differences between these two functions, there are also many similarities. Just as all quadratic equations are functions, all cubic equations are also functions.

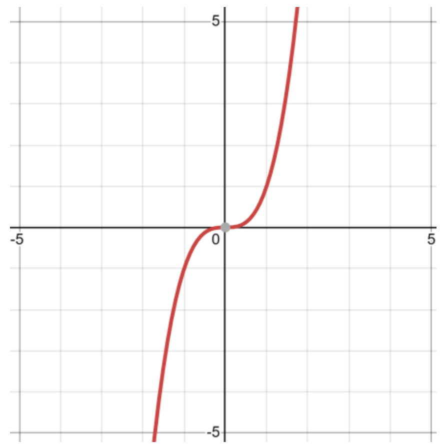
Section 7.9 Horizontal and Vertical Transformations of Cubic Equations

Looking Back 7.9

The vertex form of a cubic equation is $y = a(x-h)^3 + k$. The a represents a stretch or compression of the equation. The h represents the horizontal shift of the equation. The k represents the vertical shift of the equation. These represent transformations from the parent function, which is centered at the origin. There are no horizontal or vertical shifts from $(0, 0)$ in the graph of $y = x^3$. As we learned in Practice Problems 7.8, (h, k) represents the vertex of a quadratic equation, which is a minimum point if the graph opens upward (when a is positive) or a maximum point if the graph opens downward (when a is negative). The end behavior for a quadratic equation is positive infinity when $a > 0$ and negative infinity when $a < 0$.

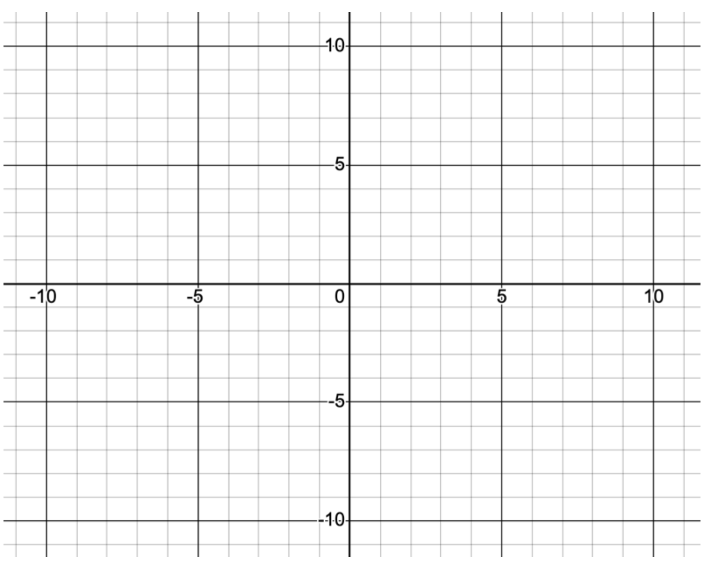
In the vertex form of a cubic equation, (h, k) represents the point of inflection. This is where the graph changes from concave up to concave down or concave down to concave up. There is no maximum or minimum point in a cubic equation, only a change in direction of the graph. If $a > 0$, the end behavior goes from negative infinity to positive infinity. If $a < 0$, the end behavior goes from positive infinity to negative infinity.



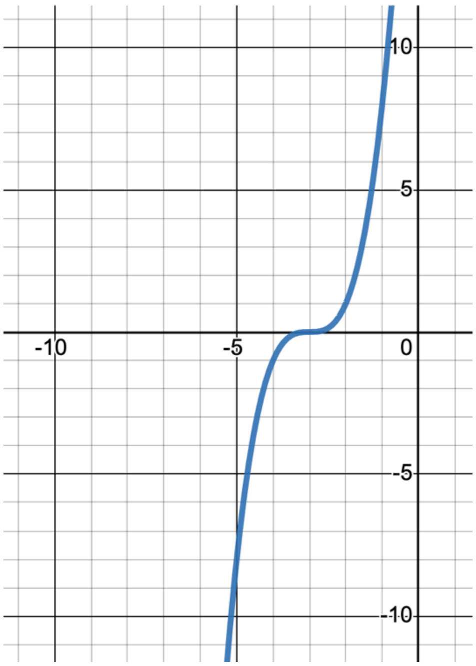


Looking Ahead 7.9

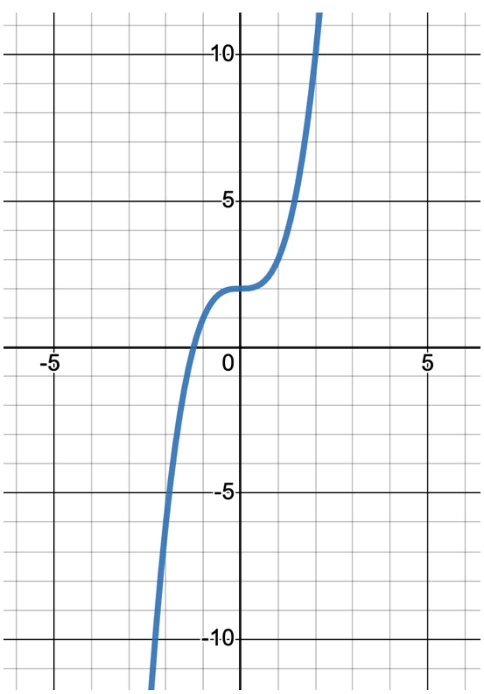
Example 1:	Sketch the graph of $y = (x-3)^3$.



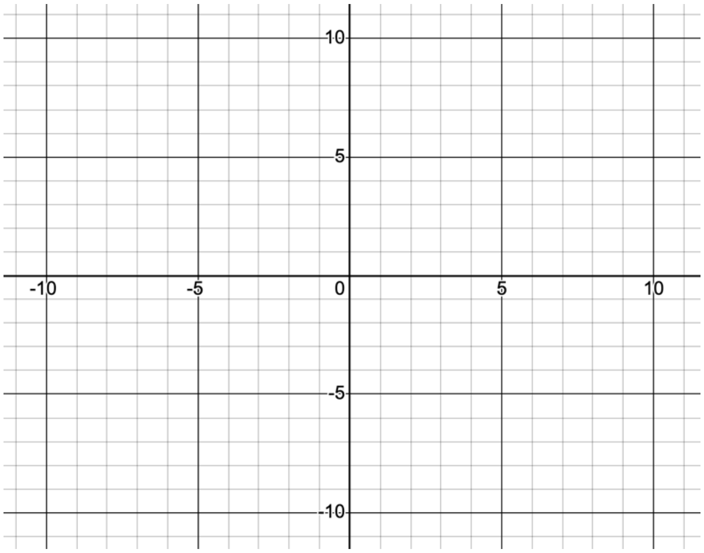
Example 2:	What is the equation for the graph below if it is a cubic equation and a is positive 1?



Example 3:	What is the equation for the graph below if it is a cubic equation and a is positive 1?



Example 4: Sketch the graph of the equation $y = x^3 - 2$. Let $a = 1$.



Example 5: Use the multiplication rule to simplify the equation below. Let $a = 3$ and $b = 6$.

$$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$$

Example 6: Simplify the expression below.

$$-4\sqrt[3]{9a} \cdot 2\sqrt[3]{3a^6}$$

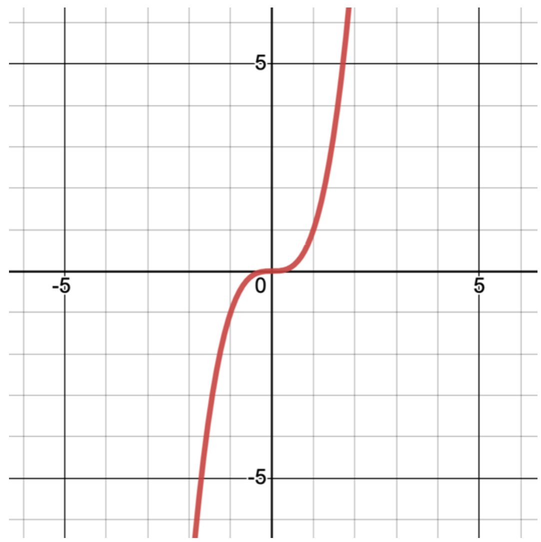
Section 7.10 Stretches and Compressions of Cubic Graphs

Looking Back 7.10

We have already learned that for the parent function $y = x^3$, the graph starts at negative infinity $(-\infty)$ for y when x goes to negative infinity $(-\infty)$ and goes to positive infinity $(+\infty)$ for y as x goes to positive infinity $(+\infty)$. This is the range of the function. The graph of $y = -x^3$ starts at positive infinity $(+\infty)$ for y when x goes to negative infinity $(-\infty)$ and goes to negative infinity $(-\infty)$ for y as x goes to positive infinity $(+\infty)$. This is also the range of the function.

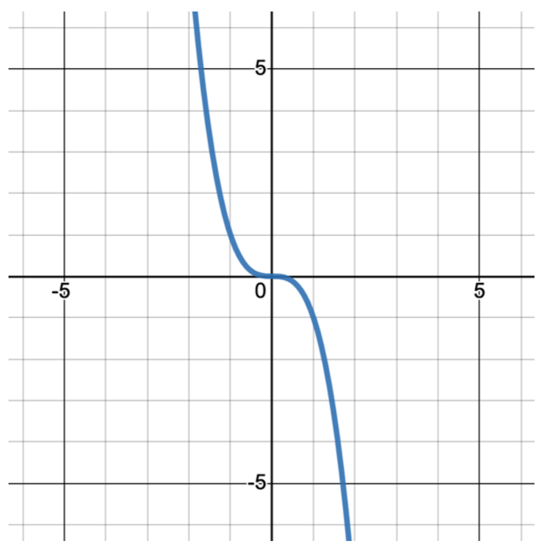
The table and graph for $y = x^3$ are shown below.

x	y
-3	-27
-2	-8
-1	-1
0	0
1	1
2	8
3	27



The table and graph for $y = -x^3$ are shown below.

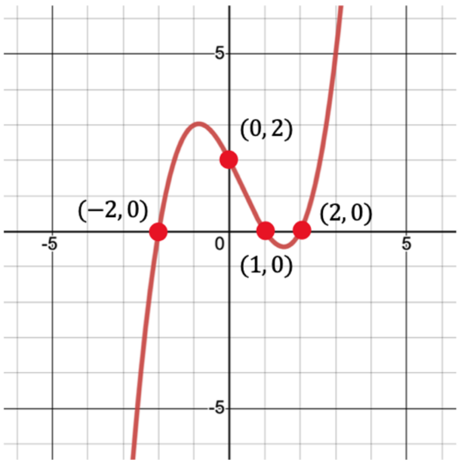
x	y
-3	27
-2	8
-1	1
0	0
1	-1
2	-8
3	-27



Do you notice what happens to the signs of the values of the output as x increases?

Looking Ahead 7.10

Example 1:	Write the standard form of the equation graphed below.



The x-intercepts are

____

The y-intercept (another point on the graph) is

____

Write the equation of the graph in factored form:

$y = a(\_\_\_\_\_\_\_\_\_\_)(\_\_\_\_\_\_\_\_\_\_)(\_\_\_\_\_\_\_\_\_\_)$

Substitute the other point (the $y -$ intercept) in for x and y in the factored form of the equation and solve for a.

Example 2:	Using the factored form of the equation and the value of a from Example 1, expand the equation into standard form.

Example 3:	Simplify the expression below using the Division Property. Rationalize the denominator.

$$\frac{2\sqrt{5x}}{\sqrt{8}}$$

Example 4:	Simplify the expression below using the Division Property.

$$\frac{x\sqrt[4]{32}}{\sqrt[4]{2x^8}}$$

Section 7.11 Even and Odd Functions

Looking Back 7.11

Some functions are even and some functions are odd. It would appear that $y = x^2$ is an even function because it is to the second degree. Actually, an even function has a graph that is symmetric with respect to the y-axis. The parent quadratic function is symmetric with respect to the y-axis; therefore, it is an even function. Algebraically, this means $f(-x) = f(x)$.

To determine if the parent quadratic function is symmetric with respect to the y-axis algebraically, $f(-x) = f(x)$, means to plug $-x$ (the opposite of x) in for the input and simplify. Do we get the same output as when we substitute x for the input? If $f(-x) = f(x)$, then we end up with the same output, whether or not the input is x or $-x$ (the opposite of x). Remember, $f(x)$ ("f of x") is the function notation for the output (y). This means $f(x) = y$. So, $f(-x)$ ("f of the opposite of x") means $f(-x) = y$.

In the function $f(x) = x^2$, plug in $-x$ for x:

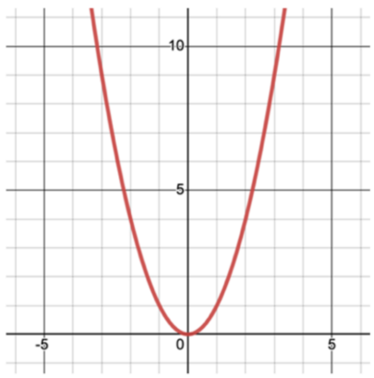
$$f(-x) = (-x)^2$$

If $x = 3$, then $(-3)^2 = 9$

If $x = -3$, then $[-(-3)]^2 = 3^2 = 9$

Both give the same solution, 9

x	$f(x)$
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9



x	$f(-x)$
-3	
-2	
-1	
0	
1	
2	
3	

In the function $f(x) = -x^2$, plug in $-x$ for x:

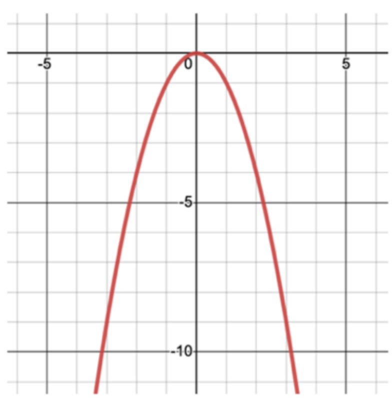
$$f(-x) = -(-x)^2$$

If $x = 3$, then $-(-3)^2 = -(9) = -9$

If $x = -3$, then $-[-(-3)]^2 = -[3]^2 = -9$

Both give the same solution, -9

x	$f(x)$
-3	-9
-2	-4
-1	-1
0	0
1	-1
2	-4
3	-9

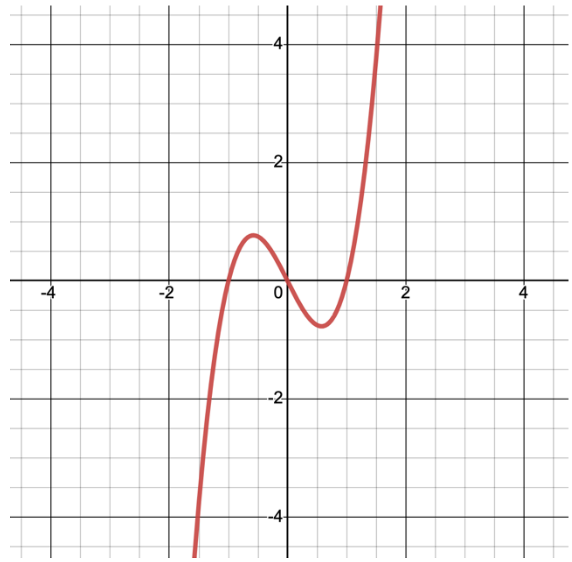


x	$f(-x)$
-3	
-2	
-1	
0	
1	
2	
3	

Both of the functions above are symmetric in the y-axis and both of the functions are even functions.

Looking Ahead 7.11

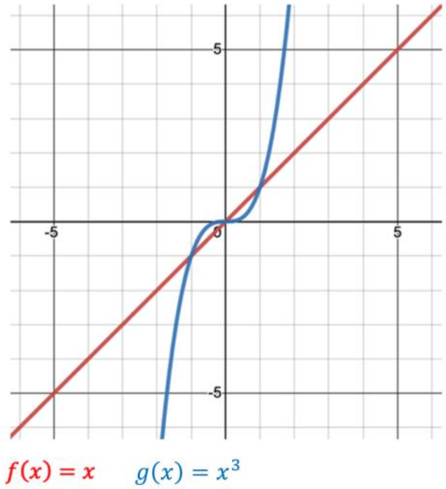
To determine if the equation $f(x) = 2x^3 - 2x$ is even or odd, look at the graph below to the right. The exponents of x^3 and x^1 are both odd, but does that mean the function is odd? Odd functions are symmetric with respect to the origin. The function $f(x) = 2x^3 - 2x$ is symmetric with respect to origin; therefore, it is an odd function.



If a function is odd, then $f(-x) = -f(x)$. To determine if a function is odd, substitute $-x$ for x in the function and simplify. If the simplified form of the function is the same as the original function, it is even. If the simplified form of the function is the opposite of the original function, it is odd. This means the sign of each term has been changed to its opposite. If you do not get an $f(x)$ or $-f(x)$ after simplifying the function, then it is neither an even nor odd function.

Example 1:	Substitute $-x$ in for x in the equation $f(x) = 2x^3 - 2x$. Is this an even or odd function?

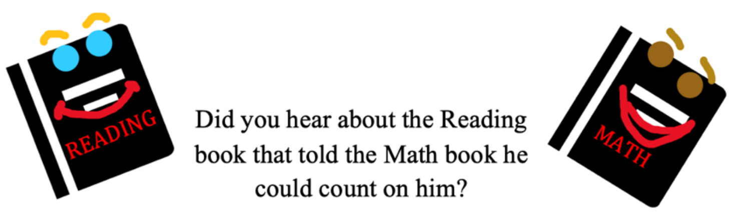
Example 2: Look at the graph of $f(x) = x$ and $g(x) = x^3$. Demonstrate that they are odd from the graph and the function.



Example 3: Add the radicals below. (The roots must be the same.)

$$6\sqrt[4]{x} + 2\sqrt[4]{x}$$

Example 4: Use the Distributive Property to simplify $\sqrt{2}(10 + \sqrt{2}) - 3\sqrt{2}$.



Section 7.12 Modeling a Cubic Relationship with "The Box Problem"

Looking Back 7.12

We can use cubic equations to model and solve real-world problems. Let us suppose that we ship unit cubes to mathematics teachers and want to use the box with the largest volume so we can send the most cubes at one time. The constraints are that the original cardboard for the net of the box is 22 cm by 18 cm and the finished dimensions must be integers (no decimals or fractions). What are the dimensions of the box that will hold the most cubes given the dimensions of the original cardboard? Do not worry about the lid. Lids will be made in another area of your manufacturing space.

We will be using centimeter grid paper to build these boxes. Record the height, length, width, and volume of the box in a table. Here is how to get started.

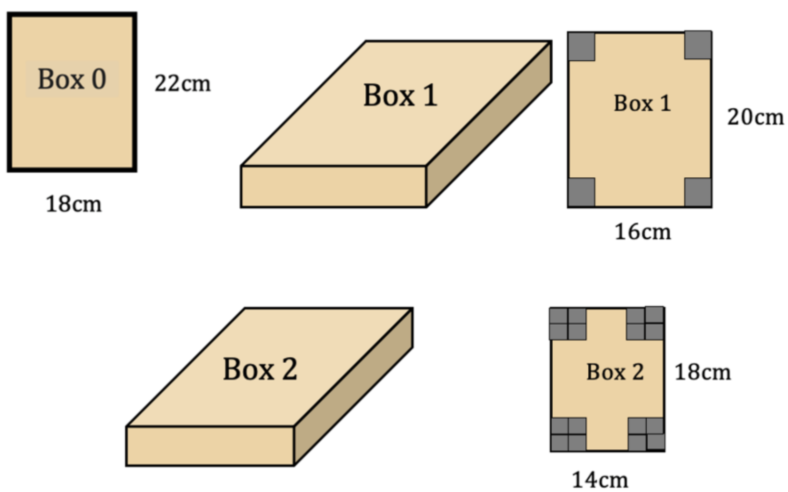
Cut out one square from each corner of the 22 cm by 18 cm grid paper. Fold up the corners and tape the box together.

What are the new dimensions of the box?

What is the volume of the box?

We will call a box with a height of 1cm Box 1. The values for Box 1 are found in the table. Notice when the paper is flat the height of the box is 0 cm and the length and width are 22 cm × 18 cm. The volume of the box is 0 cm because $V = l \cdot w \cdot h$ and $V = 22 \cdot 18 \cdot 0 = 0$cm. We call this Box 0 (0cm height). These values are written on the table for you.

h	l	w	V
0	22	18	0
1	20	16	320
2	18	14	504
3	16	12	
4	14	10	
5	12	28	
6	10	6	
7	8	4	
8	6	2	



Notice, for the box to be 2cm high, you must cut out a 2×2 square from each corner of the box. That is 2cm along the width and 2cm along the length. You are subtracting 4cm from the original width of the box.

Box 2 dimensions:

$l - 4 = 22 - 4 = 18$cm

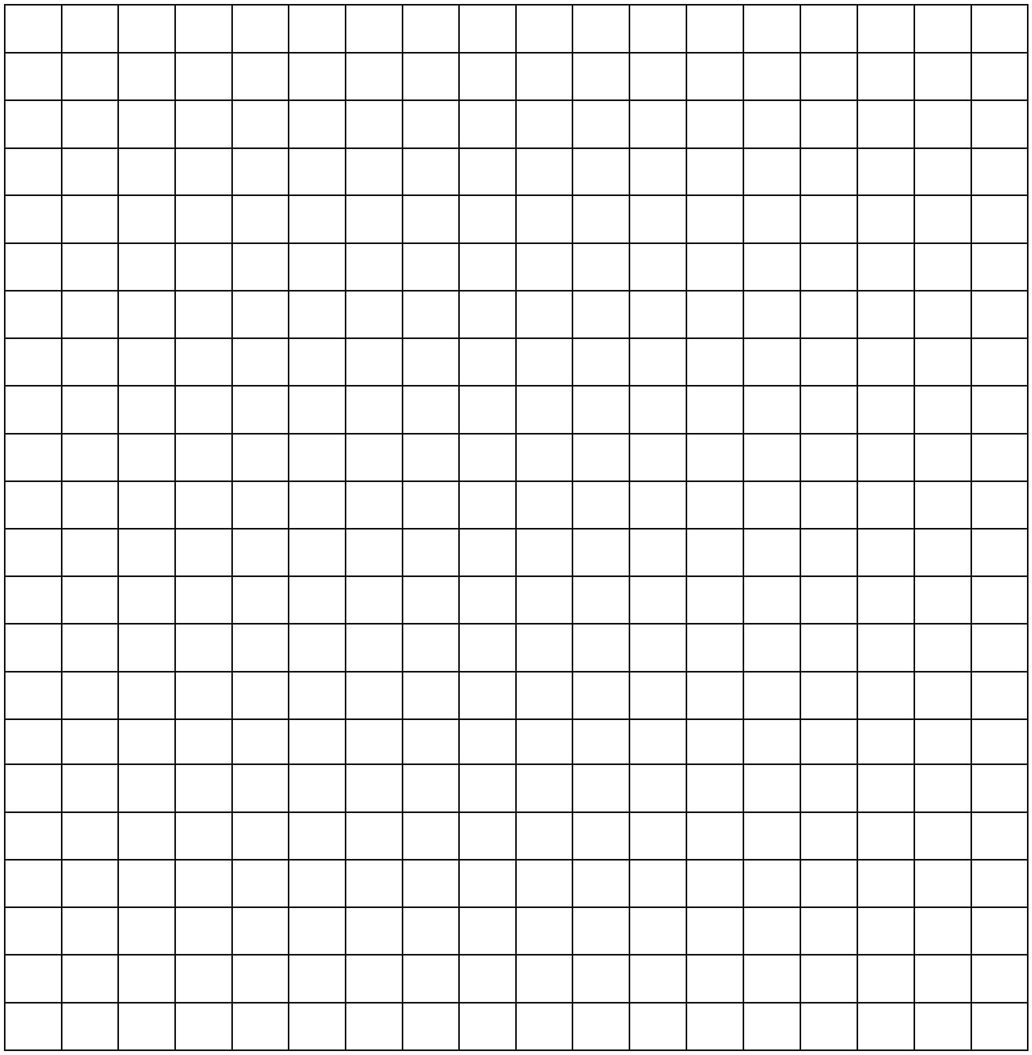
$w - 4 = 18 - 4 = 14$cm

Keep building the boxes 1 unit higher each time by cutting out perfect squares from each corner:

Box 1: $1 \times 1 = 1$cm squared from each corner

Box 2: $2 \times 2 = 4$cm squared from each corner

Box 3: $3 \times 3 = 9$cm squared from each corner



You can probably see a pattern as you fill in the new height, length, and width. Complete the previous table and then compare it to the finished table in the Practice Problems. Next, answer the questions about the box problem. You will be able to make a completed graph from the table and find the equation that models the situation. When you do, you will see that sometimes the data appears to model one type of equation, but when more data is investigated, you will see the data actually models a different type of equation.

This is a problem in the theory of evolution: Not all the data is analyzed; not all the data is available; assumptions are made. Even the age of the earth is quite varied among evolutionists.

Young earth creationists can be more specific from the data in the Bible. There are unbroken genealogies from Adam (the first man) to Jesus Christ (God among man). Sometimes the genealogies in the Bible may seem cumbersome along with ages and specific locations; however, they actually give archaeologists and genealogists much needed information about our history.

No one alive now has ever witnessed creation or evolution and it cannot be recreated. Still, creation can be believed based on the evidence in the Bible.

Looking Ahead 7.12

Exponents can be written as radicals and radicals can be written as exponents:

$$\sqrt[3]{x^3} = (\sqrt[3]{x})^3 = (\sqrt[3]{x})(\sqrt[3]{x})(\sqrt[3]{x}) = \sqrt[3]{x^3} = x$$

With fractional exponents:

$$x^{\frac{3}{3}} = x^1 = x.$$

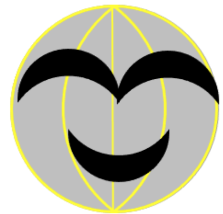
A power to a power means we multiply the exponents so $(x^3)^n$ or $(x^n)^3$ can be written: x^{3n}.
If $x^{3n} = x^1$ then we want to know what n is so that $3n = 1$. Solve for n.

That works; $x^{3(\frac{1}{3})} = x^1 = x$. The reciprocal of 3 or $(\frac{3}{1})$ is $\frac{1}{3}$. Radical signs can be represented as fractional exponents.

Example 1: Rewrite $\sqrt{y} \cdot \sqrt[3]{y}$ with fractional exponents. Then use the rules of exponents to simplify the expression. Rewrite the fractional exponent as a radical.

Example 2:	Prove $\sqrt[3]{x^3} \cdot \sqrt[3]{x^3} = x^2$ using fractional exponents.

Example 3:	Rewrite $\frac{\sqrt{m^5}}{\sqrt{m^3}}$ with fractional exponents. Simplify the expression using the rules of exponents.



Did you hear about the dollar who liked the moon more when he found out she had four quarters?

Example 4: How many solutions does the quadratic equation $x^2 - 2x + 1 = 0$ have? If it has real number solutions, solve it using the quadratic equation.

$a =$ ____________ $b =$ ____________ $c =$ ____________

The discriminant can save time if there is no solution. It also divides the work of the quadratic formula into two parts, which makes it easier to use.

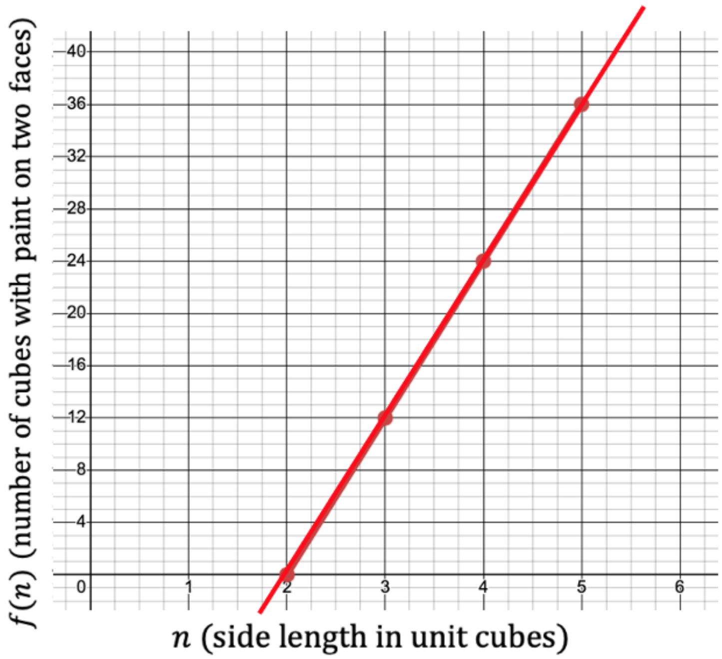
Section 7.13 Revisiting "The Dipped Cube"

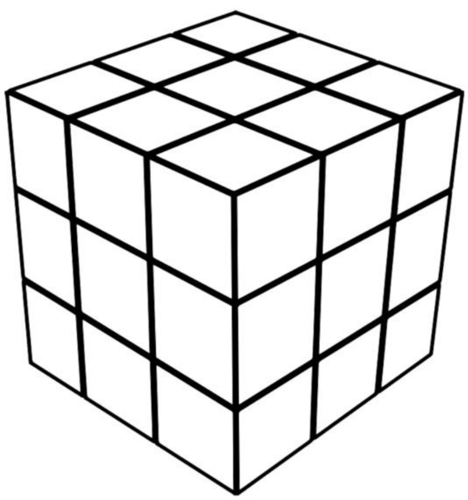
Looking Back 7.13

When working with linear numbers, we solved a problem involving a cube dipped in paint. You were asked to count the number of cubes that had paint on two faces for a $2 \times 2 \times 2$ unit cube, a $3 \times 3 \times 3$ unit cube, a $4 \times 4 \times 4$ unit cube, and a $5 \times 5 \times 5$ unit cube when dipped in paint. The table and graph of the solution are shown below.

$\boldsymbol{n}$ (Side length in unit cubes)	$\boldsymbol{f(n)}$ (Number of cubes with paint on two faces)
2	0
3	12
4	24
5	36

The equation is $f(n) = 12n - 24$.





Color in the cubes that are painted on only two faces.

How many cubes are painted on only two faces?

There are ________________ edges.

The side length is ________________.

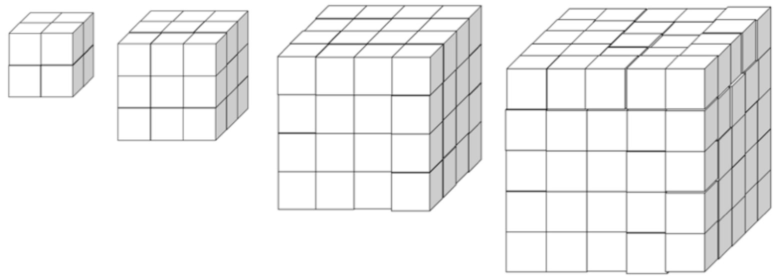
Check this in the equation $f(n) = 12n - 24$.

If we had a $10 \times 10 \times 10$ cube, we could substitute 10 in the formula and find that $f(10) = 12(10) - 24 = 120 - 24 = 96$ cubes.

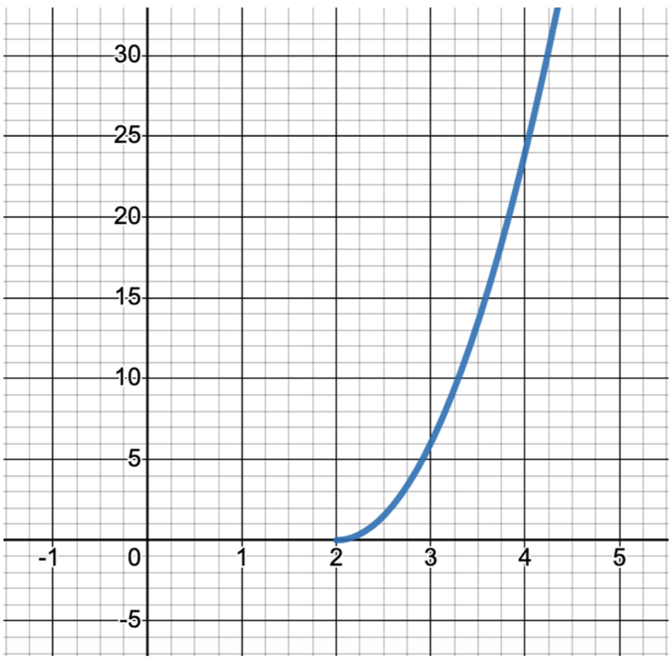
Looking Ahead 7.13

Example 1: Fill in a table, and make a graph and equation of a cube that has paint on only one face when dipped in paint. Use the cubes below the chart to help complete the chart.

$\boldsymbol{n}$ (Side length in unit cubes)	$\boldsymbol{f(n)}$ (Number of cubes with paint on one face)
2	
3	
4	
5	



This is modeled by a quadratic equation: $f(n) = 6(n-2)^2$

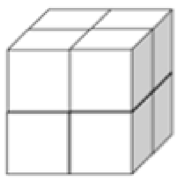


Example 2: Expand the equation $f(n) = 6(n-2)^2$ to the standard form of a quadratic equation.

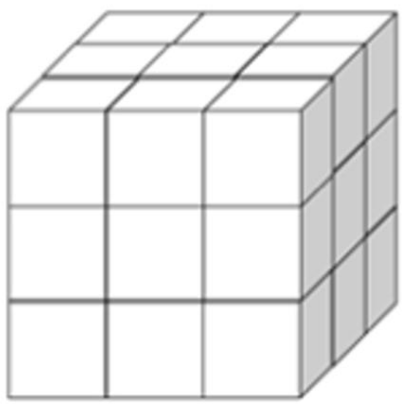
Example 3: Use the standard form to find the number of cubes painted on one side in a $10 \times 10 \times 10$ cube.

Example 4: Use the equation $f(n) = 6(n-2)^2$ to find $f(10)$. Did you get the same solution as in Example 3? Which was easier to use, the standard form or the vertex form?

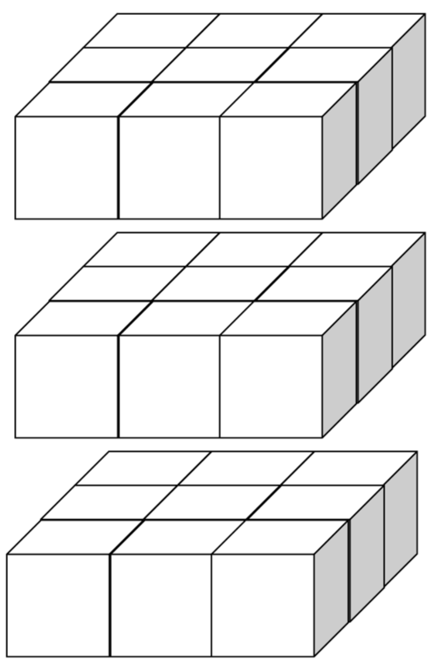
We will now investigate the number of unit cubes that are painted on zero faces for a $10 \times 10 \times 10$ cube and an $n \times n \times n$ cube.



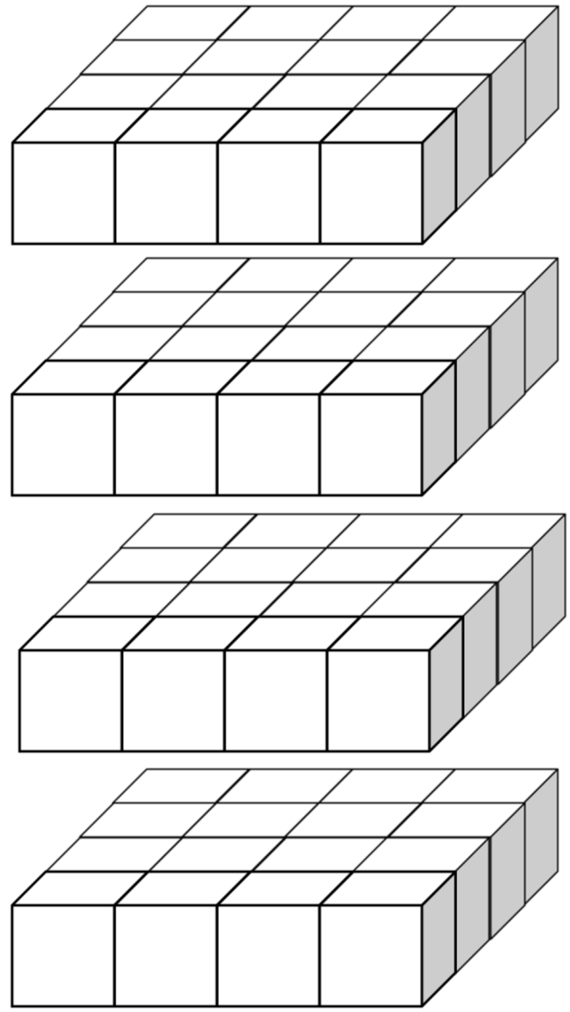
In the 2×2 cube to the left, there are zero unit cubes that have paint on zero faces when dipped in paint because every cube is a corner cube and has paint on three faces only.



In a $3 \times 3 \times 3$ cube, how many unit cubes have paint on zero faces when dipped in paint?
Let us take it apart and see.



Taking the cubes apart shows that a $3 \times 3 \times 3$ cube has one cube in the middle that has zero sides that are painted. It is a $1 \times 1 \times 1$ cube.

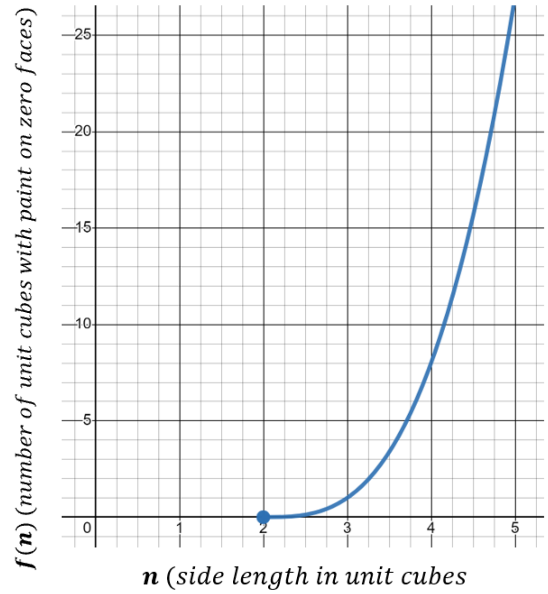


Taking the cube apart shows that $4 \times 4 \times 4$ cube has

________________cubes in the upper middle layer of a $4 \times 4 \times 4$ cube that is painted on zero faces when dipped in paint. There are ________________ cubes in the lower middle layer of a $4 \times 4 \times 4$ cube that is painted on zero faces when dipped in paint.

Example 5: Complete the table and graph for the number of cubes that are painted on zero faces when cubes of different side lengths are dipped in paint.

$\boldsymbol{n}$ (Side length in unit cubes)	$\boldsymbol{f(n)}$ (Number of cubes with paint on one face)
2	
3	
4	
5	



The graph appears to be quadratic. However, in investigating the table, the total numbers are perfect cubes so the equation is cubic. We do not have Quadrant III simply because there are no negative side lengths.
The outside layers have at least one face exposed to paint. The inside layers can be represented by $n - 2$, subtracting one of each external side exposed to paint. They form a cube that is represented by $(n-2)^3$, making the equation $f(n) = (n-2)^3$ in factored form.

Example 6: Find the expanded form of the cubic equation representing the table and graph in Example 5.

Example 7: In a $10 \times 10 \times 10$ cube, how many unit cubes would be painted on zero faces when dipped in paint?

Let us finish our review of exponents by solving equations involving exponents and radicals because one can be rewritten as the other. So, when you are solving for x to the second power, take the square root (of both sides of the equation) to get x to the first power.

Example 8: Solve $x^2 = 81$. There are two solutions. Be sure to check your solutions.

Example 9: Solve $x^3 = -64$. There is only one solution. Be sure to check your solution.

Example 10: Solve $\sqrt{x} + 5 = 15$. Find the square of both sides in the end, which is the inverse of a square root. Be sure to check your solution.

Example 11: Solve $(2x - 1)^3 = 27$. Take the cube root in the beginning to undo the cubic. Be sure to check your solution.

Did you hear about the Jack-o-Lantern that divided his circumference by his diameter and became pumpkin pie?