Module 8 Exponential Functions

Section 8.1 Exploring Exponential Equations

Looking Back 8.1

We have learned about exponents and solving equations involving exponents. Let us review the terminology used to discuss exponents

Exponent
Base → 2^4

The base is the number being multiplied by itself repeatedly. The exponent is the number of times the base is being multiplied by itself repeatedly:

$$2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$$

If x is the base and the exponent is a constant, the equation represents a power function.
Name the following power functions:

$y = x$ is the parent ___________________________ function; these are to the first power

$y = x^2$ is the parent ___________________________ function; these are to the second power

$y = x^3$ is the parent ___________________________ function; these are to the third power

$y = x^2$ $y = x \cdot x$	$y = x^3$ $y = x \cdot x \cdot x$
x is multiplied by itself two times	x is multiplied by itself three times
If x is 1, $x^2 = 1 \cdot 1 = 1$	If x is -1, $x^3 = -1 \cdot -1 \cdot -1 = -1$

The base (x) changes as the input values change, causing the output (y) to change as well. The number of times the base is multiplied does not change. It depends on the power/exponent.

Looking Ahead 8.1

In an exponential function, the base is positive and the x is in the exponent. That is why they are called exponential functions. When x is the exponent, it is called an exponential function. The base is multiplied by itself x times. As the x grows, the power grows too. This is called exponential growth. It starts slow and then grows faster. Why do you think the base cannot be 0 or 1? Any number to the 0 power is 1, except 0 because 0^0 is indeterminant. We will explore a base of 1 in the Example 1 of Section 8.2.

Example 1: In the exponentials below, identify the base and the exponent and tell which is the constant that stays the same and which is the variable that changes.

$y = 2^x$ $\qquad$ $y = 3^x$

Example 2: Exponential functions are of the form $y = ab^x$. Using the exponential function $y = 3^x$, what is the y when $x = 2$? What is the output (y) when the input is $x = 4$? Numerical constants are represented by a and b. What does a represent? What does b represent?

To graph an exponential function, you must know the base. In this module, we will investigate different bases to see which are allowable and which are not. Remember the rule for negative exponents: When an exponent is negative, we take the reciprocal of the base, and the exponent becomes positive. If there is an initial value, it stays in the numerator. For x^{-1}, the base can be made into a fraction with 1 in the numerator and the base in the denominator. The exponent of the base becomes positive. It can be written: "$x^{-1} = \frac{1}{x}$."

Let us investigate 2^{-1}.

Using cancellation: $\frac{2^3}{2^4} = \frac{2\cdot2\cdot2}{2\cdot2\cdot2\cdot2} = \frac{1}{2}$

Using the rule for dividing exponentials: $\frac{2^3}{2^4} = \frac{1}{2^{4-3}} = \frac{1}{2}$

Using the transitive property, if $\frac{2^3}{2^4} = \frac{1}{2}$ and $\frac{2^3}{2^4} = 2^{-1}$, then $\frac{1}{2} = 2^{-1}$

Example 3: Simplify the equations below so there are no negative exponents.

a) $y = 2^{-3}$

b) $y = 3^{-4}$

Example 4: Simplify the equations so there are no negative exponents.

a) $y = 2^{-x}$

b) $y = 3^{-x}$

Section 8.2 Investigating Exponential Bases

Looking Back 8.2

As of now, we have compared the exponential functions with base 2 and base 3: $y = 2^x$ and $y = 3^x$ are called increasing exponentials. As x gets larger, y gets larger. As x approaches infinity, y approaches infinity.

We have also investigated $y = 2^{-x}$ and $y = 3^{-x}$. In these exponential functions, as x gets larger, y gets smaller. As x approaches infinity, y approaches 0.

There are many applications of these exponential functions in the real-world. Their patterns are evident in God's creation. Population growth and decline can be modeled by exponential equations. Banking and investments require exponential equations to determine profits and loss and the amount that is compounded and how it is compounded.

We have also seen that when the exponent is negative, the base becomes a fractional denominator. The exponential output is getting smaller as the input is getting larger. In other words, the output is decreasing as the input is increasing.

We have explored negative, zero, and positive exponents. We have investigated positive bases (specifically 2 and 3). In this section, we will investigate negative bases and exponentials with a base of 1.

Looking Ahead 8.2

Example 1: Is $y = 1^x$ an exponential equation? What happens to y (the output) when x (the input) increases? What happens to y (the output) when x (the input) decreases?

What is 1^{-1}?

What is 1^0?

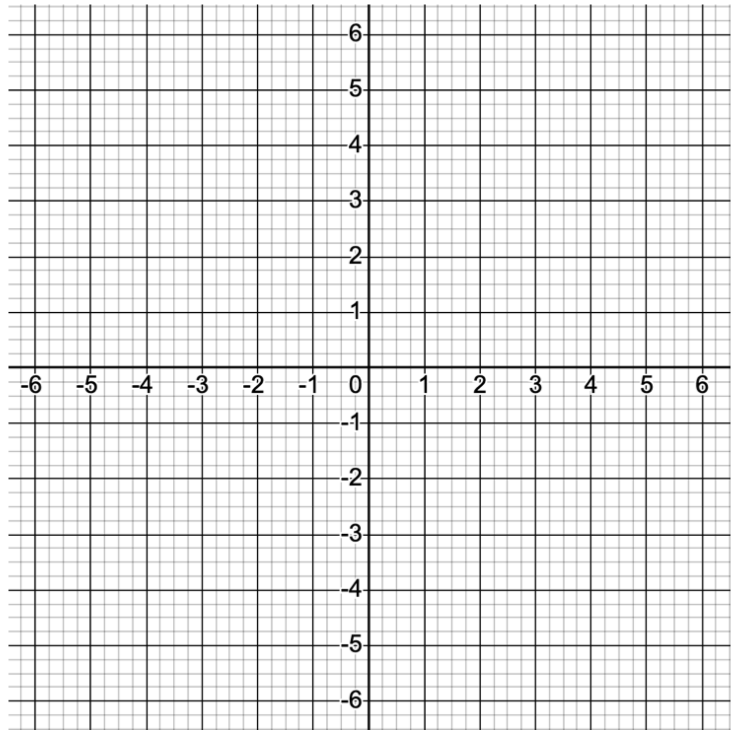
What is $1^{\frac{1}{3}}$?

What is $1^{\frac{1}{2}}$?

What is 1^{256}?

Example 2: The exponential equation in Example 1 is always equal to 1 because it is the identity element of multiplication. No matter how many times you multiply 1 the product is always 1. Make a table and graph for the exponential of 1 or $y = 1^x$.

x	y
-1	
0	
1	
2	

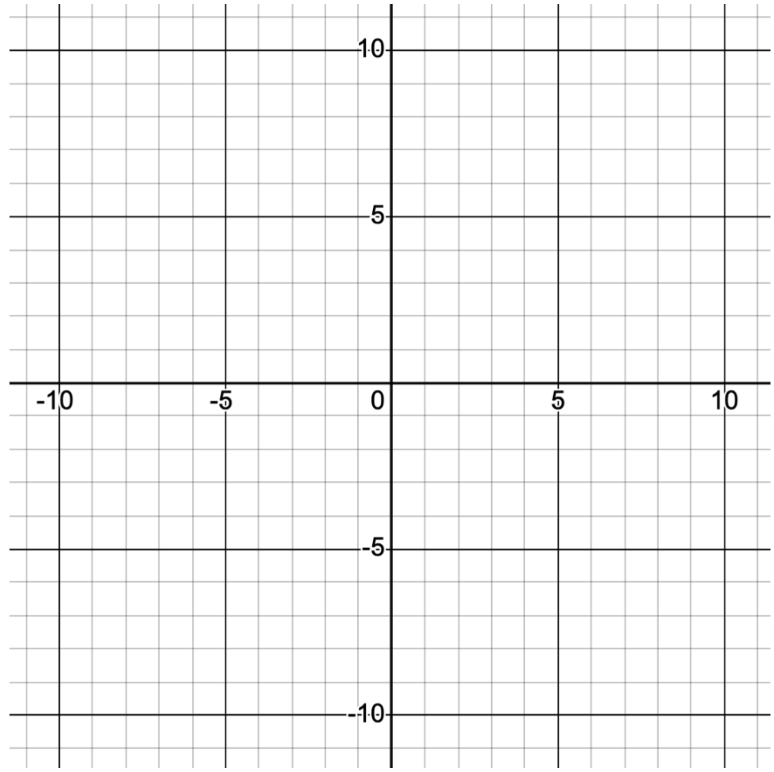


The graph is a ____________________. This is the equation $y = 1$. It does not model exponential growth or decay.

Therefore, the base of an exponential equation cannot be __________.

Example 3: What do you think the graph of $y = (-2)^x$ will look like? Complete the table and draw the graph.

x	y
-3	
-2	
-1	
0	
1	
2	
3	



Who can lift more weight?

Section 8.3 Geometric Sequences

Looking Back 8.3

Linear equations have a constant ratio. This ratio gives a common difference, which is called the slope or the constant rate of change.

If n represents the term number and $a(n)$ represents the value of the term number in an arithmetic sequence, then the recursive formula for the arithmetic sequence is as follows:

$$a(n) = a(n-1) + d\ldots$$

… in which $a(n-1)$ is the value of the previous term and $a(n)$ is the value of the term and d is the common difference.

If n is a term in the sequence, say the 4^{th} term, then $n-1$ is the term just before it: that would be the 3^{rd} term. The common difference (d) is being added each time. This is the slope in a linear equation. The arithmetic sequence below is as follows:

$$8, 12, 16, 20, 24, \ldots$$

Four is being added each time. It is the common difference, $d = 4$. There are five terms listed. The sixth term can be found if the previous term is known. You simply add 4 to 24.

$$a(6) = a(6-1) + 4$$
$$a(6) = a(5) + 4$$

$a(5)$ is the value of the fifth term, which is 24

$a(6)$ is the value of the sixth term, which is 28

Term Value	8	12	16	20	24	28
Term Number	1	2	3	4	5	6

$$a(6) = 24 + 4$$
$$a(6) = 28$$

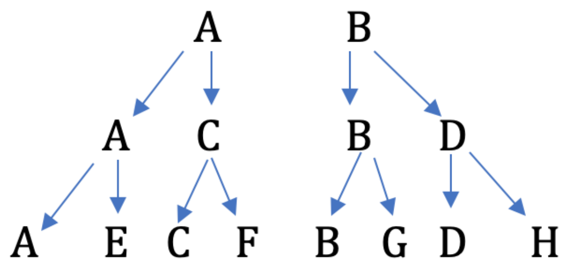
Looking Ahead 8.3

Let us review a problem similar to Problem 7 and 8 of Section 8.2.

The good news: "God is alive," which the two friends learned in Bible Study, comes from Ephesians 2:8-10: "But God being rich in mercy, because of His great love for us, even when we were dead in our transgressions, made us alive together with Christ (by Grace you have been saved through faith) and raised up with Him and sealed in the heavenly places, in Christ Jesus, in order that in ages to come He might show the surpassing riches of His grace in kindness towards us in Christ Jesus. For by grace, you have been saved through faith, and that not of yourselves, it is the gift of God; not as a result of works, that no one should boast. For we are His workmanship, created in Christ Jesus for good works, which God prepared beforehand, that we should walk in them."

After Bible Study, the young friends had lunch with two friends and told them the life-changing message. All the friends had to do to be saved from death, which is separation from God, was to believe (have faith). If they accepted this truth, this gift from God, they would become Christians, Christ followers, and live eternally in Heaven with God.

The young friends believed and each left lunch and told another friend on the way home. Before dinnertime, the two friends that heard the message each told another person on their way home. This pattern continues at dinner. If this pattern continues, happening three more times before bedtime, how many young friends will have heard the good news by the end of the day?



Sunday School Class

At Lunch

On the Way Home

At Dinner

Example 1: Let us make a table to see the results if the good news pattern continues from dinnertime and three more times until bedtime. Let n, the term number, represent each time the good news is passed on from one friend to another. Let us say Sunday School is where the story is told at the start. Then decide if this is an arithmetic sequence. Can we show a common difference? Let $t(n)$ represent the total number of people that have heard the good news.

	$\boldsymbol{n}$ (Term number)	$\boldsymbol{t(n)}$ (Number of people)
Sunday School	1	
Lunch	2	
On the Way Home	3	
Dinner	4	
1^{st} Time Before Bed	5	
2^{nd} Time Before Bed	6	
3^{rd} Time Before Bed	7	

Example 2: Show that Example 1 is a geometric sequence and find the common ratio of its data.

$\boldsymbol{n}$ (Term number)	$\boldsymbol{t}(\boldsymbol{n})$ (Number of people)
1	
2	
3	
4	
5	
6	
7	

The value of each term is being doubled. We call this a

___________________________ ___________________________ or

___________________________ ___________________________.

A ratio is a __________________________ between two numbers.

So, this factor or multiple comes from a common

_____________________.

Each step can be compared to the step before:

$4:2$ as $8:4$ as $16:8$ as $32:16$ as $64:32$ as $128:64$

These can be written as equivalent fractions:

$$\frac{4}{2}=\frac{8}{4}=\frac{16}{8}=\frac{32}{16}=\frac{64}{32}=\frac{128}{64}$$

Each fraction simplifies to ___________. The value of each term is divided by the value of the previous term. This is called a ___________________________ __________________________.

Section 8.4 Recursive Formulas for Geometric Sequences

Looking Back 8.4

In this section, we want to compare the recursive formula for an arithmetic formula with the recursive formula for a geometric sequence. Before we do, below is a quick review of arithmetic sequences and geometric sequences.

Arithmetic Sequence	Geometric Sequence
- Common difference - Additive graph (the same number is being added each time) - Additive property, which is the common difference	- Common ratio - Multiplicative graph (the exponent tells how many times the base is multiplied by itself) - Multiplicative property, which is the common ratio

Looking Ahead 8.4

In the Practice Problems of Section 8.3, the word "recursive" was used again to represent formulas for an arithmetic sequence. Recursive formulas come about when something happens over and over again or repeats itself. To find the value of any term using the recursive arithmetic formula, $a(n) = a(n-1) + d$, the previous term must be known.

In a geometric sequence, $g(n)$ is the value of the term n and $g(n-1)$ is the previous term. However, a common difference is not being added each time, but a common ratio is being multiplied each time, r being the common ratio. To find the value of any term in a geometric sequence using the recursive formula, the previous term must be known.

Example 1: Find the common ratio and recursive formula for the geometric sequence below. Use the formula to find the value of the sixth term in the sequence, $g(6)$.

$$5, 15, 45, 135, 405, \ldots$$

Example 2: Find the common ratio and recursive formula for the geometric sequence below. Use the formula to find the value of the fifth term of the sequence, $g(5)$.

$$25, 75, 225, 675, \ldots$$

Section 8.5 Explicit Formulas for Geometric Sequences

Looking Back 8.5

As has been previously stated, using recursive formulas can be quite difficult as we need to know the total value of any previous term to find the next term. When the term number is large, making a table can take very long just as the tree diagram was rather long for the Good News problem.

Explicit formulas can be used to find any term directly. Explicit means "clearly or precisely expressed." The precise calculation is much quicker using an explicit formula as opposed to a recursive formula. Earlier in our studies, we found the explicit formula for an arithmetic sequence:

$$a(n) = a(1) + (n - 1)d\ldots$$

… in which n is the term number, $a(n)$ is the value of the n^{th} term, $a(1)$ is the value of the first term, $n - 1$ is the previous term or one less than the number for the n^{th} term, and d is the common difference.

This can also be written:

$$a_n = a_1 + (n - 1)d$$

Example 1:	Using the arithmetic sequence below, find the 8th term in the sequence, n, $a(1)$, and d.

$$5, 10, 15, 20, 25, \ldots$$

Looking Ahead 8.5

In a geometric sequence such as $6, 30, 150, 750$, the first term is 6 and the common ratio is 5.

The first term is g_1 or g_1r^0

The second term, g_2, is $g_1 \cdot r = g_1r^1$

The third term, g_3, is $g_1 \cdot r \cdot r = g_1r^2$

The fourth term, g_4, is $g_1 \cdot r \cdot r \cdot r = g_1r^3$

The fifth term, g_5, is $g_1 \cdot r \cdot r \cdot r \cdot r = g_1r^4$

The common ratio is multiplied by the first term one less time than the term number. If the term number is n, the number of times the common ratio is multiplied by the first term is $n - 1$. Therefore, $g_n = g_1r^{n-1}$.

For the sequence, the 8th term would be $g_8 = 6 \cdot (5)^{8-1} = 6 \cdot (5)^7 = 468{,}750$

Example 2: Find the first term, the common ratio, the explicit formula, and the value of the 13th term for the geometric sequence below.

$$3, \ 30, \ 300, \ 3{,}000, \ 30{,}000, \ldots$$

Example 3: Find the first term, the common ratio, the explicit formula, and the value of the 10th term for the geometric sequence below.

$$15, \ 45, \ 135, \ 405, \ldots$$

Section 8.6 Exponential Growth

Looking Back 8.6

We have been investigating geometric sequences for exponential functions. We have found that there is a common ratio. When solving real-world problems in which something grows exponentially, a growth rate or a growth factor is used. (Growth rate and growth factor will be discussed later in this module.) Exponential equations can model growth in problems such as bacterial growth, population growth, or investment growth.

In this section, we will do an experiment involving growth. We will use the experiment to model a real-world problem.

Looking Ahead 8.6

Find a plastic container. Make sure it has a lid. You will use this to represent growth stages by pouring M&M®'s in the plastic container, which will represent growth Stage 0, Stage 1, Stage 2, etc. and shaking the container. The M&M®'s will represent "Meemer" bugs that grow according to a growth factor shown in the table below. The shake of the container is what affects the growth of the Meemers. In this experiment, the growth rate is the same but the initial population changes.

Meemer Bugs		
Color	**Growth Factor**	**Initial Population**
Blue	For every blue Meemer with the *m* side up, add 1 blue	Start with 1 blue Meemer
Brown	For every brown Meemer with the *m* side up, add 1 brown	Start with 2 brown Meemers
Green	For every green Meemer with the *m* side up, add 1 green	Start with 3 green Meemers

1. Put the number of Meemers in your container to represent the initial population of Meemer bugs (1 blue; 2 browns; 3 greens). These have been put in the table for you for Stage 0 (below). There is no growth before the experiment begins.

2. Put the lid on your container and shake it a few times. Set the container down on a flat surface. Take the lid off. If the blue Meemer has the *m* showing, add another blue Meemer to the container to account for the growth factor. If *m* is not showing, do not add another blue Meemer to the container; the Meemer cannot grow without exposure to *m*. Do the same with the brown and green Meemers: for each showing, add another. (If there are 2 brown Meemers showing *m*, add 2 brown Meemers to the container; if there are 3 green Meemers showing *m*, add 3 green Meemers to the container.)

3. Before you put the lid back on and shake the container, record your data in the table below, accounting for any new Meemers added to the container. This is Stage 1 because the container has been shaken once.

4. Put the lid back on and then repeat this process eight more times. Complete the table below as you go.

5. When you have finished the table below, answer the questions which follow.

	Meemers		
Stage of Growth	**Total Number of Blue**	**Total Number of Brown**	**Total Number or Green**
0	1	2	3
1			
2			
3			
4			
5			
6			
7			
8			
9			

Below is an example of possible results for this experiment as well as possible answers to the questions.

	Meemers		
Stage of Growth	**Total Number of Blue**	**Total Number of Brown**	**Total Number or Green**
0	1	2	3
1	2	2	5
2	2	3	8
3	4	4	13
4	6	6	20
5	12	8	28
6	20	12	32
7	30	17	44
8	49	25	65
9	70	43	95

a) After 10 shakes, which color would you expect to have the greatest population?

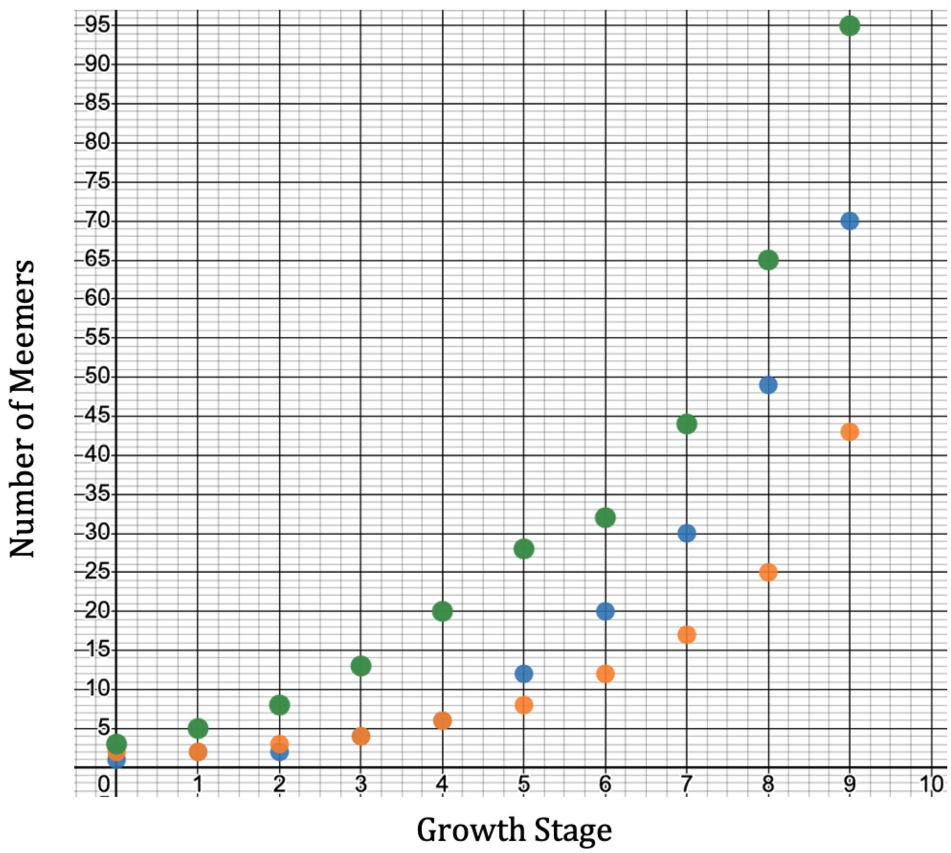
b) After 10 shakes, which color would you expect to have the least population?

c) What two factors seem to affect the population of Meemer bugs?

d) Did the experiment turn out as expected? Why or why not?

e) What is the probability the blue Meemer but will have the *m* side up after each shake? What is the probability the brown Meemer bug will have the *m* side up after each shake? What is the probability the green Meemer bug will have the *m* side up after each shake?

f) Make a graph using the color of Meemers. Let x be the number of shakes or Growth stage and y be the total number of Meemers after each stage of growth. What type of graph does it appear to be?



Section 8.7 Exponential Decay

Looking Back 8.7

We have done two Meemer experiments (with M&M®'s) in the previous section. In this section, we will investigate further.

In the first experiment, the growth factor was the same for all three colors of Meemers, but the initial population was different for each color. Therefore, the color with the largest initial population had the most growth overall. The graph appeared to be exponential, which is represented by a curve that grows slowly at first and then appears to grow faster. The curve starts low and then begins to get steep. This makes sense; doubling 1 Meemer gives a total of 2 Meemers, but doubling 132 Meemers gives a total of 264 Meemers, which is much higher on the graph. The difference between 264 and 132 is much larger than the difference between 1 and 2.

In the second experiment, the initial population was the same for all three colors of Meemers, but the growth factor for each color was different. It did not matter whether the *m* was showing or not, the yellow Meemers doubled each time, the orange Meemers tripled each time, and the red Meemers quadrupled each time. This outcome was much more predictable and precise than the first experiment because it did not rely on probabilities. The percent of growth was the growth rate because the Meemers experience exponential growth.

Looking Ahead 8.7

The experiments in this section will start with a large initial population of Meemers and decrease each time. The decay factor is an *m* virus that is causing the Meemers to die.

For this experiment, you will again need your plastic container. The shake stage will represent each stage of the virus. Again, M&M®'s will represent Meemers. It might be good to use a large bag of mini-M&M®'s as they have more M&M®'s per bag than a snack or fun-size bag. Follow the summarized directions in the table below and steps below to perform the experiment.

Meemer Bugs		
Color	**Growth Factor**	**Initial Population**
Blue	For every blue Meemer with the *m* side up, remove 1 blue	Start with 200 blue Meemers
Brown	For every brown Meemer with the *m* side up, remove 1 brown	Start with 150 brown Meemers
Green	For every green Meemer with the *m* side up, remove 1 green	Start with 100 green Meemers

1. Put the number of Meemers in your container to represent the initial population (200 blue; 150 brown; 100 green). These are in the table labeled Stage 0, which represents the population of Meemers before any shake of the container.
2. Put the lid on your container and shake the container. Take the lid off. For every blue Meemer that has *m* showing, remove it from the container. For every blue Meemer that does not have the *m* showing, leave it in the container. Follow the directions for the brown and green Meemers as well.
3. Before putting the lid back on, count the number of each color in the container and record your data in the table for Stage 1.
4. Put the lid back on the container and repeat the process five more times to represent six stages of growth. Complete the table as you go along.
5. When you have completed the experiment and filled in the table, answer the questions given.

Stage of Growth	Total Number of Blue Meemers	Total Number of Brown Meemers	Total Number of Green Meemers
0	200	150	100
1			
2			
3			
4			
5			
6			

Questions:

a) After 6 shakes, which color would you expect to have the least population?

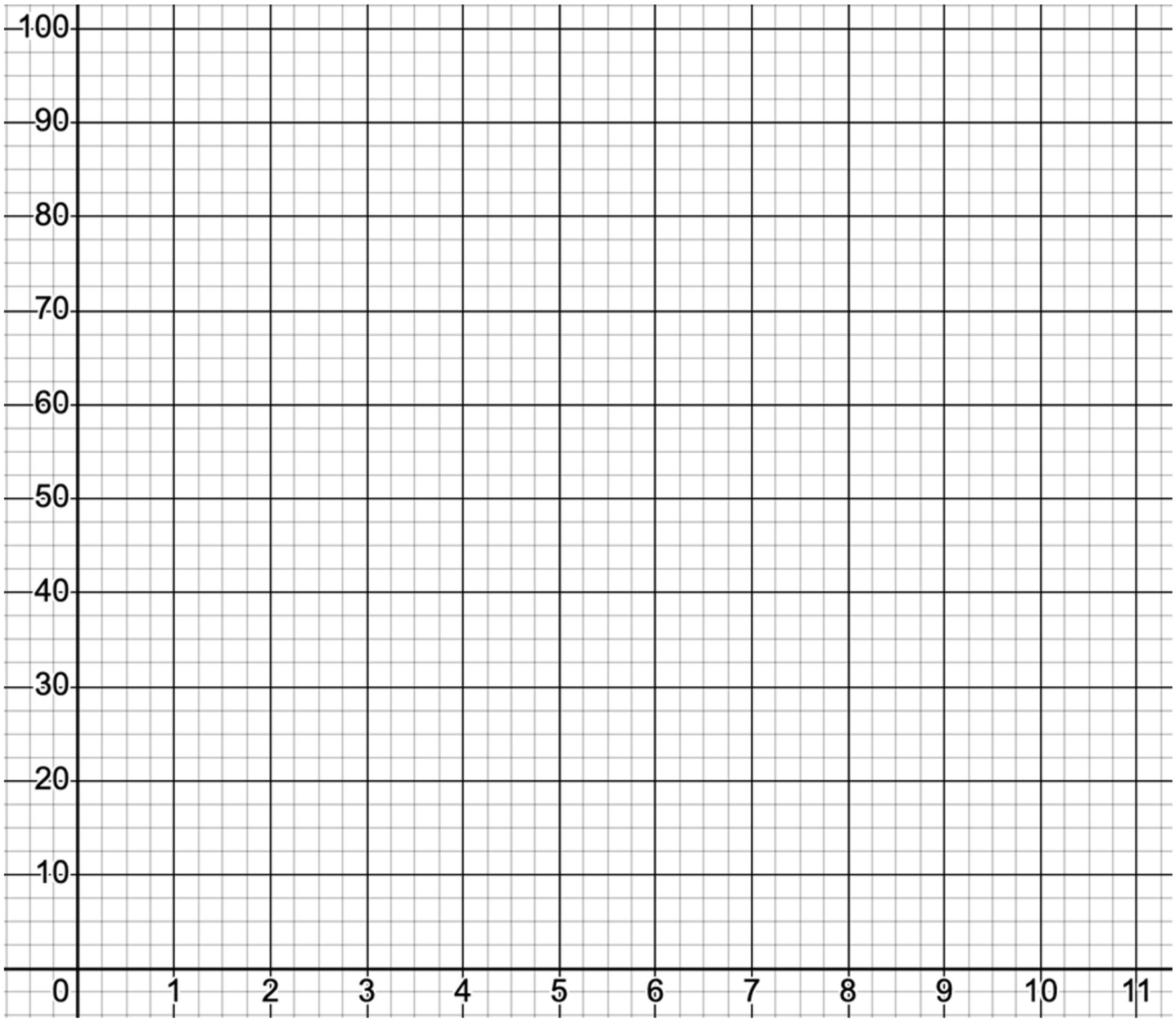
b) After 6 shakes (decay stages), which color would you expect to have the greatest population?

c) What two factors seem to affect the population of Meemers the most?

d) Did the experiment turn out as expected? Why or why not?

e) What is the probability a Meemer will land *m* side up for any color?

f) Make a graph. Let x represent the shake number (virus exposure), and y represent the total Meemer population. Use three different colors to show the total population of each color of Meemer.



Below is a sample experiment through five stages of the virus growth beginning with the initial population of 100 blue Meemers, 90 brown Meemers, and 80 green Meemers.

Stage of Growth	**Total Number of Blue Meemers**	**Total Number of Brown Meemers**	**Total Number of Green Meemers**
0	100	90	80
1	27	35	35
2	17	17	13
3	11	7	8
4	5	4	5
5	3	4	3

Questions:

a) After 6 shakes, which color would you expect to have the least population?

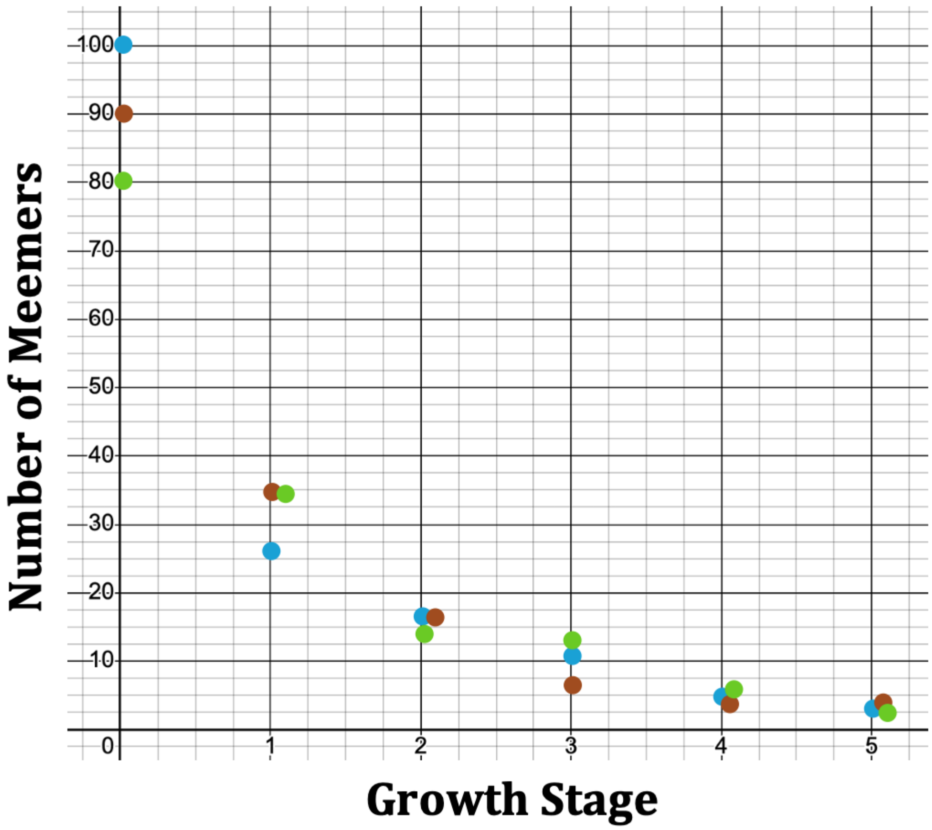
b) After 6 shakes (decay stages), which color would you expect to have the greatest population?

c) What two factors seem to affect the population of Meemers the most?

d) Did the experiment turn out as expected? Why or why not?

e) What is the probability a Meemer will land m side up for any color?

f) Make a graph. Let x represent the shake number (virus exposure), and y represent the total Meemer population. Use three different colors to show the total population of each color of Meemer.



It appears to be exponential but decreasing instead of increasing.

Section 8.8 The General Exponential Equation

Looking Back 8.8

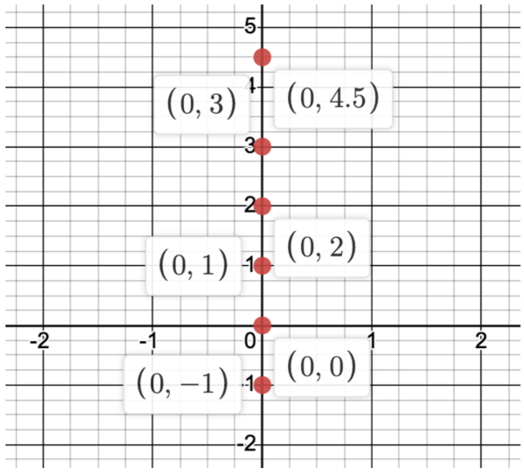
From the previous Practice Problems section, we saw that in an exponential equation, the explicit formula is the initial number or population multiplied by the growth factor (or decay factor), to a power. The power represents the number of times the growth factor (or decay factor) is repeatedly multiplied by itself. The power can represent time, hours, days, weeks, years, etc. The general equation for an exponential equation is the following:

$$y = ab^x$$

In this equation, a is the initial population, b is the growth factor, and x is the time.

Looking Ahead 8.8

Just as in linear, quadratic, and other equations, the y-intercept is where the equation crosses the y-axis. This one happens when x is 0 because any value on the y-axis has an x-coordinate of 0.

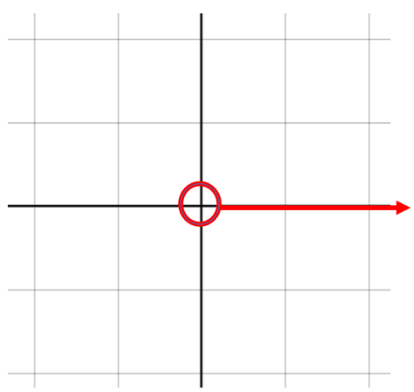


When x is 0 in an exponential equation, then you have the equation $y = ab^0$. Any number to the 0 power except 0 is 1; therefore, you have $y = a(1)$ or $y = a$. Because a is the initial number (start value), the y-intercept is always the initial value (start value). This is similar to the linear equation $y = b$ (b is the y-intercept).

How is it that the y-intercept of an exponential equation is never 0?

If $a = 0$, there is no initial number or population and 0 doubled, tripled, or quadrupled gives you 0. This is similar to the example of why the base could never be 0, which we investigated earlier in this module. In result, you simply get a straight line not including 0 because 0^0 is undefined, or like $\frac{0}{0}$, indeterminate.

Below is $y = 0^x$.

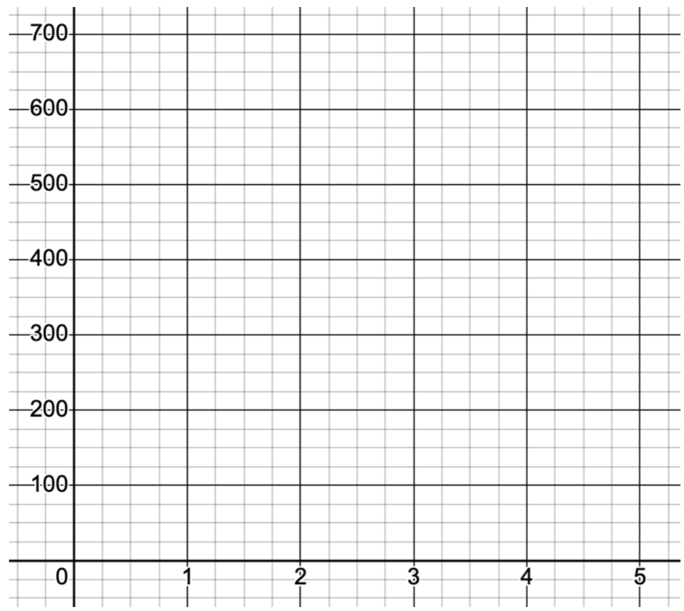


So, for an exponential equation, $y = ab^x$ and $a \neq 0$. Unlike the quadratic equation $y = x^2$, which has the y-intercept at the origin, then $y = 2^x$ has a y-intercept of $(0, 1)$ and $y = 3^x$ has a y-intercept of $(0, 1)$ as well.

Example 1: Buddy, Dawn, Solomon, Elijah, and Ezra start a family business and have 21 other employees the first year. The company triples in size each year. How many employees will their company have at the end of 3 years? Write an equation and make a table and graph to verify your solution.

Using $y = ab^x$ in which a is the initial number of employees, x is the number of years, and y is the total number of employees, complete the table and graph.

x	y
0	
1	
2	
3	



Example 2: How would the equation and outcome from Example 1 change if the company doubled in size each year?

Example 3: How would the equation and outcome from Example 1 change if the initial number of employees were just the family members (Buddy, Dawn, Solomon, Elijah, and Ezra)?

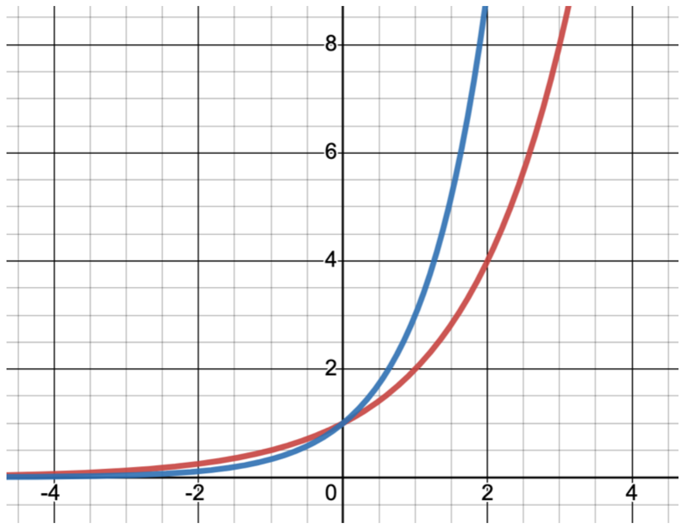
Example 4: Given the table below, find the initial value a, and the growth factor b, and then write the exponential equation that represents the table.

x	y
0	5
1	10
2	20
3	40
4	80

Section 8.9 Transformations of Exponential Equations

Looking Back 8.9

Transformations of exponential equations include horizontal and vertical shifts as well as horizontal and vertical stretches and compressions. We have already seen the graphs of $y = 2^x$ and $y = 3^x$.



These graphs are represented by the equations $y = 1 \cdot 2^x$ and $y = 1 \cdot 3^x$.

Again, both have a y-intercept of 1 because $a = 1$. Both cross the y-axis at 1, but the higher the base, b, the steeper the curve. It looks like a vertical stretch or horizontal compression.

Looking Ahead 8.9

The horizontal shift occurs on the x-axis. The vertical shift occurs on the y-axis. Because the x is in the exponent of the exponential, the h shift (horizontal shift) appears in the exponent. Again, the k (vertical shift) is the constant term; it is not in the exponent.

The parent function: $y = ab^x$
(in which $a = 1$ and $b > 1$ or $0 < b < 1$)

With a horizontal shift: $y = ab^{x-h}$

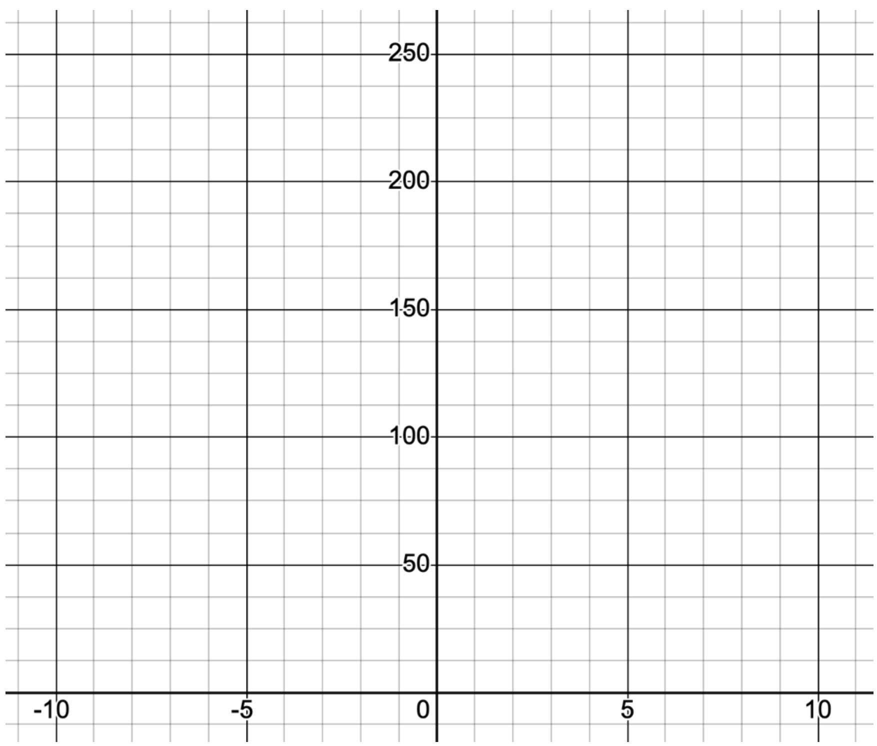
With a vertical shift: $y = ab^x + k$

With a horizontal and vertical shift: $y = ab^{x-h} + k$
(in which a is the initial value, b is the growth factor, x is the input (domain), and y is the output (range))

Example 1:	Write the exponential equation that has a base of 5 and has been shifted right 3 and up 4.

Example 2: In the equation $y = 3(4)^{x+2} + 1.5$, find the initial value, the growth factor, the horizontal shift, and the vertical shift from the parent function.

Example 3: Sketch a graph of the exponential function in Example 2.



Section 8.10 Compound Interest

Looking Back 8.10

We have investigated exponentials with a base of 2 and exponentials with a base of 3. These are of the form $y = 2^x$ and $y = 3^x$. In each equation the initial value is the same: 1. This is because the equations can be written $y = 1(2)^x$ and $y = 1(3)^x$. The numbers 2 and 3 may also be called growth factors because factors are numbers, variables, or expressions multiplied to obtain a product. In $y = 2^x$, 2 is being multiplied over and over to get a product (output). In $y = 3^x$, 3 is being multiplied over and over to get a product (output). In both cases, x is the input, the number of times the base is multiplied by itself. The tables for $y = 2^x$ and $y = 3^x$ are shown below.

$y = 2^x$	
x	y
0	1
1	2
2	4
3	8
4	16
5	32

$y = 3^x$	
x	y
0	1
1	3
2	9
3	27
4	81
5	243

The numbers 2 and 3 are growth factors because the y-values in each equation are growing. When 1 becomes 2 in the equation $y = 2^x$, it grows by 100% because $1 + 1 = 2$.

Growth factor and growth rate may seem interchangeable, but a factor usually refers to a decimal number (in this case, 2.00) while a growth rate usually refers to a percent (in this case, 100%).

When 1 becomes 3 in the equation $y = 3^x$, it grows by 200% because $1 + 2 = 3$. The growth factor here is 3.00 and the growth rate is 200%. If the growth rate is 10%, then the growth factor is 1.10.

$$\text{Growth Factor} = 100\% + \text{Growth Rate or } 100\% + 10\% = 1.00 + 0.10 = 1.10$$

If the rate of decay is 5%, then the growth factor is 0.95 and…

$$\text{Decay Factor} = 100\% - \text{Decay Rate or } 100\% - 5\% = 1.00 - 0.05 = 0.95$$

If the factor represents growth, it is a number more than 1, or more than 100%. It is a number that gets added to 100%. A growth factor between 0 and 1 is a decimal or fraction that is less than 1, or less than 100%. This represents decay (decrease). It is a number that gets subtracted from 100%.

An interest rate of 5% for a savings account means the amount in the savings account appreciates (grows) and will be 105% of the original value after a given amount of time; 100% represents the initial amount of money in the savings account and the 5% growth is included. This can be written as the decimal 1.05.

A 5% decrease in the interest rate means the amount in the savings account depreciates and will only be 95% of what it originally was after a given amount of time: $100\% - 5\%$ is now 95% growth. This can be written as the decimal 0.95.

Looking Ahead 8.10

Let us investigate an example to see if we can determine a general equation that works for compound interest problems as described above.

Example 1: Sienna puts $\$100.00$ in a saving account that earns 5% interest each year. Complete the table for the amount of money in the savings account for the first five years.

Year 1	$\$100 \times 0.05 = \$5.00$	$\$100 + \$5.00 = \$105.00$
Year 2	$\$105 \times 0.05 = \$5.25$	$\$105 + \$5.25 = \$110.25$
Year 3	$\$110.25 \times 0.05 = \$5.50$	$\$110.25 + \$5.50 = \$115.75$ (Rounded down)
Year 4	$\$115.75 \times 0.05 = \$5.75$	$\$115.75 + \$5.75 = \$121.50$ (Rounded down)
Year 5	$\$121.50 \times 0.05 = \$6.07$	$\$121.50 + \$6.07 = \$127.57$ (Rounded down)

The total amount is the principal added to the principal multiplied by the interest rate each time (year):

$$\text{Amount} = \text{Principal} + \text{Principal} \times \text{Interest Rate}$$

Using the distributive property, this becomes:

$$\text{Amount} = \text{Principal}(1 + \text{Interest Rate})$$

This is the process for the first year. For the second year, this becomes:

$$\text{Amount} = \text{Principal}(1 + \text{Interest Rate}) \times (1 + \text{Interest Rate})$$

Using exponents, this becomes:

$$\text{Amount} = \text{Principal}(1 + \text{Interest Rate})^2$$

If $A = $ Amount and $P = $ Principal, $r = $ Interest Rate and $t = $ the amount of time, the general formula becomes:

$$A = P(1 + r)^t$$

Example 2: Jordan deposits $\$500$ in a bank account that pays 2% annual interest. Adreyan deposits $\$300$ in a bank account that pays 4% annual interest. Khali says Jordan will have more money saved for college than Adreyan after both have been saving for college through four years of high school because he started with more in his account. Is Khali correct? Use the formula $A = P(1 + r)^t$.

Example 3: Will Jordan always have more money in his savings account than Adreyan because he started with more money in his account? Will this always be true? Use the formula $y = P(1+r)^t$. What happens after 27 years (when $t = 27$)?

Just as in evolution, looking at some data for a small period of time may not always be reliable in the long run. We have learned this previously when we solved the Box Problem. Data must be investigated over long periods of time to provide somewhat reliable models. There is not reliable data to support millions of years for the age of earth. Assumptions have to be made.

The Bible does provide reliable data on the age of man. In Matthew 1:1-17, we have a genealogy of Jesus Christ. This Scripture tells us there are fourteen generations between Abraham and David and fourteen generations between David and the deportation into Babylon, and fourteen generations between the deportation into Babylon and the birth of Christ. The Bible lists these generations. The data is given in terms of genealogies in the book of Matthew and throughout the Old Testament with names, births, deaths, and years of life. Genealogies and mathematics can help us calculate how long man has lived on earth. These genealogies can be added together to put the age of Earth around 6,000 to 10,000 years old because the Bible says Man was created on the sixth day.

Section 8.11 Population Growth

Looking Back 8.11

There are many real-life situations in the world God created that can be modeled by exponential growth. One such example is population growth. There is a growth rate that can be estimated for cities, states, or countries that can help one make predictions about how many people will populate an area in a given year. These predictions help with city planning or determining how much produce is needed to feed people. These plans, based on reliable predictions, can benefit the people living in a community at any certain time. In general, population growth can be predictable, but can also vary greatly depending on factors such as famine, war, disease, or extreme poverty.

Example 1: If a town has a population of 15,000 people and grows at a rate of 2%, how many people will be in the town after 5 years? 10 years? Note, 2% is the decimal 0.02.

Let us calculate the population each year for the first 5 years.

Year 1	$15{,}000 \times 0.02 = 300$ $15{,}000 + 300 = 15{,}300$
Year 2	$15{,}300 \times 0.02 = 306$ $15{,}300 + 306 = 15{,}606$
Year 3	$15{,}606 \times 0.02 = 312$ $15,606 + 312 = 15{,}918$ (Rounded down)
Year 4	$15{,}918 \times 0.02 = 318$ $15{,}918 + 318 = 16{,}236$ (Rounded down)
Year 5	$16{,}236 \times 0.02 = 325$ $16{,}236 + 325 = 16{,}561$ (Rounded down)

The total population is the initial population added to the initial population multiplied by the growth rate each time (year):

$$\text{Population} = \text{Initial Population} + \text{Initial Population } \times \text{Growth Rate}$$

Using the distributive property, this becomes:

$$\text{Population} = \text{Initial Population}(1 + \text{Growth Rate})$$

This is the process for the first year. For the second year, this becomes:

$$\text{Population} = \text{Initial Population }(1 + \text{Growth Rate}) \times (1 + \text{Growth Rate})$$

Using exponents, this becomes:

$$\text{Population} = \text{Initial Population }(1 + \text{Growth Rate})^2$$

If $P = \text{Total Population}$, $I = \text{Initial Population}$, $r = \text{Growth Rate}$, and $t =$ the amount of time, the general formula becomes:

$$P = I(1 + r)^t$$

This looks very much like the general equation for compound interest. The same principles are at work here. The exponential equation for Example 1 in which y represents the total population and x represents the number of years is:

$$y = 15{,}000(1.02)^x$$

Looking Ahead 8.11

We will be investigating population growth for natural growth (due to birth and death rates) and actual growth (including migration: people moving in and out of an area) for Belgium. We will be using the *2010 World Population Data* from the Population Reference Bureau. (https://www.prb.org/10wpds_eng-pdf/)

Example 2: Migration occurs when foreign-born residents move into a country and natural born citizens move out of a country. The "Net Migration Rate" in Belgium for 2010 was 4 per 1,000, $\frac{4}{1,000}$. What percent of growth does this represent?

Example 3: The rate of natural increase is due to births and deaths and can be calculated by births minus deaths. In 2010, the birth rate of Belgium was 11 people per 1,000 and the death rate was 9 people per 1,000. What percent of growth does this rate of natural growth represent?

Example 4: Belgium had a population of 10,800,000 people in 2010. Using the natural rate of increase and the rate due to migration, predict the population of Belgium in 2025.

Let us try to predict the population using the equation below:

$$\boldsymbol{p = (10{,}800{,}00) \cdot (0.2)^t}$$

total population **initial population** **growth rate**

The year 2025 is 15 years after 2010. The value of t is 15 because t represents the time in years.

Substitute $t = 15$ in the equation:

$$p = (10{,}800{,}000) \cdot (0.2)^{15}$$

$$p = 0.000354$$

This is not the total population. It does not even represent one whole person. What is wrong with using the above equation for total population? As with banking or bacteria, a 1 must be added to the growth rate to include the initial population. Let us try another equation to predict the population:

$$p = (10{,}800{,}000) \cdot (1.2)^{15}$$
$$p = 166{,}396{,}000$$

The expected population for Belgium in 2025 is actually 11,800,000 according to the Population Reference Bureau. Our prediction is far too large. What went wrong this time? Look at our growth rate: It is 0.2%, which is 0.002 as a decimal number. Let us a try a third equation:

$$p = (10{,}800{,}000) \cdot (1.002)^{15}$$
$$p = 11{,}128{,}600$$

This is a much closer prediction, but this is too small. What went wrong this time? The growth due to natural increase and migration was not considered.

There is a simple way to predict actual growth, which considers natural growth and migration:

$$\text{Births} - \text{Deaths} + \text{Migration} = \text{Actual Growth}$$

For Belgium, this is $\frac{11}{1{,}000} - \frac{9}{1{,}000} + \frac{4}{1{,}000} = \frac{2}{1{,}000} + \frac{4}{1{,}000} = \frac{6}{1{,}000}$. The actual growth rate is 6 people per 1,000 and the growth rate is 0.006. The actual equation for expected population is as follows:

$$\boldsymbol{p_a = (10{,}800{,}000) \cdot (1.006)^t}$$

initial population **growth rate**

p_a means actual population

$$p_a = (10{,}800{,}000) \cdot (1.006)^{15}$$
$$p_a = 11{,}813{,}9900$$

This is approximately 11.8 million, which is the expected population of Belgium by the Population Reference Bureau.

Section 8.12 Comparing Power and Exponential Functions

Looking Back 8.12

A power function is of the form $y = x^2$. The base is x and the exponent is some constant that represents a power or exponent. The value input for x is multiplied by itself that number of times.

$$y = x^2 \qquad y = x^3$$
$$y = x \cdot x \qquad y = x \cdot x \cdot x$$

An exponential function is of the form $y = 2^x$. The base is some constant that represents the number being multiplied x times for any input of x. The variable exponent is x. The power is not a constant but a variable. That is why it is called an exponential equation.

$$y = 2^x \qquad y = 3^x$$
$$y = 2 \cdot 2 \cdot 2 \ldots (x \text{ times}) \qquad y = 3 \cdot 3 \cdot 3 \ldots (x \text{ times})$$

Let us investigate further in order to compare and contrast power and exponential functions.

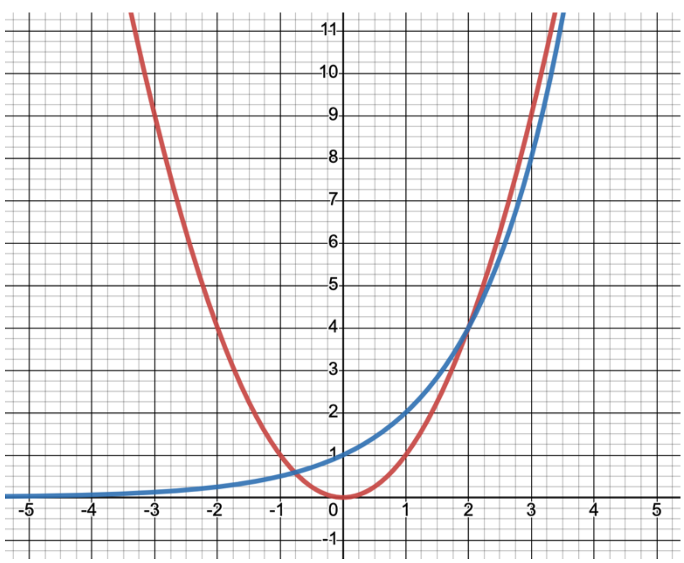
Looking Ahead 8.12

Example 1: Using a domain of integers that are $-3 \le x \le 3$, the tables for $y = x^2$ and $y = 2^x$ are shown below. Fill in the tables.

x	$y = x^2$
-3	
-2	
-1	
0	
1	
2	
3	

x	$y = 2^x$
-3	
-2	
-1	
0	
1	
2	
3	

Example 2: Look at the graph of $y = x^2$ in red and $y = 2^x$ in blue and answer the questions that follow.



a) What can be said about the graphs of $y = x^2$ and $y = 2^x$ in Quadrant I when x increases?

b) For what values of x is y greater for $y = 2^x$ than $y = x^2$?

c) At any point is 2^x equal to x^2?

d) Will $y = x^2$ continue to be greater than $y = 2^x$ as x increases to infinity?

e) Which of the two equations will not go through the origin? Why?

These graphs look very similar in Quadrant I. However, while there are some characteristics that are similar, others are different, which can be determined by observation.

This is also true of the Theory of Evolution and the Theory of Creation. Both are called theories because they cannot be recreated and cannot be observed from the origin. Ultimately, the belief in evolution and the belief in creation are faith-based. Neither can be proven scientifically.

Our belief in God is faith-based, but our faith is based on evidence. Only God can make something from nothing: light, dark, land, earth, sun, moon, and sky appear out of the *void*. Genesis 1:1 tells us: "*In the beginning, God created the heavens and the earth, and the earth was formless and void…"*

Furthermore, Romans 1:18-20 states:

"*For the wrath of God is revealed from heaven and against all ungodliness and unrighteousness of men, who suppress truth in unrighteousness because that which is known about God is evident within them: for God made it evident to them. For since the creation of the world His invisible attributes, His eternal power and divine nature, have been clearly seen, being understood through what has been made, so that they are without excuse."*

Men throughout history have observed that man begets man, horse, begets horse, and dog begets dog. Though there are over 100 new breeds of dogs in the last century, dogs always remain dogs, different breeds, but still dogs. There is no evidence to suggest that fish become birds. It has not been observed in the history of man. If man has interpreted the fossil record to demonstrate that, then perhaps man has misinterpreted his findings.

Genesis 1:12 tells us: "*And the earth brought forth vegetation, plants yielding after their kind, and trees bearing fruit with seed in them, after their kind; and God saw that it was good."* This is what has been observed by man throughout history regarding God's order of creation.

Section 8.13 Investigating Logarithms

Looking Back 8.13

To end this module on exponentials, we are going to learn about logarithms in this section. Logarithms are the exponent that a base must be raised to in order to give a certain number.

For example, in $2^x = 8$, we know that x is 3 because $2^3 = 8$, $2 \cdot 2 \cdot 2 = 8$. If we write it as a logarithm, $\log_2 8 = x$, and input $\log_2 8$ into the calculator using the logarithm key, we get 3. For this example, we do not need to use the calculator because we can do this math in our heads. However, an example such as $2^x = 9$ is quite difficult to do in our heads. We know that x is some irrational number that is just bigger than 3 but less than 4.

We can use the calculator to guess and check:

$2^{3.2} \approx 9.18959$ (too big)

$2^{3.05} \approx 8.28212$ (too small)

$2^{3.17} \approx 9.00047$ (very close)

This would take very long to get exactly 9. However, we can use the logarithm key to solve it quickly. If we type the following into our calculators:

$$\log_2 9 =$$

This will give us $x \approx 3.16993$. Logarithms help us find unknown exponents quickly. We can simply convert $b^x = y$ to $\log_b y = x$ and use the calculator to solve for x.

Example 1:	Solve for x in the exponential equations below. Round your solution to the hundredths place.

a) $4^x = 43$

b) $3^x = 14$

c) $9^x = 1000$

Looking Ahead 8.13

Now, let us use logarithms and Newton's Law of Cooling to solve a real mystery and discover who committed random acts of kindness by delivering coffee.

Newton's Law of Cooling states that the rate at which an object cools is proportional to the difference in temperature between the object and the object's surroundings. In other words, a glass of hot water will cool faster in a cold room than in a hot room.

$$T_C = T_E(b)^t + T_A$$

$T_C =$ Coffee Temperature (120° is opitimal)

$T_E =$ Excess Coffee temperature at time 0 (Starting Coffee Temperature − Air Temperature)

$T_A =$ Air Temperature

$b =$ cooling rate

$t =$ time

The best temperature for a cup of coffee is between 120° and 140°. So, we will say the optimum temperature for a cup of Joe (coffee) is 120°. The normal brewing temperature for coffee is 185°.

A friend at work brought Joe a cup of Joe and put it on his desk. The room temperature was 72°. At 1:00 PM, Joe came to his desk and saw the coffee. He checked its temperature to make sure it was not too hot. It was 143.7°. Joe ran some errands and came back to his desk at 1:07 PM. The coffee had cooled to a temperature of 122.8°. Joe decided it was just right to drink. He took a break from work to enjoy his brew. Find out what time the coffee was dropped off so Joe can thank his friend.

Example 2: Follow the steps to solve the Coffee Cup Mystery described above to find out who did the random act of kindness for Joe.

Step 1: Find T_E. Subtract the room temperature from the temperature of the cup of coffee at 1:00 PM.

Step 2: Let the elapsed time between the first temperature reading of the coffee and the second temperature reading of the coffee be t. Let T_C be the temperature reading of the coffee at 1:07 PM. Find the values for:

$$t =$$

$$T_C =$$

$$T_A =$$

$$T_E =$$

Step 3: Substitute the values found in Step 2 into the equation and use roots (inverses of exponents) to solve for b, the cooling rate:

$$T_C = T_E(b)^t + T_A$$

Step 4: Use the normal brewing temperature of coffee for T_C and the value of b, the cooling rate you found in Step 3, and use logarithms to solve for t, the elapsed time from when the coffee was first delivered until the present.

Step 5: Add the time you got in Step 4 to 1:00 PM, which is the time the temperature was first taken. If it is a negative number, you will subtract it. This is the time the coffee was delivered.

Step 6: Using your equation for Step 4, let y_1 or $f(x_1)$ be the left side of the equation and let y_2 or $f(x_2)$ be the right side of the equation. Let time be x. Graph the two functions on the same coordinate grid on your calculator. Does this confirm the solution you found in Step 5 for the time? Find the point of intersection.
On your calculator, use “menu”- “analyze”- “graph”- “intersection” to find the exact point or go to the menu and trace the graph until you reach the point of intersection.
Go to the menu to set the viewing window to $[-50, 50]$ for x and $[-10, 250]$ for y.

Step 7: Here are the times Joes’ friends had breaks at work:

Marilyn: 12:30-12:40
Kimmy: 12:45-12:52
Nanny: 12:50-1:00

Who committed the random act of kindness and brought Joe the cup of coffee?