

## Algebra 2 Module 2 Matrices

### Section 2.1 Logic Matrices

#### Looking Back 2.1

A matrix is used to construct things in computer science. It is a rectangular array of numbers in which each entry in the matrix is in each row and column, called a cell. A matrix is named by its rows (first) and columns (second). A 2 x 3 matrix (read “two by three”) is shown below.

$$\begin{array}{c} 2 \text{ rows} \\ \left[ \begin{array}{ccc} a & b & c \\ d & e & f \end{array} \right] \\ 3 \text{ columns} \end{array}$$

This matrix has two rows going across and three columns going up and down. It is a set of numbers in brackets, which represent something. The numbers in the cell may represent light, intensity or density of objects in trials of a science experiment or the coefficients of an equation. So, matrices are arrays of numbers that represent data. These can be used in mathematics to solve problems. For example, they can be used to solve systems of equations like those in the last module.

Algebraic operations, such as multiplication, addition and subtraction, can be performed on matrices. You will learn all about these in this module.

Matrix is singular for one rectangular array of numbers, and matrices is the plural form for more than one rectangular array of numbers.

Let’s begin this module with some fun matrix problems, which use table arrays and the processes of deduction to solve.

#### Looking Ahead 2.1

Matrix logic problems begin with a series of clues. The categories you are exploring become the titles of the rows and columns.

Brenda and Mark have five children that each volunteer for a different role at church. The roles include singing in the choir, playing the piano, working in the nursery, leading youth group and cleaning the church. Each child volunteered for a different position, which means that none will work the same job. Read the clues below to find out who does what.

1. Jessica does not like cleaning her room or any rooms.
2. Lauren would rather be with adults than children.
3. The nursery is next to the youth room, so Emily and Olivia walk together to their jobs.
4. The piano is inside the youth room, so Olivia sets up the music for her sister and brother before going to her volunteer work.
5. Jessica enunciates the words while practicing with Matthew.

The two categories that make up the rows and columns of the matrix are name and volunteer position. The jobs are given, and the names can be found in the clues. It does not matter which category goes in the rows or columns, but if name is chosen for the rows then volunteer position must be chosen for the columns.

	<b>Singing in the Choir</b>	<b>Playing the Piano</b>	<b>Working in the Nursery</b>	<b>Leading the Youth Group</b>	<b>Cleaning the Church</b>
<b>Jessica</b>					
<b>Lauren</b>					
<b>Emily</b>					
<b>Olivia</b>					
<b>Matthew</b>					

Place “x’s” in the boxes that are false. For example, if you conclude that Jessica does not sing in the choir, put an “x” in the box under “Singing in the Choir” and next to “Jessica.”

Place “o’s” in the boxes that are true. Once an “o” is placed in a box, that job belongs to that child and “x’s” can be placed in the remaining spaces for that row and column. No child will have more than one position.

Clue 1 tells us that Jessica does not clean the church. Put an “x” in that column in the first row, Jessica’s row.

Clue 2 tells us that Lauren does not work with children. Put “x’s” in the boxes for “Working in the Nursery” and “Leading the Youth Group” next to Lauren’s name.

Clue 3 tells us that Emily and Olivia do work in the nursery and with the youth. We do not know who does which job yet, so we cannot put an “o” in either, but we can put “x’s” in all the others, which we now know they do not do.

Clue 4 tells us that Emily works in the youth room, so Olivia must work in the nursery. Clue 4 also tells us that Matthew either sings or plays the piano since he is the only brother.

Clue 5 makes it clear that Jessica sings, so Matthew must play piano and Lauren is the only one left, so she must do the cleaning.

Have fun solving more matrix logic problems like this in the “Practice Problem” section. Follow the steps to complete the table logic.

#### Matrix Logic Problem Steps

1. Decide the two categories that make up the rows and columns after reading the problem and clues.
2. Put the one category title in the rows of the matrix and put the other category title in the columns of the matrix.
3. Put an “o” in all the boxes that are true. Cross out all the other boxes in the same row and column as an “o.”
4. Put an “x” in all the boxes that are false.
5. If you cannot determine for certain if a clue given is true or false information, come back to it later after reading the other clues.
6. Clues are related. Put notes inside the spaces in case you must come back to them later.

Section 2.2 Analyzing Data with a MatrixLooking Back 2.2

A matrix is a rectangular array of numbers. It is usually enclosed in brackets. Each number in the matrix is called an element. Each element in a cell is labeled by its row location and column location. A generic cell labeled  $A_{i,j}$  indicates that the matrix is A, the row number is represented by  $i$  and the column number is represented by  $j$ .

Suppose a grocery store counted its weekly chip sales. There are four flavors of chips that the grocery sells: sour cream and onion, barbeque, original, and nacho cheese. This can be represented in a  $1 \times 4$  matrix (read “one by four”) because it has one row and four columns.

	<b>Sour Cream and Onion</b>	<b>Barbeque</b>	<b>Original</b>	<b>Nacho Cheese</b>
Weekly Sales	[350	2,261	728	777]

The number of weekly sales for the sour cream and onion chips is in Row 1, Column 1. If  $C$  is the matrix name chips, then the cell number for sales of sour cream and onion chips is  $C_{1,1}$ . The subscript has the row first followed by the column.

The barbeque sales are in cell  $C_{1,2}$ . In one week, 2,261 barbeque chips sold. Also, in one week, 728 packages of original chips sold. This information is in cell  $C_{1,3}$  (1 is for the first row and the 3 is for the third column).

What would be the cell number for the Nacho Cheese chip sales?

The information could also be stored in a  $4 \times 1$  matrix.

**Weekly Sales**

$$\begin{bmatrix} 350 \\ 2,261 \\ 728 \\ 777 \end{bmatrix}$$

Looking Ahead 2.2

Example 1: A matrix may also be used to represent store chip sales for more than one week.

	<b>Sour Cream and Onion</b>	<b>Barbeque</b>	<b>Original</b>	<b>Nacho Cheese</b>
<b>Weeks</b>	[350	2,261	728	777]
	245	1,990	607	781]
	285	2,104	650	776]
	313	2,002	684	750]

- a) What is the cell number that represents Week 3 sales for barbeque chips?
- b) How many barbeque chips sold in Week 3?
- c) What are the total sales of barbeque chips in all four weeks?
- d) Which chips sold the least in the four weeks?
- e) How many more barbeque chips sold than those which sold the least?

<p>Example 2: Because the manufacturer knows that some chips are more popular than others, they put packs of multiple flavors together to sell to grocery stores.</p>
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A snack pack includes the following chips:

- 4 packs of sour cream and onion chips
- 3 packs of barbeque chips
- 3 packs of original chips
- 2 packs of nacho cheese chips

A family pack includes the following chips:

- 6 packs of sour cream and onion chips
- 5 packs of barbeque chips
- 3 packs of original chips
- 2 packs of nacho cheese chips

A travel pack includes the following chips:

- 8 packs of sour cream and onion chips
- 6 packs of barbeque chips
- 6 packs of original chips
- 2 packs of nacho cheese chips

A grocery store orders 115 snack packs, 222 family packs, and 50 travel packs. How many of each type of chip is needed to complete the order?

Example 3: Use an array to organize the chip information and use it to determine if the number in Example 2 is accurate. The types of packs will be the row heading and the individual chip pack needed will be the column heading.

Packs	Sour Cream and Onion	Barbeque	Original	Nacho Cheese
Snack Pack	4	3	3	2
Family Pack	6	5	3	2
Travel Pack	8	6	6	2

Example 4: The following array represents hamburger and cheeseburger costs at the restaurants Happy Jack's, Linda Lou's, and Merry Meredith's. Answer the following questions given the information in the matrix.

Burger	Happy Jack's	Linda Lou's	Merry Meredith's
Hamburger	\$ 2.95	\$ 3.99	\$ 3.50
Cheeseburger	\$ 3.05	\$ 4.25	\$ 3.99

Questions:

- Are the matrix dimensions  $2 \times 3$  or  $3 \times 2$ ?
- If  $C$  represents a cell, what cost is in  $C_{3,3}$ ? Is that a hamburger or cheeseburger cost? Name the restaurant.
- True or False: another matrix representing the same information can be displayed as shown below:

Restaurants	Hamburgers	Cheeseburgers
Happy Jack's	\$ 3.05	\$ 2.95
Linda Lou's	\$ 4.25	\$ 3.99
Merry Meredith's	\$ 3.99	\$ 3.50

- List the dimensions of the above matrix array.
- How could you change the array above so that it is equivalent to the one given in Example 4.

Section 2.3 Matrices with Graphs and Charts

Looking Back 2.3

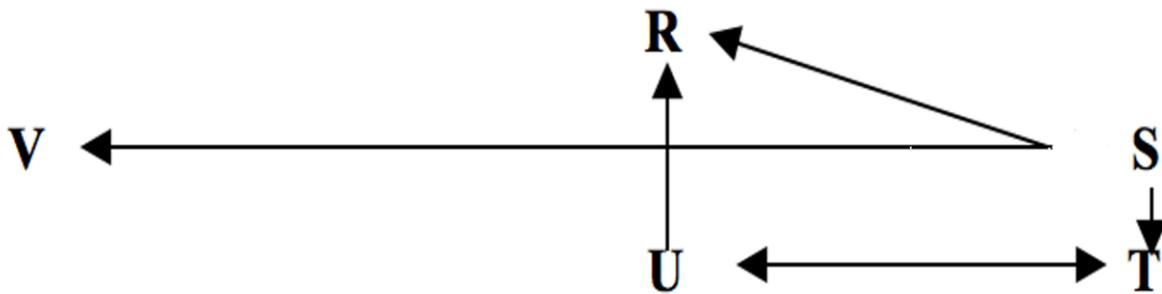
Matrices are used in many areas of problem solving. Graph Theory is a field that uses graphs to problem solve. You used graphs to work with matrices and previously to solve probability problems.

In this section, graphs will again be used with matrices to solve computer communication problems, chemistry problems and population problems.

Looking Ahead 2.3

Computers today are often shared among users. Just as when you talk to one another using computers, desktops can be shared. Sometimes, when a computer company is called upon for help, the service technician has access to your computer and you can see him or her moving around on your screen to repair it, but you cannot see his or her computer screen on your personal laptop or device.

Example 1: Mr. Pool sets up a network of five computers in which some have access to others directly and some do not. Set up a matrix using a “0” if a computer cannot see the screen of another. Use a “1” if the screen of the user can be viewed directly, or through another user only. Put a “2” if the screen can be viewed directly and through another user.



		R	S	T	U	TO USER
						V
	R					
	S					
	T					
	U					
FROM USER	V					

Use the above matrix to answer the following questions:

- a) Which user has the most access to the screens of others?
- b) Which user(s) has the least access to the screens of others?
- c) Which user is most likely a computer technician?
- d) If there were violation of privacy issues regarding computer sharing, which user(s) would be the most likely to be eliminated as violator(s)?

Example 2: In Chemistry you are introduced to the periodic table of elements. The symbol for Hydrogen is H. It is a nonmetal that begins the periodic chart. The top number is the atomic number, telling the number of protons. The bottom number is the atomic mass in units where 1,000 amu =  $1.66 \times 10^{-24}$  g.

1 <b>H</b> 1.01
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The first row of transition metals on the periodic table are listed from columns 3B to 7B. Make a matrix of these transition metals with the rows being the element and the columns being the atomic number and the atomic mass in units. Following the tables below, answer the questions.

21 <b>Sc</b> 45.0	22 <b>Ti</b> 47.9	23 <b>V</b> 50.9	24 <b>Cr</b> 52.0	25 <b>Mn</b> 54.9
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	Atomic Number	Amu
<b>Scandium</b>		
<b>Titanium</b>		
<b>Vanadium</b>		
<b>Chromium</b>		
<b>Manganese</b>		

Questions:

- a) What do you notice about the rate of change in the atomic numbers?
- b) What do you notice about the rate of change of the atomic mass units of the elements?
- c) Does alphabetical order of elements play an important role in the organization of the periodic table of elements?

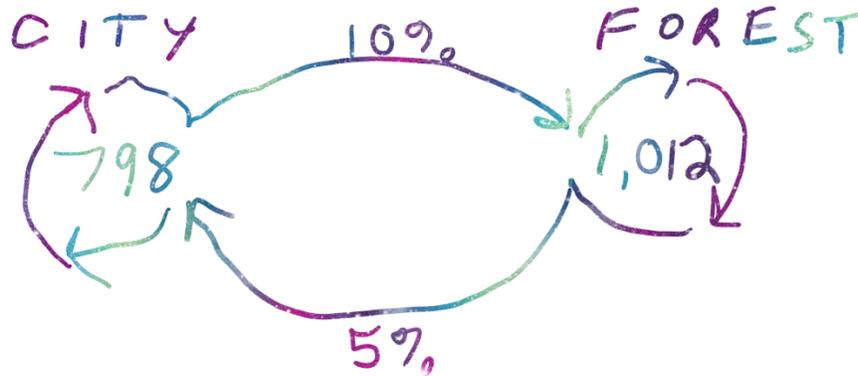
Looking at the Periodic Table of Elements, it is no surprise that the elements of the universe are very orderly. Because of this order, spaces were later filled in for the chart as elements were discovered. God is a God of order and He created us in His image. We like to order and organize things. Chemists organized chemicals by looking at similar physical and chemical properties. Less than 60 of the 112 named elements were discovered by the beginning of the 19<sup>th</sup> century. A Russian chemist, Dmitry Mendeleev, laid the groundwork for the periodic table of elements. It is one of the most useful patterns of nature and its structure was laid when God created the universe. However, it was discovered by Dmitry and others over years of systematic work.

Mendeleev organized the table of elements by their atomic weights and other properties. He was encouraged by his mother to pursue divine and scientific truth and live by Proverbs 25:2, which says, "It is the glory of God to conceal a thing, but the honor of kings to search out a matter."

He saw the table in a vision. He left spaces blank, in faith believing that elements would follow this pattern to be placed there. Scandium is one such element that is Element 21 from Example 2. He used mathematical interpolation based on the properties of elements on the chart. Element 101, Mendelevium, was discovered after his death and named in his honor.

**Example 3:** The deer population is changing. As city growth increases the deer population in the city decreases by 10%. The deer relocate to the forest. As forests become overpopulated with deer then 5% of the forest deer relocate to the city. This continuous cycle can be represented by a continuous graph and a matrix.

Given an initial city deer population of 798 deer and an initial forest population of 1,012 deer, find the population totals after the third new housing subdivision is built. The construction of each new subdivision sets in motion the deer relocation of the two populations, city deer and forest deer.



Subdivision I:  $1,012 \cdot 0.05 = 50.6$

50 deer leave the forest and relocate to the city

$798 \cdot 0.10 = 79.8$

79 deer leave the city and relocate to the forest

The forest now has  $1,012 - 50 + 79 = 1,041$  deer.

The city now has  $798 - 79 + 50 = 769$  deer.

Put these in a matrix to represent the data. Let 0 represent the initial deer population and 1 represent the deer population with the addition of one subdivision.



Subdivision II:  $1,041 \cdot 0.05 = 52.05$

$$769 \cdot 0.10 = 76.9$$

52 deer leave the forest and relocate to the city

76 deer leave the city and relocate to the forest

The forest now has  $1,041 - 52 + 76 = 1,065$  deer.

The city now has  $769 - 76 + 52 = 745$  deer.



Subdivision III:  $1,065 \cdot 0.05 = 53.25$

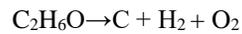
$$745 \cdot 0.10 = 74.5$$

53 deer leave the forest and relocate to the city

74 deer leave the city and relocate to the forest



Let's look at one more example that involves some Chemistry. There are three basic types of chemical reactions: decomposition, formation, and combustion. The chemical equation that describes the decomposition of ethanol ( $C_2H_6O$ ) is as follows:



Decomposition breaks the compound down into its component elements of carbon, hydrogen, and oxygen (both of which are a homonuclear diatomic with a subscript of 2). This is an unbalanced equation. There are two carbons on the reactants side, but only one element of carbon on the products side. Algebra can be used to balance this. The coefficient of the carbon element is 1 and the subscript is 2 on the reactants side. When the products of the coefficient and subscript of each element are equal, then the equations are balanced.

The balanced equation is as follows:

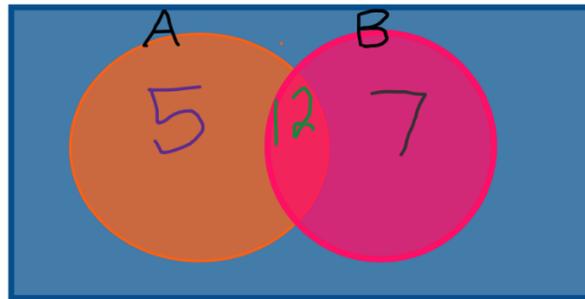


Example 4: Use matrices to demonstrate that the decomposition of ethanol is balanced. If the equations balanced then the products of the reactants and the products of the products must be equivalent matrices.

Section 2.4 Matrices and Punnett SquaresLooking Back 2.4

There are certain inherited traits that you have that are determined by your genetic code. Every individual has genetic tendencies based on DNA, which is stored in a chromosome. The DNA is the storehouse of information that tells each gene what protein to produce.

Some genetic tendencies you may have are facial dimples. Smile in the mirror and you can see if you have one or more of these dimples. Another tendency is hairline; pull your hair back to see if you have a widow's peak (which looks like a "W" or "M") or if you do not (your hairline is relatively straight). These traits can be represented by a Venn diagram, which you learned about in Algebra.



Facial Dimples

Widows Peak

In this group of 24 members...

- 5 have facial dimples
- 7 have a widow's peak
- 12 have both facial dimples and a widow's peak

If 17 members ( $5 + 12$ ) have facial dimples, 7 members ( $24 - 17$ ) of the group do not have the facial dimples. Because  $A$  represents the group that have facial dimples,  $A'$  (read "A prime") is called the complement of  $A$ , and  $A'$  represents the group members that do not have facial dimples.

This information can be put into a matrix.

$$\begin{bmatrix} 5 \\ 12 \\ 7 \end{bmatrix}$$

The probability of  $A$  occurring is  $5/24$ ; \_\_\_\_\_ % of the group has facial dimples.

The probability of  $B$  occurring is  $7/24$ ; \_\_\_\_\_ % of the group has a widow's peak.

The probability of both occurring is  $12/24$ ; \_\_\_\_\_ % of the group has both facial dimples and a widow's peak.

There is some debate among university professors in Utah and Delaware as to whether the widow's peak is really an inherited trait. Let us look at some traits that are known to be inherited in the Looking Ahead section.

Looking Ahead 2.4

Gregor Mendel is to the field of biology as Dmitri Mendeleev is to the field of chemistry. Gregor Mendel did experiments on pea plants that would go on to earn him the title “Father of Modern Genetics.” He entered the monastery where his brilliance was recognized. His family were poor farmers and it was a sacrifice to send him to school. The monastery took over the burden.

Mendel was subsequently sent to the university in modern-day Czech Republic, where he studied mathematics, physics, and Greek. After graduating, he conducted his experiments and taught at a secondary school for more than a decade, teaching the same subjects he was studying. He taught his students with great patience and befriended them. Later, he became abbot of the monastery and pursued his spiritual passions. The monastery allowed him to use his gentleness and giftedness to discover the scientific laws of God and help others in these pursuits.

Mendel’s work was published in 1866, but Darwin’s book of scientific theories had been published in 1859 and won the minds of those in the nineteenth century and there was no place for the laws discovered by Mendel at that time. His achievements were never recognized in his lifetime, but after his death, his work was heralded.

Example 1:	Draw a matrix to demonstrate the possible outcomes for two plants carrying $TT$ and $tt$ chromosomes when bred. The large $T$ stands for the dominant trait of tallness and the small $t$ stands for the recessive trait of shortness. Punnett squares are used in Biology, but matrices are used in mathematics when studying genetics.
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Example 2:	Draw a matrix to demonstrate the possible outcomes when two plants carrying $Tt$ and $Tt$ chromosomes are bred.
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Mendel began his experiments with peas because he was a farmer. He recognized that there are two factors that determine traits. In each trait, there is a dominant factor. If it exists in the offspring, it will be expressed. In other words, dominant factors override recessive factors.

Example 3: Every cell in a human contains 23 pairs of chromosomes or 46 chromosomes in total. Twenty-two of these pairs look the same in both males and females. The chromosome that determines sex is the 23<sup>rd</sup> pair and it looks different in males and females. Males have two different chromosomes, an X and a Y. Females have two of the same chromosomes, an X and an X. The X is recessive, while the Y is dominant. Draw a matrix for the possible outcomes for a male and female and interpret the results.

There are 4 blood types in humans: A, B, AB and O. This is important to know if a person should need a blood transfusion after losing blood in an emergency.

Example 4: Use a matrix to determine all the possible combinations of A, B, and O. The Type O allele is recessive to blood types A and B. Both A and B are co-dominant since they dominate each other. What is the blood type of each outcome in the matrix?

Possible combinations:

Blood types from the combinations:

Type A and B appear to be the most common, followed by Type AB. The least common type of blood appears to be Type O.

This is another application of matrices. As you can see, matrices are used often in the fields of mathematics, science, and computer science.

Section 2.5 Adding and Subtracting MatricesLooking Back 2.5

Matrices are used to organize, store and analyze data. That data can represent costs, temperatures, test scores, genetic codes and various other topics. The rows and columns have headings or titles that give information about the data stored there. We have been putting those titles in the dimensions of the matrices much like a table.

For the rest of the module, once the information is given in a table or problem situation, the labels will be omitted. We will know from the order what each cell represents. Operations can then be performed on a matrix. Addition is one matrix operation that is computed cell by cell. This means the dimensions of the matrices must agree. Any  $i \times j$  matrix can be added to another  $i \times j$  matrix with similar data.

Looking Ahead 2.5

Example 1: Tina makes small, medium, and large pizzas for her 4-H club to sell for \$7.95, \$11.95, and \$13.95. Mike's automotive club sells small, medium and large pizzas for \$6.99, \$9.99 and \$12.99. Make a matrix called  $[P]$  for the costs of those pizzas. Make another matrix called  $[T]$  for the charge for additional toppings. Add the matrices to get the total cost of a small pizza with one topping, a medium pizza with two toppings and a large pizza with three toppings from Tina if one topping cost \$0.95, two toppings cost \$1.25, and three toppings cost \$1.95. Do the same for Mike if one topping cost \$1.30, two toppings cost \$1.80 and three toppings cost \$2.10.

Example 2: How much does a medium two-topping pizza cost if you buy it from Mike? What color is that in the matrices?

Example 3: Calculate  $[T] + [P]$ . Does the order matter when adding matrices? What property holds true for addition of matrices?

Example 4: Add  $[D] + [E]$ .

$$[D] = \begin{bmatrix} 0.4 \\ 0.2 \\ 9.8 \end{bmatrix}$$

$$[E] = [3.1 \quad 1.1 \quad 9.8]$$

Example 5: Calculate the  $-[A]$  if...

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Section 2.6 Scalar and Matrix MultiplicationLooking Back 2.6

Matrices can be added and subtracted. That makes sense because subtraction is the inverse of addition. We know that order is important in subtraction because it is not commutative for real numbers or matrices. The Commutative Property of Addition does hold true for both real numbers and matrices, so order is not important when adding.

To find the opposite of a given matrix,  $[M]$  becomes  $-[M]$ . The sign of the member of each cell changes to its opposite. If the cell has one numerical element that is positive, it becomes negative. If the cell has two binomial terms that are negative, both become positive.

When a matrix is multiplied by a number, it is called a scalar. If the matrix  $[M]$  is tripled in size, it is multiplied by three:  $3 \cdot [M]$  means the entry in each cell in the matrix is multiplied by 3. This is called scalar multiplication. Scalars will be used in future mathematics courses with vectors.

A scalar is a real number, as opposed to a vector or matrix, but it is related to both and can be used to perform operations on both.

Looking Ahead 2.6

Example 1: A potato chip manufacturing company manufactures 2,000 mini snack-packs, 1,240 family packs, and 808 vacation packs each day, Monday through Friday. How many of each chip do they produce? The number of packs for each type of chip is given in the table below.

Packs	Barbecue	Original	Nacho Cheese	Sour Cream and Onion
Mini-Snack Pack	4	3	3	2
Family Pack	6	5	3	2
Vacation Pack	8	6	6	2

Example 2: Example 1 does answer the question, but only after much arranging and combining of items. A simpler way to solve the problem is to use matrix multiplication, which multiplies and adds all in one step.

If matrix  $[M]$  has dimensions  $p \times q$  and matrix  $[N]$  has dimensions  $q \times s$ , then the product matrix will have dimensions  $p \times s$ . The last dimension of the first matrix must be the same as the first dimension of the last matrix in order to multiply. To find the product, the first row of the first matrix is multiplied by the first column of the second matrix. The process is repeated for each row and column. The sum of all the products is placed in Cell<sub>1,1</sub>.

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \cdot \begin{bmatrix} g & j \\ h & k \\ i & l \end{bmatrix} = \begin{bmatrix} ag + bh + ci & aj + bk + cl \\ dg + eh + fi & dj + ek + fl \end{bmatrix}$$

Example 3: The potato chip manufacturer sells mini packs for \$9.50 and family packs for \$11.99 and vacation packs for \$13.00 each. They are allowing two girl scout troops to sell the potato chips in front of grocery stores as a fundraiser for their service projects. Below are the numbers of sales the girl scout troops made.

Troop Bonzai: [80 42 20]

Troop Tripletree: [93 27 31]

What are the total values of the sales for the troops of the multi-packs of chips?

Example 4: Use matrix multiplication to determine the commission made by each troop. The commission made on a mini snack-pack is \$2.00, the commission made on a family pack is \$3.09, and the commission made on a vacation pack is \$4.12.

Section 2.7 Row Reduction Method for Solving SystemsLooking Back 2.7

We know that the Commutative Property of Addition is true when working with matrices, but the Commutative Property of Multiplication is not true when working with matrices.

The Associative Property of Addition also holds true for matrices.

$$([A] + [B]) + [C] = [A] + ([B] + [C])$$

$$[A] = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix} \quad [B] = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \quad [C] = \begin{bmatrix} -5 & 3 \\ 7 & 2 \end{bmatrix}$$

$$\left( \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \right) + \begin{bmatrix} -5 & 3 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix} + \left( \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} -5 & 3 \\ 7 & 2 \end{bmatrix} \right)$$

$$\begin{bmatrix} 2+1 & 1+2 \\ -3+1 & 4+2 \end{bmatrix} + \begin{bmatrix} -5 & 3 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix} + \begin{bmatrix} 1+(-5) & 2+3 \\ 1+7 & 2+2 \end{bmatrix}$$

$$\begin{bmatrix} 3+(-5) & 3+3 \\ -2+7 & 6+2 \end{bmatrix} = \begin{bmatrix} 2+(-4) & 1+5 \\ -3+8 & 4+4 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 6 \\ 5 & 8 \end{bmatrix} = \begin{bmatrix} -2 & 6 \\ 5 & 8 \end{bmatrix}$$

There are many other properties that hold true when adding matrices that you have discovered throughout this module. They are summarized below.

*Addition Properties of Matrices*

If  $[A]$ ,  $[B]$ , and  $[C]$  are  $i \times j$  matrices and  $[O_{(i \times j)}]$  is the zero matrix, then...

$[A] + [B] = [B] + [A]$	Commutative Property of Addition
$([A] + [B]) + [C] = [A] + ([B] + [C])$	Associative Property of Addition
$[A] + [O_{(i \times j)}] = [O_{(i \times j)}] + [A]$	Identity Property of Addition
$[A] + (-)[A] = [O_{(i \times j)}]$	Inverse Property of Addition
$[A] + [B]$ is also an $i \times j$ matrix	Closure Property of Addition

There are several of these properties that hold true for scalar multiplication and are summarized below. Matrix multiplication properties will be investigated in the Practice Problems section.

*Scalar Multiplication Properties of Matrices*

If  $[A]$  and  $[B]$  are  $i \times j$  matrices,  $[O_{(i \times j)}]$  is the zero matrix, and  $m$  and  $n$  are scalars, then ...

$m[A] = [A]m$ or $n[B] = [B]n$	Commutative Property of Multiplication
$m(n[A]) = (mn)[A]$	Associative Property of Multiplication
$(m + n)[A] = m[A] + n[A]$	Distributive Property
$1 \cdot [B] = [B] \cdot 1$	Identity Property of Multiplication
$-1 \cdot [B] = -[B]$	Multiplicative Property of -1
$[O_{(i \times j)}] \cdot [A] = [O_{(i \times j)}]$ or $[O_{(i \times j)}] \cdot [B] = [O_{(i \times j)}]$	Multiplicative Property of 0
$m[A]$ or $n[B]$ are also $i \times j$ matrices	Closure Property of Multiplication

Looking Ahead 2.7

In this section, we will solve a system of equations using matrices. One method you learned in the last module for systems of two equations with two variables was called the elimination method. When matrices are used, this method is called the row reduction method. You have already used matrices and the properties of matrix addition to find missing variables when there is one unknown variable in an equation.

Example 1: Use the properties of matrix addition to find the values of  $x$  and  $y$ .

$$\begin{bmatrix} 2x + 3 \\ y - 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

A system of equations in standard form can also be written as a matrix equation by placing the coefficients of the  $x$  and  $y$  variables in one matrix being multiplied by the matrix  $\begin{bmatrix} x \\ y \end{bmatrix}$ .

Example 2: Write the system of two equations in two variables as matrix equations.

$$-3x + y = -10$$

$$x + 5y = 14$$

$$\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -10 \\ 14 \end{bmatrix}$$

The two matrices on the left can be combined to one matrix called an augmented matrix. The coefficients and variables on the left are separated from the constants on the right by a line.

Example 3: Write the system of two equations in two variables as an augmented matrix.

$$-3x + y = -10$$

$$x + 5y = 14$$

We can eliminate  $y$  in the first equation and solve for  $x$ . If  $1x + 0y = m$ , then  $x = m$ . We can eliminate  $x$  in the second equation and solve for  $y$ . If  $0x + 1y = n$ , then  $y = n$ . The solution matrix looks like this:

$$\left[ \begin{array}{cc|c} 1 & 0 & m \\ 0 & 1 & n \end{array} \right]$$

The solution matrix on the left must have all 1's along the diagonal going top left to bottom right and zeros everywhere else. The  $2 \times 2$  identity matrix is:

$$\left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$$

The solution to Example 3 is called the reduced row-echelon form because each cell of the diagonal from top left to bottom right is 1, and the remaining cells have entries of 0. The solution for  $x$  and  $y$  will be the numerical values for  $m$  and  $n$  on the right side of the augmented matrix.

The rules for equations also work for row operations in a matrix since the equations are written in matrix form. You will need to apply these rules and the method of elimination to reduce the rows to row-echelon form where the solution will be found.

*Matrix Row Operations*

1. Every number in a row can be multiplied (or divided) by any number except zero.
2. Rows can be added by adding the numbers of one row to the corresponding numbers of another row.
3. A multiple of any number in one row can be multiplied by the numbers in another row.
4. Two rows can be interchanged.

When these operations are applied to a matrix, the system of equations of the resulting matrix is equivalent to the original system of equations.

**Example 4:** Use the matrix row operations and method of elimination to solve the system of two equations and find the solution for  $x$  and  $y$ .

$$-3x + y = -10$$

$$x + 5y = 14$$

**Step 1:** Write the system of equations as an augmented matrix.

**Step 2:** Multiply Row Two by 3 and add it to Row One so that  $c_{2,1}$  becomes 0.

**Step 3:** Divide the second row by 16 to solve for  $y$ .

**Step 4:** Multiply Row Two by -1 and add Row One so that  $c_{1,2}$  becomes 0.

**Step 5:** Divide Row One by -3 for  $c_{1,1}$  to become 1 and solve for  $x$ .

When the final matrix has 1's along the diagonal going top left to bottom right and 0's in the other two cells, the solution is in the far-right column.

**Check:**

Section 2.8 The Identity MatrixLooking Back 2.8

The fraction  $\frac{1}{4}$  is the multiplicative inverse of 4 because  $\frac{1}{4} \cdot 4 = 1$ . When you multiply any number by its multiplicative inverse, you get 1, which is the identity element for multiplication. Therefore, when you multiply a matrix by its inverse matrix, if it exists, then you get the identity matrix.

The reason  $\frac{1}{4} \cdot 1 = \frac{1}{4}$  is because any number multiplied by the identity element for multiplication is the number itself. The number remains the same; it does not change its identity when multiplied by the identity element. Therefore, when you multiply a matrix by the identity matrix, which will be called  $[I]$ , the matrix does not change; again, it remains the same.

$$[A] \cdot [I] = [A]$$

Looking Ahead 2.8

Using the row reduction method, the 2 x 2 identity matrix in the previous section became the following:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The 1's are along the diagonal from upper left to lower right and the 0's are in the remaining cells.

Example 1: Multiply the matrix  $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$  by the 2 x 2 identity matrix.

$$[I] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 2: Find the variables  $w, x, y, z$  that make the matrix multiplication true.

$$\begin{bmatrix} 1 & 3 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 4 & 3 \end{bmatrix}$$

Example 3: Find the variables  $a$ ,  $b$ ,  $c$ ,  $d$ ,  $e$ , and  $f$  that make the matrix multiplication true.

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 1 & 5 \end{bmatrix}$$

Example 4: Find the variables  $a$ ,  $b$ ,  $c$ ,  $d$ ,  $e$ , and  $f$  that make the matrix multiplication true.

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 1 & 5 \end{bmatrix}$$

Please note: All identity matrices are square matrices. Why do you think that is the case?

Section 2.9 Inverse MatricesLooking Back 2.9

Inverses undo the effect of any mathematical changes. The inverse of 3 is -3 when adding because -3 is the opposite of 3;  $3 + (-3) = 0$ , and 0 is the identity element for addition. Adding inverses gives the identity element for addition.

The inverse of 3 is  $\frac{1}{3}$  when multiplying. This is because 3 and  $\frac{1}{3}$  are reciprocals;  $3 \cdot \left(\frac{1}{3}\right) = 1$ , and 1 is the identity element for multiplication. Multiplying inverses gives the identity element for multiplication.

This is also true for matrix multiplication. Multiplying a matrix by its inverse equals the identity matrix.

The inverse matrix for  $[A]$  is written  $[A]^{-1}$  and is read, "the inverse matrix of  $A$ ."

Therefore...

$$[A][A]^{-1} = [I]$$

And ...

$$[A]^{-1}[A] = [I]$$

Only square matrices can have inverses because both products  $[A][A]^{-1}$  and  $[A]^{-1}[A]$  must exist and give the identity matrix. In the last section, we found that the identity matrix must be a square matrix.

Looking Ahead 2.9

Example 1: Find the inverse matrix for  $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ .

The inverse matrix must also be a 2 x 2 square matrix that equals  $[I]$  when multiplied by the following matrix:

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 2: Find the inverse of the following matrix:

$$\begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2a + 3c & 2b + 3d \\ 5a + 6c & 5b + 6d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 3: Find  $\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}^{-1}$

This is also asking for the inverse, but inverse notation is used rather than words. Check your solution when finished.

2.10 Determinants and Inverse MatricesLooking Back 2.10

Not all square matrices have inverse matrices. For example,  $\begin{bmatrix} 3 & 3 \\ 5 & 5 \end{bmatrix}$  does not have an inverse matrix.

If we used the same process as in the last section, we will try to find an inverse matrix,  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , such that when it is multiplied by  $\begin{bmatrix} 3 & 3 \\ 5 & 5 \end{bmatrix}$  the product is  $[I]$ , the identity matrix  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ .

Let's try it.

?

$$\begin{bmatrix} 3 & 3 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

?

$$\begin{bmatrix} 3a + 3c & 3b + 3d \\ 5a + 5c & 5b + 5d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

When we set up the equations to solve for the variables, we find there is no solution.

$$\begin{array}{ll} 3a + 3c = 1 & 3b + 3d = 0 \\ 5a + 5c = 0 & 5b + 5d = 1 \end{array}$$

When you solved systems of two equations with two first degree variables in Algebra 1 and the lines were parallel, no solution was the answer.

Looking Ahead 2.10

The inverse of a square matrix is shown below.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{d}{ad - bc} & \frac{-b}{ad - bc} \\ \frac{-c}{ad - bc} & \frac{a}{ad - bc} \end{bmatrix}$$

This can only be true when  $ad - bc \neq 0$ . If it were equal, we would be dividing by 0 in each of the element cells, which is not possible.

Example 1: Show that given a 2x2 matrix,  $[A][A]^{-1} = [I]$  and  $[A]^{-1}[A] = [I]$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} =$$

Example 2: Find the inverse of  $\begin{bmatrix} 1 & 2 \\ 6 & 5 \end{bmatrix}$ .

Let  $a = 1$ ,  $b = 2$ ,  $c = 6$  and  $d = 5$ .

If  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , then the determinant of A is  $ad - bc$ . A determinant is abbreviated det. A and looks like matrix brackets, but with only straight lines (which means it looks more like absolute value bars!).

$$\det. A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

If the determinant of any matrix is zero, then there is no inverse matrix.

Example 3: Evaluate  $\begin{vmatrix} -3 & 0 \\ 4 & -5 \end{vmatrix}$ . To evaluate means to find the value of the determinant.

The determinant of a 3 x 3 matrix is more complicated because the diagonal lines go outside of the matrix and wrap around.

$$\text{Let } B = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$\det. B = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} - \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 - a_2 b_1 c_3 - a_1 b_3 c_2$$

Example 4: Find the determinant of M if  $M = \begin{bmatrix} 4 & 3 & -1 \\ 1 & -1 & 3 \\ 2 & 5 & 6 \end{bmatrix}$ . This means  $\begin{vmatrix} 4 & 3 & -1 \\ 1 & -1 & 3 \\ 2 & 5 & 6 \end{vmatrix}$  is a single numerical value.

Now that you know how to find determinants and what they are, there is a much simpler way to find inverses when given a matrix.

$$\text{Let } N = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

All the denominators are the determinant so scalar multiplication could be used to write a new theorem as shown below:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \frac{1}{\det N} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 5: Find the inverse of  $\begin{bmatrix} 3 & 5 \\ 1 & 8 \end{bmatrix}$  using the new theorem.

$$a = 3 \quad b = 5 \quad c = 1 \quad d = 8$$

$$\begin{vmatrix} 3 & 5 \\ 1 & 8 \end{vmatrix} = 3(8) - 5(1) = 24 - 5 = 19$$

Section 2.11 Using Technology with MatricesLooking Back 2.11

Throughout this module, you have added and subtracted matrices, multiplied matrices, and multiplied by scalars. In Geometry, you will learn that when multiplied by a matrix, scalars can dilate or shrink a figure if the matrix represents the vertices of a figure. Matrices are often transformed in just such a way to be used in computer graphics.

Now that you have learned about inverse matrices and how to solve systems of equations with matrices, we are going to use technology to help us. Using a graphing calculator will make finding inverses of 3 x 3 matrices and solving systems of three equations much easier and quicker.

It is important that you learned to do these things firstly by hand. In that way, you understand where things come from and what they mean. This understanding will help you input the data into the calculator correctly, so no errors occur.

To work with matrices, we will be using the Navpad® in the center of the calculator that can move right (→), left (←), up (↑), or down (↓) around the screen. The homepage (⏏) is where all the applications for the calculator are found. The “Enter” button is in the middle of the Navpad® or on the lower right-hand side of the calculator. The catalog, also on the right, is an open book that houses the templates, including those for matrices.

Looking Ahead 2.11

Move the Navpad® to #5 in the catalog to move to the 2 x 2 matrix template. Press “Enter.” This will appear:  $\begin{bmatrix} & \\ & \end{bmatrix}$ . You can use the “Tab” button to move around inside the matrix and around the screen. The fifth click of the “Tab” button will take you outside the matrix. Press the left arrow of the Navpad® to get back in.

Enter the cell elements one by one and press “Tab” after each to move to the next cell or move around using the Navpad®.

$$\begin{bmatrix} 2 & 1 \\ 3 & -5 \end{bmatrix}$$

The minus sign (−) is on the black key near the plus sign (+). The negative sign is on the key with the numbers next to the period. It is a gray key. We will be using the calculator to solve the matrix problems.

Example 1: Multiply $\begin{bmatrix} 2 & 1 \\ 3 & -5 \end{bmatrix} \cdot \begin{bmatrix} 1 & -4 \\ 3 & 0 \end{bmatrix}$
---

Once you have inserted all the elements in the left matrix, press “tab” or the right arrow on the Navpad® (→) to move outside or to the right of the first matrix. Press the “X” sign for multiplication (next to division) to let the calculator know what you intend to do.

Go back to the catalog to retrieve a second 2x2 matrix template. Press the catalog button. Move to the matrix  $\begin{bmatrix} & \\ & \end{bmatrix}$ . Click “Enter.” Fill in the cells one by one using the “Tab” button and press “Enter.” If you put in an incorrect number, move to the right of it and press the delete button ( $\overset{del}{\leftarrow}$ ) to clear it. Press “Enter”, and the solution matrix will appear on the right side of the screen.

$$\begin{bmatrix} 2 & 1 \\ 3 & -5 \end{bmatrix} \cdot \begin{bmatrix} 1 & -4 \\ 3 & 0 \end{bmatrix} \qquad \begin{bmatrix} 5 & -8 \\ -12 & -12 \end{bmatrix}$$

Example 2: Multiply the scalar (or scale factor) 2 by the matrix  $\begin{bmatrix} 1 & -4 \\ 1 & 2 \end{bmatrix}$ .

On the calculator page, type in “2” followed by the multiplication button. A dot will appear on the screen. Press “Catalog,” click on the 2x2 matrix template, and hit “Enter.” It will appear on the home screen next to the 2.

$$2 \cdot \begin{bmatrix} \phantom{0} & \phantom{0} \\ \phantom{0} & \phantom{0} \end{bmatrix}$$

Insert the numbers using “Tab” on the Navpad® to move from cell to cell and press “Enter.” Be sure and use the negative sign (-), not the subtraction sign (−), in front of the 4.

$$2 \cdot \begin{bmatrix} 1 & -4 \\ 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & -8 \\ 2 & 4 \end{bmatrix}$$

Example 3: Find the inverse of  $\begin{bmatrix} 2 & 1 \\ 3 & -5 \end{bmatrix}$ .

Input the matrix just as you have done previously. When the data is in each of the four cells press “Tab” or the right arrow (→) or the Navpad® to move to the right of the matrix. Press the up arrow to insert an exponent. Press the negative sign and 1. Press “Enter”, and the inverse will appear on the right side of the screen.

$$\begin{bmatrix} 2 & 1 \\ 3 & -5 \end{bmatrix}^{-1} \qquad \begin{bmatrix} \frac{5}{13} & \frac{1}{13} \\ \frac{3}{13} & -\frac{2}{13} \end{bmatrix}$$

Example 4: Multiply  $\begin{bmatrix} 2 & 1 \\ 3 & -5 \end{bmatrix}$  by its inverse. If it is correct, you should get the identity matrix.

$$\begin{bmatrix} 2 & 1 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} \frac{5}{13} & \frac{1}{13} \\ \frac{3}{13} & -\frac{2}{13} \end{bmatrix}$$

To insert the fractions, use the fraction template, which is found above the division sign key (÷). Press the control button on the left side firstly. The “Ctrl” button allows you to access the secondary commands that are above the keys. The primary commands, or default, are on the keys. Once you press “5” in the numerator, press “Tab” to move the denominator and press “13.”

$$\begin{bmatrix} 2 & 1 \\ 3 & -5 \end{bmatrix} \cdot \begin{bmatrix} \frac{5}{13} & \frac{1}{13} \\ \frac{3}{13} & -\frac{2}{13} \end{bmatrix}$$

You must insert the fraction template in each cell once you “Tab” there. Press “Ctrl” and “÷” each time and the fraction template will appear in the cell. If at any time you hit a button incorrectly and another screen appears, press the escape key at the top left (“Esc”). It will take you back to where you were. Don’t panic! You can always “Esc.” When complete, click “Enter.”

Example 5: Test to see if the identity matrix is its own inverse.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Follow the instructions to walk through the calculator steps to solve the [Practice Problems](#). Then go back to previous problems you did by hand and use the graphing calculator to make sure you get the same answers.

Section 2.12 Deciphering Codes Using Inverse MatricesLooking Back 2.12

You worked on Cryptograms in Algebra 1 by finding the numbers associated with words. The message became a mathematical problem.

Cryptology is the area of mathematics that creates ciphers, or codes, that have hidden meanings, and it often uses mathematical symbols. Cryptoanalysis deals with breaking codes and figuring out the message.

In earlier times, these were created and solved by hand. In present day, these are designed by computers and solved by using computers.

Lester Hill earned a bachelor's degree in mathematics from Columbia University in 1911 and then joined the US Navy Reserves in 1916 during World War I.

After the war, Lester earned his doctorate from Yale University. He published articles in 1926 and 1927 in the *Telegraph and Telephone Age* about detecting mathematical errors in code transmissions. He wrote about codes on checks.

In 1929 and 1931, he published articles in *The American Mathematical Monthly* about a code he had invented often referred to as the *Hill Cipher*. This cipher uses modular arithmetic and matrix algebra to encipher a string or block of letters. In 1932, Hill received a patent for his *Message Protector*.

Despite all of this, for most of his life, Lester Hill was a professor at Hunter College in New York City. Around age 60, Hill's health began to decline, and he died about a year later. Lester left behind him many advances in his fields of study.

After Hill's death, David Kahn, author of *The Codebreakers*, visited his widow for an interview. She gave Kahn many manuscripts of Lester's that she still had, and they are now housed at the National Cryptologic Museum in Maryland. This is the National Societies Agency (NSA) gateway to the public. Perhaps one day you may be able to visit there yourself!

In this section, we will use a modified form of *Hill's Cipher* with matrices to break codes.

Looking Ahead 2.12

There are 26 letters in the alphabet. Assign each letter a number.

$A = 1$	$N = 14$
$B = 2$	$O = 15$
$C = 3$	$P = 16$
$D = 4$	$Q = 17$
$E = 5$	$R = 18$
$F = 6$	$S = 19$
$G = 7$	$T = 20$
$H = 8$	$U = 21$
$I = 9$	$V = 22$
$J = 10$	$W = 23$
$K = 11$	$X = 24$
$L = 12$	$Y = 25$
$M = 13$	$Z = 26$

Example 1: What would be the coded message for the message "TIME IS ON?" Put the corresponding letters of the word in square matrices from left to right and top to bottom: *TIME* | *IS ON* in two 2 x 2 matrices.

$$\begin{bmatrix} T & I \\ M & E \end{bmatrix} \text{ and } \begin{bmatrix} I & S \\ O & N \end{bmatrix}$$

**Step 1:** Find the code number that is equal to each letter and substitute it in that cell of each matrix.

**Step 2:** The sender of a message and the receiver must know a coding matrix that is a secret between the two. Make a coding matrix with a determinant of + 1 or - 1.

**Step 3:** Multiply each number of the matrix for TIME IS ON by the coded matrix.

**Step 4:** Now change the numbers back to letters.

There seems to be a problem because 39, 46, and 47 are not on our chart. They are greater than 26. This is where an arithmetic sequence is helpful. Go back to the initial chart and start all over again.

<i>A</i> = 1     27	<i>N</i> = 14     40
<i>B</i> = 2     28	<i>O</i> = 15     41
<i>C</i> = 3     29	<i>P</i> = 16     42
<i>D</i> = 4     30	<i>Q</i> = 17     43
<i>E</i> = 5     31	<i>R</i> = 18     44
<i>F</i> = 6     32	<i>S</i> = 19     45
<i>G</i> = 7     33	<i>T</i> = 20     46
<i>H</i> = 8     34	<i>U</i> = 21     47
<i>I</i> = 9     35	<i>V</i> = 22     48
<i>J</i> = 10    36	<i>W</i> = 23     49
<i>K</i> = 11    37	<i>X</i> = 24     50
<i>L</i> = 12    38	<i>Y</i> = 25     51
<i>M</i> = 13    39	<i>Z</i> = 26     52

This is where Hill used Modulo Arithmetic;  $27 = 1 \pmod{26}$  since  $\frac{27-1}{26} = 1$ . Given the number 27 in a cycle of 26 numbers, the cycle is completed one time with 1 left over. The remainder is 1. Since 1 is A, 27 is also A. So, for the number 39,  $39 \div 26 = 1$ , remainder 13. Since 13 is M, 39 is M also.

The arithmetic sequence could also be used to find negative numbers.

Term Number:	-2	-1	0	1	$n$
A:	-51	-25	1	27	$26n + 1$
B:	-5	-24	2	28	$26n + 2$

Therefore, -51, -25, 1 and 27 are all coded to A and -50, -24, 2 and 28 are all coded to B.

**Step 5 (or Step 4 using Modulo Math):** Use the Modulo Math to change each of the numbers back to letters.

This is the message that you as the sender would send to the receiver. To decode or decipher the message, the receiver would use the reverse process of the previous four steps.

**Example 2:** Act as the receiver and decipher the code METSONMU.

**Step 1:** Put the message into two 2 x 2 matrices left to right and top to bottom.

$$\begin{bmatrix} M & E \\ T & S \end{bmatrix} \text{ and } \begin{bmatrix} O & N \\ M & U \end{bmatrix}$$

**Step 2:** Replace the letters of each matrix with the matching numbers.

**Step 3:** Each matrix would then be multiplied by the inverse coded matrix since the receiver is undoing the operations (or working in reverse).

**Step 4:** Now these numbers must be changed back to letters. The message is TIME IS ON!

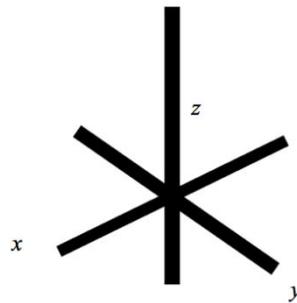
Section 2.13 Contour MapsLooking Back 2.13

In Module 1 of this text, you were introduced to the  $xyz$ -coordinate plane. You also solved systems of equations in three variables using substitution and elimination. In this module, you solved a system of three equations with three variables using matrices. In this final section, we will review the three-dimensional plane and how it can be evaluated on a two-dimensional surface, such as a topographical or contour map. The Practice Problems section today will be limited to building a contour map and locating certain parts of the landscape. It will make more sense when you have gone through the lesson.

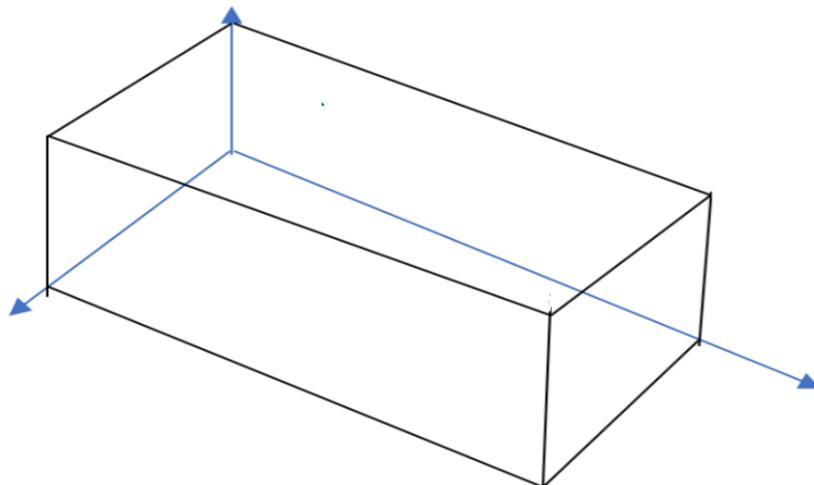
Topographic maps show large scale detail. Contour lines are used to show relief or height of a land surface.

In this lesson, you will use sticks to make a three-dimensional model, which is described below.

Put two sticks in the form of a cross on the floor. Put one standing upright where the two sticks on the floor intersect. If the upright stick continued through the floor, this would be an accurate representation of the three-dimensional  $xyz$ -plane.

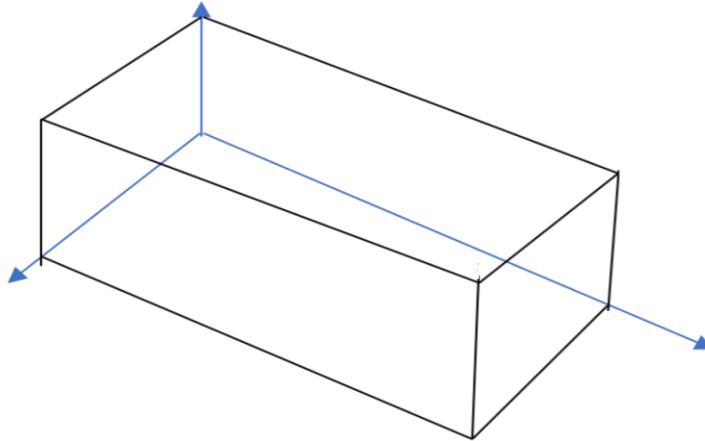


The corner of a room is a good example of the three-dimensional  $xyz$ -plane.

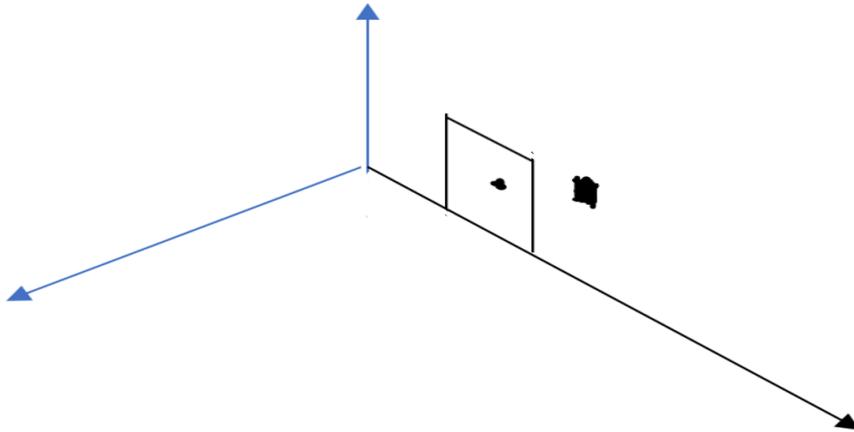


Let the corner by the door at Point D be the origin  $(0, 0, 0)$ . The room is 8 feet long, 6 feet wide and 4 feet high. It is a Hobbit's house!

Example 1: List the coordinates of all the corners of the Hobbit's house. Remember to move on the  $x$ -axis firstly, then on the  $y$ -axis (and those parallel to it), followed by the move on  $z$  (which can only move up or down).



Example 2: The light switch is 2 meters from the corner of the room nearest the door and is 1.5 meters above the floor. What is the ordered triple for the light switch?



Example 3: The doorknob is 0.5 meters to the left of the light switch. What is the ordered triple for the doorknob?

Example 4: What is the distance from Point D to Point F in the room?

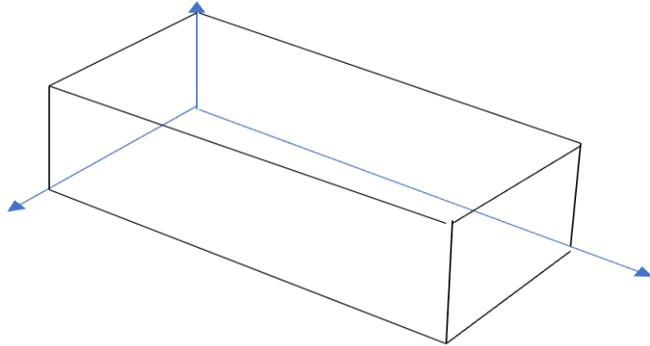
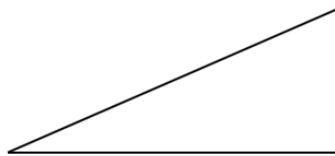
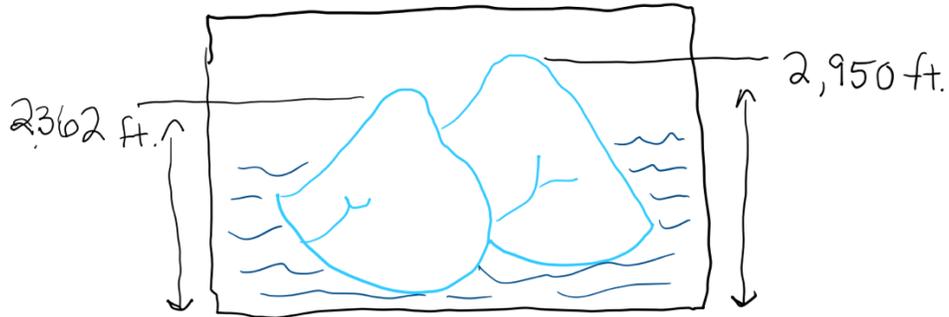


Figure DGF is a right triangle, so the Pythagorean Theorem and the distance formula will help to find the distance.



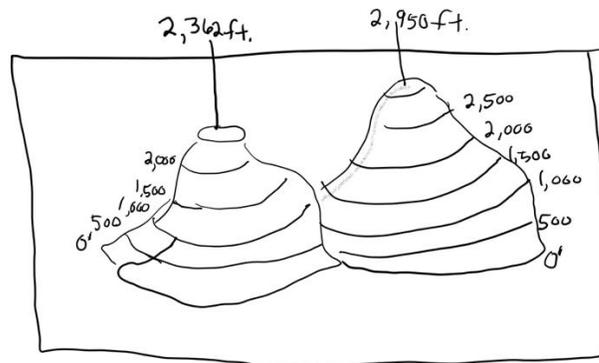
Looking Ahead 2.13

Now let us look at contour maps. Here is a side view of an iceberg in an ocean:



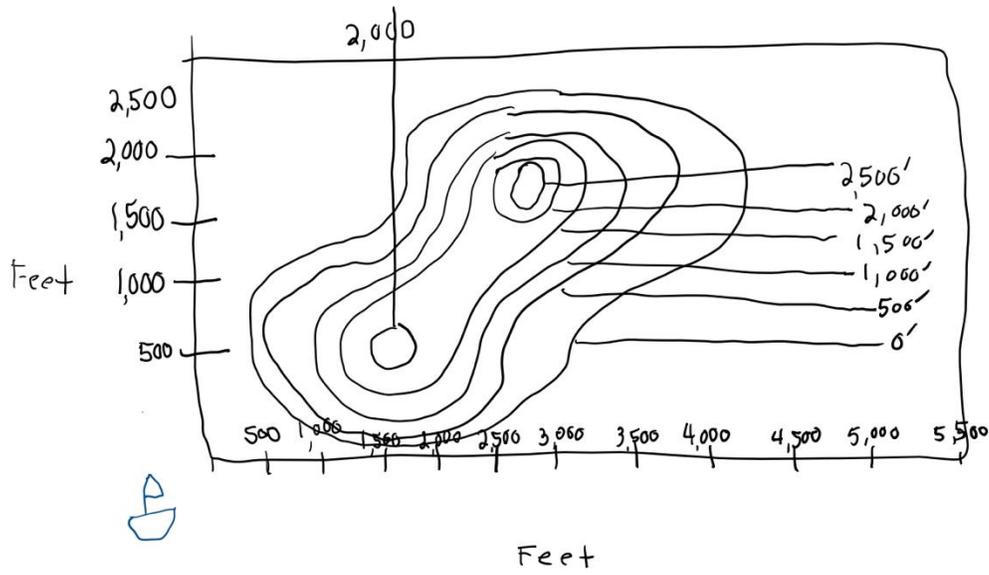
The highest peak is 2,950 feet above sea level. The elevation of the smaller tip of the iceberg is 2,362 feet above sea level. It is difficult to see where the water line and iceberg meet because three dimensions are drawn on a two-dimensional space.

Drawing contour lines helps us visualize how much of the iceberg would be visible if the water rose or fell.



The distance between contour lines is 500 feet. The symbol 0' is sea level. The elevation at the top of each peak is the highest point of the iceberg above sea level. There will also be some below sea level, which is not visible or located on the map.

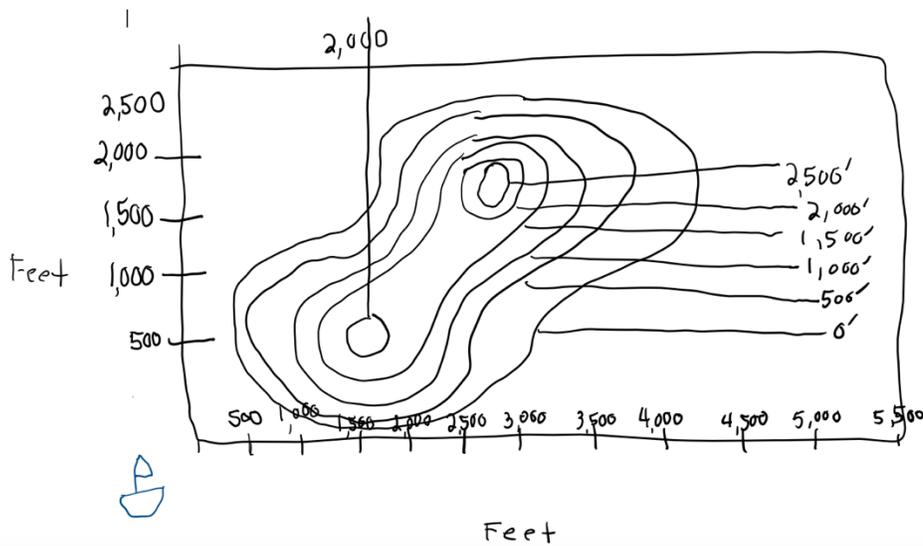
The best way to view the front, back, and side terrain, as well as the elevations of the iceberg is from an aerial view, which is from overhead and looks down upon the peaks. The contour lines all the way around each iceberg can be seen on this topographical map below:



Each contour line is the elevation that would be the z-coordinate in the xyz-plane.

If a boat is at  $(0, 0, 0)$  we could locate it and if the 500' distances were drawn on the x and y-axis, then the elevation for the ordered triple for the highest peak would be approximately  $(2,750, 1,750, 2,500)$ . From the boat, move 2,750 right on the x-axis, then 1,750 up on the paper following the y-axis. The contour line closest to it is 2,500! We know the height of the peak to be 2,950 feet so the point is more approximately  $(2,750, 1,750, 2,950)$ .

**Example 5:** Using the contour map in the figure below, find the ordered triple for the lower peak.



Today, instead of doing problems, you will build a three-dimensional contour map. You will need corrugated cardboard and large graph paper. Instructions are given in the [Practice Problems](#) section.