Module 4 Squares and Square Roots

Section 4.1 Exponents Revisited

Looking Back 4.1

We have learned about exponents, square roots, and cube roots in previous modules. Moving forward, we used exponents, square roots, and cube roots with operations, equations, and inequalities.

In this module, we will investigate other roots and dig deeper with radicals. Let us begin with a review of exponents. There are names for the parts of an exponential expression:

Exponent

Base → $7^3 = 343$ ← **Standard Form Solution**

The expression 7^3 means that 7 is multiplied by itself 3 times as in $7 \times 7 \times 7$. The base tells us what we are multiplying and the power tells us how many times we are multiplying the base. When a number is raised to the second power it is said to be "squared." If a base is raised to the third power it is said to be "cubed." Beyond cubed, we say a number is raised: "to the 4th power; to the 5th power; to the 6th power," and so on.

Looking Ahead 4.1

Example 1: Do you get the same solution when you simplify $(-1)^2$ and -1^2?

Example 2: What do you notice when you simplify the exponents below?

$-(3)^4$ $(-4)^2$ $(-2)^6$

Example 3: What do you notice when you simplify the exponents below?

$(-3)^5$ $(-4)^3$ $(-2)^7$

When the base is negative and the exponent is even, the solution is positive. When the base is negative and the exponent is odd, the solution is negative.

Example 4: Simplify the exponents below.

1^0 25^0 $(-14)^0$ x^0

When the exponent is 0, the solution is always 1 for any variable and for any real number except 0^0 because 0^0 is indeterminate.

Example 5: Simplify the expressions below using the properties of exponents.

a) $x^2 \cdot x^5$

b) $y^4 \cdot y^3 \cdot y$

c) $m^3 \cdot m^3 \cdot m^3$

d) $q^3 \cdot r^3$

When the bases are the same, we add the exponents. Do not forget that $y = y^1$ and $x = x^1$. If there is no exponent, the expression is to the 1st power.

Example 6: Simplify the expressions below using the properties of exponents.

a) $t^2 \cdot t^{-3}$

b) $t \cdot t \cdot t^{-2}$

c) $m^{-4} \cdot m$

d) $x^{-1} \cdot y^{-1} \cdot z^{-1}$

Example 7: Simplify the expressions below using the properties of exponents.

a) $(t)^3$

b) $(m^3)^2$

c) $(x^3)^4$

d) $(x^1y^1)^2$

The power to a power rule tells us that when you are finding the power of an exponential, you multiply each exponent being multiplied inside the parenthesis by the exponent outside the parenthesis.

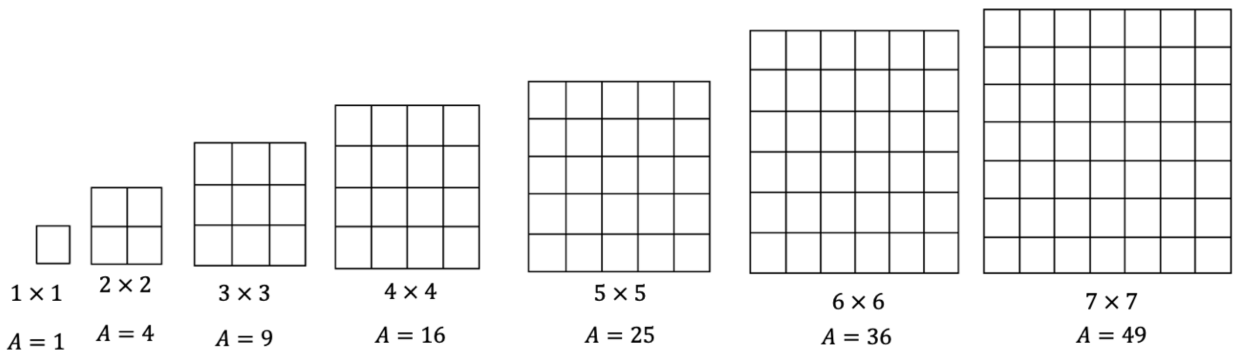
Section 4.2 Perfect Squares

Looking Back 4.2

In this section, we will combine what we know about factors and exponents to investigate perfect squares. A square has a length equal to its width. A perfect square is a number that can be expressed as the product of an integer by itself or as the second exponent of an integer. In a perfect square, the sides are positive integers.

We know how to find the area of different figures. To find the area of a square, we multiply side by side and both sides are the same, so we can write it as "s^2" in which s represents the side.

Looking at the squares below, we find that if the side is 1 square unit, the area is 1^2, which is equal to 1; if the side is 2 square units, the area is 2^2, which is equal to 4; if the side is 3 square units, the area is 3^2, which is equal to 9; etc.



Looking Ahead 4.2

A perfect square is a number with two identical factors. One factor pair repeats. For example, the factors of 18 are: 1, 2, 3, 6, 7, 9, 18. We can stop checking factor pairs at 6 because it pairs with 3 to multiply and result in 18; we already have 3 and 6.

The factor pairs of 16 are: $1, 2, 4, 8, 16$. We can stop checking as 4 because it pairs with itself to result in 16.

Below is a list of perfect squares from 1 through 625:

$1^2 = 1$ $\quad 2^2 = 4$ $\quad 3^2 = 9$ $\quad 4^2 = 16$

$5^2 = 25$ $\quad 6^2 = 36$ $\quad 7^2 = 49$ $\quad 8^2 = 64$

$9^2 = 81$ $\quad 10^2 = 100$ $\quad 11^2 = 121$ $\quad 12^2 = 144$

$13^2 = 169$ $\quad 14^2 = 196$ $\quad 15^2 = 225$ $\quad 16^2 = 256$

$17^2 = 289$ $\quad 18^2 = 324$ $\quad 19^2 = 361$ $\quad 20^2 = 400$

$21^2 = 441$ $\quad 22^2 = 484$ $\quad 23^2 = 529$ $\quad 24^2 = 576$

$25^2 = 625$

When we square a number, or multiply a number by itself, we get a perfect square. The perfect square is the area (or inside) of the square in square units.

Example 1: Make a square made up of 1 by 1 squares that has 81 units. How many rows are there and how many columns are there in your square?

Example 2: Below is a multiplication table from 1 to 25. Take a look at the diagonal going from top left to bottom right. What do you notice about the numbers along the diagonal?

X	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
2	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50
3	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60	63	66	69	72	75
4	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64	68	72	76	80	84	88	92	96	100
5	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100	105	110	115	120	125
6	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108	114	120	126	132	138	144	150
7	7	14	21	28	35	42	49	56	63	70	77	84	91	98	105	112	119	126	133	140	147	154	161	168	175
8	8	16	24	32	40	48	56	64	72	80	88	96	104	112	120	128	136	144	152	160	168	176	184	192	200
9	9	18	27	36	45	54	63	72	81	90	99	108	117	126	135	144	153	162	171	180	189	198	207	216	225
10	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190	200	210	220	230	240	250
11	11	22	33	44	55	66	77	88	99	110	121	132	143	154	165	176	187	198	209	220	231	242	253	264	275
12	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216	228	240	252	264	276	288	300
13	13	26	39	52	65	78	91	104	117	130	143	156	169	182	195	208	221	234	247	260	273	286	299	312	325
14	14	28	42	56	70	84	98	112	126	140	154	168	182	196	210	224	238	252	266	280	294	308	322	336	350
15	15	30	45	60	75	90	105	120	135	150	165	180	195	210	225	240	255	270	285	300	315	330	345	360	375
16	16	32	48	64	80	96	112	128	144	160	176	192	208	224	240	256	272	288	304	320	336	352	368	384	400
17	17	34	51	68	85	102	119	136	153	170	187	204	221	238	255	272	289	306	323	340	357	374	391	408	425
18	18	36	54	72	90	108	126	144	162	180	198	216	234	252	270	288	306	324	342	360	378	396	414	432	450
19	19	38	57	76	95	114	133	152	171	190	209	228	247	266	285	304	323	342	361	380	399	418	437	456	475
20	20	40	60	80	100	120	140	160	180	200	220	240	260	280	300	320	340	360	380	400	420	440	460	480	500
21	21	42	63	84	105	126	147	168	189	210	231	252	273	294	315	336	357	378	399	420	441	462	483	504	525
22	22	44	66	88	110	132	154	176	198	220	242	264	286	308	330	352	374	396	418	440	462	484	506	528	550
23	23	46	69	92	115	138	161	184	207	230	253	276	299	322	345	368	391	414	437	460	483	506	529	552	575
24	24	48	72	96	120	144	168	192	216	240	264	288	312	336	360	384	408	432	456	480	504	528	552	576	600
25	25	50	75	100	125	150	175	200	225	250	275	300	325	350	375	400	425	450	475	500	525	550	575	600	625

Example 3: Answer the questions below.

What is a repeat (identical) factor of the perfect square 36? Where do you find these numbers on the multiplication table?

What is a repeat (identical) factor of 25? Where do you find these numbers on the multiplication table?

What is a repeat (identical) factor of 81? Where do you find these numbers on the multiplication table?

Example 4:	Tell whether or not the numbers below are perfect squares and explain why.

a) 25

b) 18

c) 99

d) 100

e) -9

Example 5:	Tell whether or not the variable expressions below are perfect squares.

a) x^2

b) y^3

c) m^4

d) s^5

e) g^6

f) w^7

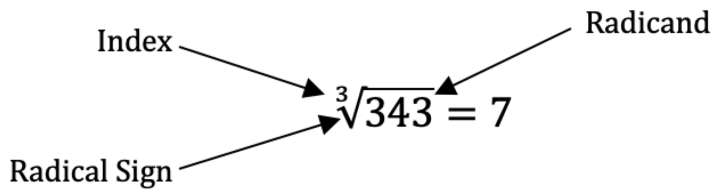
Just as squares are perfect, our heavenly Father is perfect. Deuteronomy 32:4 says: “His (God’s) work is perfect!” And God is at work perfecting us. Ephesians 2:10 says that: “We are His workmanship, created in Christ Jesus for good works.” Philippians1:6 says: “He who began a good work in us will perfect it until the day of Christ Jesus.”

Section 4.3 Perfect Squares and Their Square Roots

Looking Back 4.3

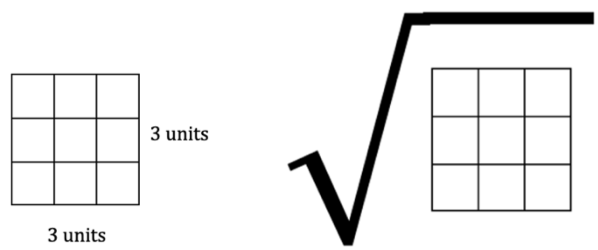
We found perfect squares by multiplying two identical factors together. We have seen that this works for numbers and variables. Sometimes, we want to work in reverse. Therefore, in this section, we are going to look back at those perfect squares from Section 4.2 and learn about the inverse of a perfect square. The inverse of a perfect square is called the square root.

Below are the names of the parts of a radical expression:



Looking Ahead 4.3

Below is a geometric representation of a perfect square number and a square root.



If there is no index number, it is a square root. For a square root we can write $\sqrt[2]{9}$ or $\sqrt{9}$ in which 2 is the index and 9 is the radicand. For the fourth root, we write $\sqrt[4]{16}$ in which 4 is the index and 16 is the radicand.

If we are finding a square root then we are looking for two identical factors. If we are finding a fourth root then we are looking for four identical factors. The index is the number of identical factors of the radicand.

Example 1:	What is the index number for each radical below?

a) A square root

b) A sixth root

c) An eighth root

d) A tenth root

Example 2: Name the index and radicand for each radical below.

a) $\sqrt{900}$

b) $\sqrt[2]{121}$

c) $\sqrt[4]{256}$

d) $\sqrt[6]{64}$

Example 3: How many identical factors are we looking for in each radicand below?

a) $\sqrt[5]{32}$

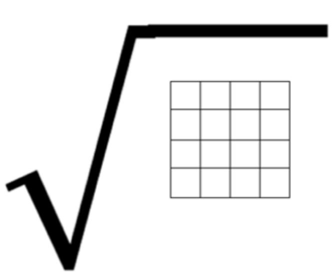
b) $\sqrt[3]{125}$

c) $\sqrt{169}$

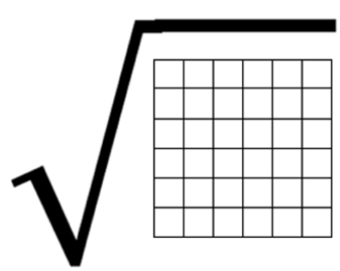
d) $\sqrt[4]{16}$

Example 4: Write the square roots below in mathematical notation and find the solution.

a)



b)



The principal square root of a number is the positive square root of the number. For example, $\sqrt{49} = 7$.

If there is a negative sign in front of the radical it is the opposite of the principal square root, which is negative. For example, $-\sqrt{49} = -7$.

If there is a "$\pm$" (read: "plus or minus") sign in front of the radical, that means there are both positive and negative solutions. For example, $\pm\sqrt{49} = \pm 7$.

Example 5: Find the square roots of the expressions below. Give the principal square root unless directed otherwise.

a) $\sqrt{81}$

b) $-\sqrt{144}$

c) $\pm\sqrt{64}$

d) $\sqrt{-100}$

Notice that the perfect square area under the radical is always positive and the square root that is the side length of that perfect square is always positive as well. However, a perfect square number can be found two ways. The first way is to multiply a positive number by the identical positive number, such as $10 \times 10 = 100$. The second way is to multiply a negative number by the identical negative number, such as $-10 \cdot -10 = 100$. Therefore, there are really two solutions to $\sqrt{81}$, which are $(+9)$ and (-9) or ± 9. However, there is only one solution if the principal square root is asked for: $(+9)$. Notice also that no two identical positive or negative integers when multiplied together will ever give a negative product under the radical.

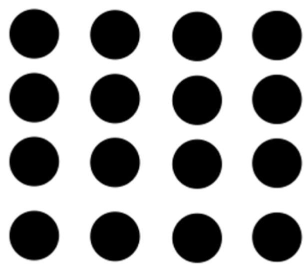
Section 4.4 Non-Perfect Squares and Square Roots

Looking Back 4.4

What if a number is not a perfect square number? Can we find the square root of it? In some cases, we need only to find an estimate of the square root of a number.

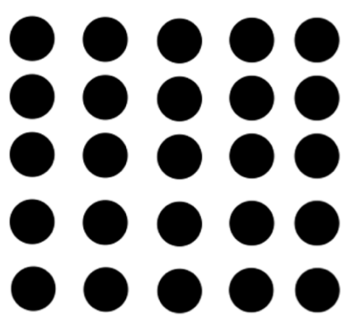
What do you think non-perfect squares look like? Let us call perfect squares "upright squares" and let us call non-perfect squares "tilted squares."

Example 1: Use the dot grid below to find a tilted (non-perfect) square that has 2 square units. Each horizontal or vertical line between dots is 1 unit. Each 1×1 square is 1 square unit.



What is the length of the side of a square with an area of two square units?

Example 2: Draw a non-perfect (tilted) square with 8 square units. What is the length of each side?



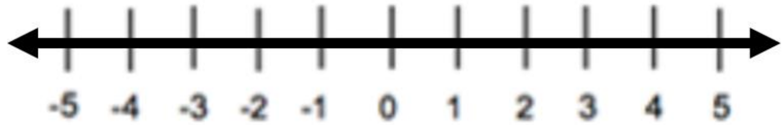
These side lengths are identical factors that are not rational but irrational numbers. The decimal portion does not repeat nor end; if the decimal portion does repeat or end, only God knows where that is!
We will explore these in more detail in Section 4.11 when we investigate approximating irrational square roots.

Looking Ahead 4.4

Let us make a square root ruler now. First, take a piece of paper and fold it in half longwise. Then fold it in half again. Mark lines from top to bottom every inch from left to right.

1. Fill in the blanks with the digits 0-10 along the first row.
2. Write each digit as a square for the second row.
3. Write each number as a square root radical with two solutions for the third row.
4. Find the two square roots of the radical for the fourth row.

Example 3: Write the perfect squares below the square root on the number line.



The integers on a number are the solutions to the square roots of perfect square numbers. We can use this to approximate the square roots of numbers between perfect square numbers.

Example 4: Use the number line and square root ruler to approximate $\sqrt{17}$.

Example 5: Use the number line and square root ruler to approximate $\sqrt{12}$.

Example 6: Answer the questions below for each of the expressions.

a) What two perfect squares would the expression lie between?
b) What integers would the expression lie between?
c) Give the decimal approximation for the number.

$\sqrt{37}$ $\sqrt{91}$ $\sqrt{50}$

Example 7: Graph and label the solutions from Example 3 on the number line below.

Section 4.5 Simplifying Non-Perfect Squares

Looking Back 4.5

When the square root symbol is used with a square that is not a perfect number, then it will have a decimal approximation for a solution. For example, $\sqrt{2}$ can be rounded to the decimal number 1.4 using the number line method from the previous section, or by using a calculator and rounding. The decimal value is an approximation, not an exact solution.

In this section, we are going to put together what we know about prime factorization and perfect squares to simplify square roots. The solutions will be exact solutions in simplified form. They will not be decimal approximations; therefore, the solutions will still have the square root symbol.

Looking Ahead 4.5

Example 1:	Simplify $\sqrt{26}$.

Example 2:	Simplify $\sqrt{48}$.

Notice that you can multiply numbers by numbers and roots by roots. You can multiply radicals by other radicals as long as the index number is the same in both: $\text{square roots} \times \text{square roots} = \text{square roots}$;
$\text{cube roots} \times \text{cube roots} = \text{cube roots}$, etc.

Example 3:	Simplify $\sqrt{18}$.

Because we are trying to find square roots, we are looking for perfect squares. If we are trying to find cube roots, then we would be looking for perfect cubes.

Example 4:	Simplify $\sqrt{75}$.

Example 5:	Simplify $\sqrt{200}$ and check your solution.

Example 6:	Simplify $\sqrt{x^4}$ and check your solution.

Example 7:	Simplify $\sqrt{y^3}$ and check your solution.

Example 8:	Simplify $\sqrt{m^4n^3}$ and check your solution.

Section 4.6 Numbers with Square Roots

Looking Back 4.6

It is difficult to see how small or large an exact form of a square root number really is until it is converted to a decimal approximation. With a decimal approximation such as 1.413 ... it is easy to see that it is between the integers 1 and 2.

It takes a little more thought to understand that $\sqrt{73}$ is between the integers 8 and 9. We must first know that $\sqrt{73}$ (the square root of 73) is between $\sqrt{64}$ (the square root of 64) and $\sqrt{81}$ (the square root of 81).

If we want to order square root numbers to compare them, it is easiest to first estimate the decimal approximation of the numbers. Once this is done, they can be ordered and then converted back to square roots in that order.

Looking Ahead 4.6

Example 1:	Order the numbers below from greatest to least.

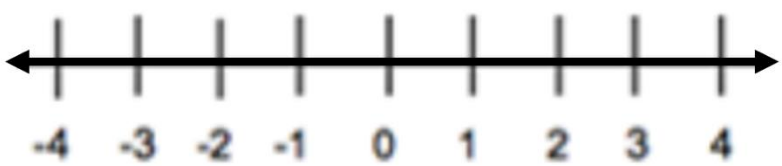
$\sqrt{5}$ $\frac{5}{3}$ $-\sqrt{14}$ -0.92

Example 2:	Order the numbers below from least to greatest.

$-\sqrt{3}$ $-\sqrt{15}$ $-\frac{9}{2}$ -1

Example 3:	Place the numbers below on the number line in their approximate position.

$-\sqrt{16}$ $\sqrt{2}$ 3.3 $-\frac{5}{2}$



Example 4:	Compare the numbers below using inequality or equality symbols $(>, <, =)$.

a) $2\sqrt{3}$ __________ $\sqrt{8}$

b) $\sqrt{16}$ __________ 1.4

c) $\frac{22}{8}$ __________ $\sqrt{11}$

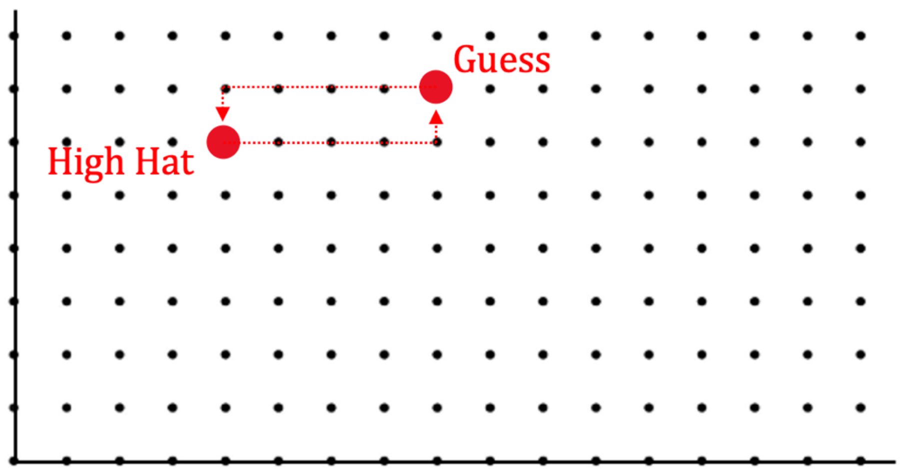
Section 4.7 Squares and the Coordinate Grid

Looking Back 4.7

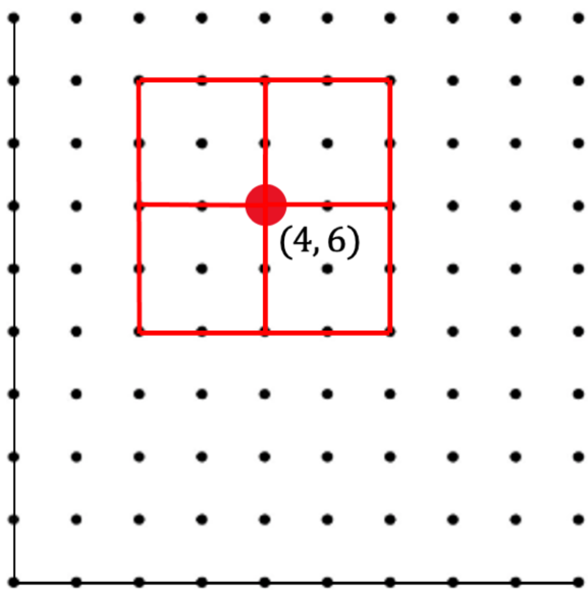
In Module 2, we played a game called "Where is High Hat Hiding?" In this game, your partner tried to guess where High Hat was hiding on a coordinate grid and if the point guessed was not correct, you gave them directional hints as to where High Hat was hiding in relation to the point guessed. If you gave them a hint as to how far the guess is from High Hat horizontally and how far the guess is from High Hat vertically, the next guess should be the point where High Hat is hiding!

In this section, it will be helpful to have colored pencils. We will be using centimeter grid squares to find lengths and areas of shapes and translating those to square roots. An understanding of square roots is needed to understand the Pythagorean Theorem.

Example 1: High Hat is hiding at $(4, 6)$ on the coordinate grid. Your partner guessed $(8, 7)$. How many spaces would you tell your partner to move horizontally and vertically to get from the guess to High Hat?



Example 2: How many square houses with an area of 4 square units could High Hat build if the point where he is must be one of the corners of the house?



High Hat could build 4 different houses with an area of 4 square units using his spot as one the four corners. Now, find the ordered pair for each corner of each house:

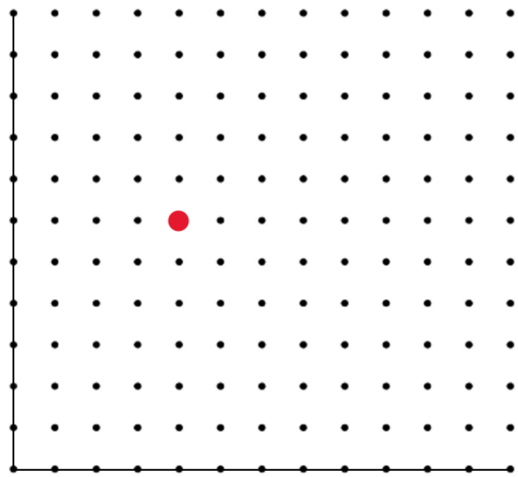
Top right corner:

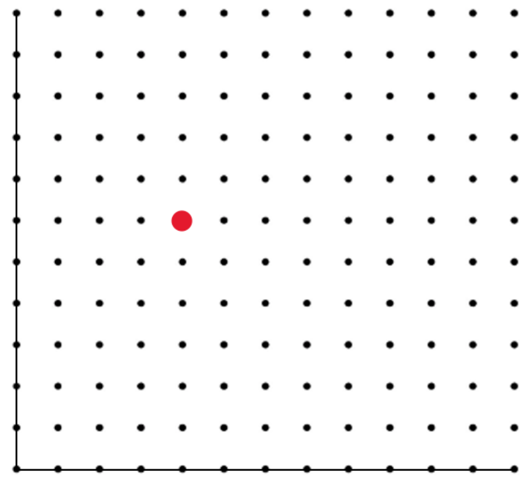
Top left corner:

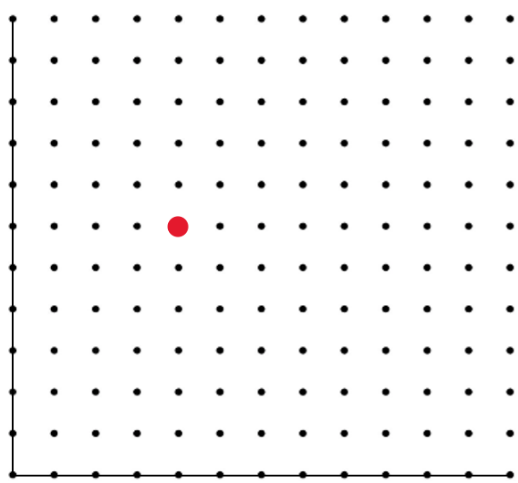
Bottom right corner:

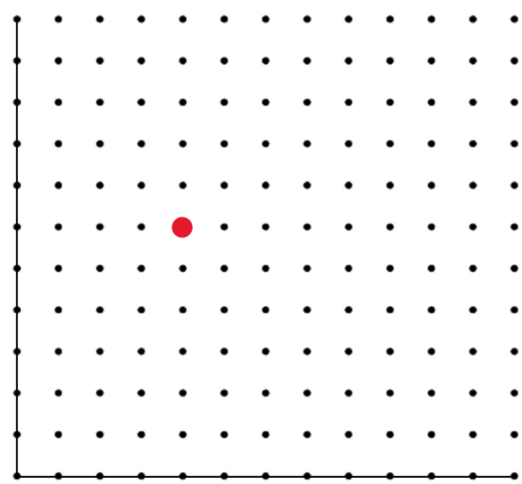
Bottom left corner:

Example 3: How many rectangular houses of 4 square units could High Hat build if the point where he presently is at, $(4, 6)$, has to lie on one of the sides? Do not include squares, just rectangles that are not squares.









Section 4.8 Finding Area Using the Chop Strategy

Looking Back 4.8

In the previous section, we found the number of High Hat's houses that were either rectangular or squares. In Problem 10 of the previous Practice Problems section, you found the area of city blocks with High Hat's distance walked.

In this section, we are going to find the area of different shapes by chopping them up into smaller shapes. The shapes we will use are those with areas whose formulas we know. Then we can add all the areas together. We will watch a video to see how to derive the areas of squares, rectangles, parallelograms, and triangles. We are going dot grid paper and the formulas below to find the areas of the various shapes. However, first let us watch a short video to review where these formulas come from.

Square: $A = s^2$

Rectangle: $A = l \cdot w$

Parallelogram: $A = b \cdot h$

Triangle: $A = \frac{1}{2} b \cdot h$ or $\frac{b \cdot h}{2}$

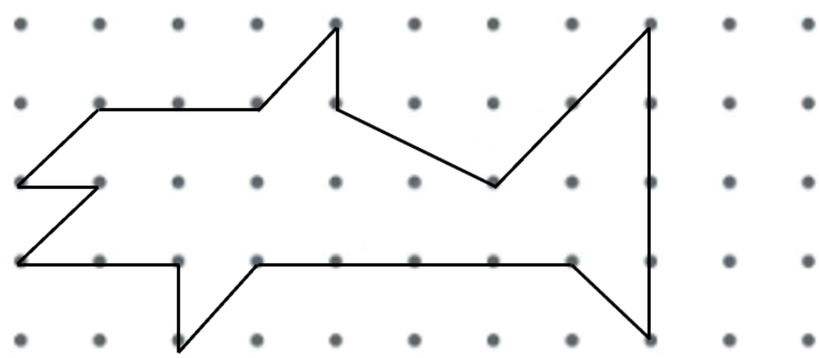
Trapezoid: $A = \frac{1}{2}(b_1 + b_2)h$ or $\frac{(b_1+b_2)h}{2}$

We will also use the T*i*-*n*spire® graphing calculator to use technology to see how the formulas relate to one another.

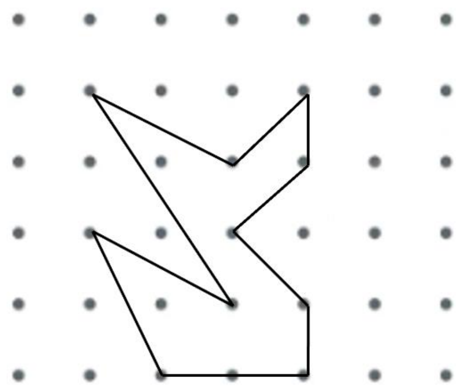
On the dot paper, there is 1 centimeter between dots horizontally and vertically. We are going to call this distance one unit. The distance between diagonal dots is longer than this and we will see how to estimate this distance in Section 4.10. If you make the smallest square possible on the grid using four dots, then you have one square unit.

Looking Ahead 4.8

Example 1:	Use the chop strategy to find the area of the shark below.



Example 2: Below is a shark standing on its head. Use the chop strategy to find the area of the shark below. There are two odd shapes at the sharks' head and tail fin (caudal fin). What strategy will you use to find these areas?



Section 4.9 Finding Area Using the Subtraction Method

Looking Back 4.9

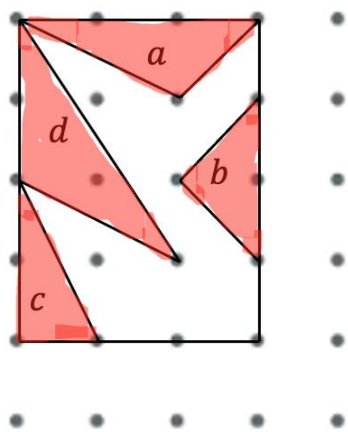
In the previous section, we looked at finding the area of a shark standing on its head. The chop strategy did not seem to be the best method to do this. The subtraction method, which we will learn in this section, works much better for a problem such as this.

Using the subtraction method, we could surround the shark with a shape for which we know the total area formula and break areas outside the shark and inside the shape into other figures for which we know the area formulas, and then subtract them from the total area of the surrounding shape. What will be left is the area of the shark.

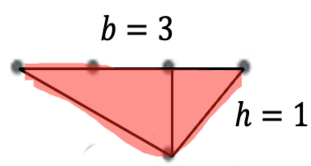
For the shark, we need to use the formula for the area of a parallelogram, which we reviewed last year: $A = b \times h$; but the height is not the slant height. The height is the altitude, which is the line from a vertex perpendicular to the base on the opposite side of the vertex.

Looking Ahead 4.9

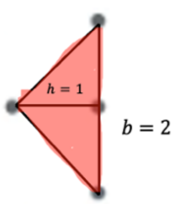
Surround the shark with a rectangle and shade the areas that are outside the shark. If we take the entire area of the surrounding rectangle and subtract out the shaded portion, we will be left with the area of the shark.



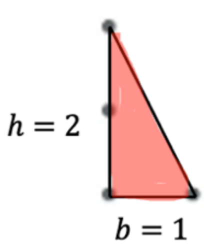
a)



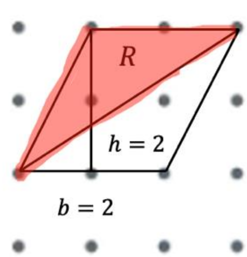
b)



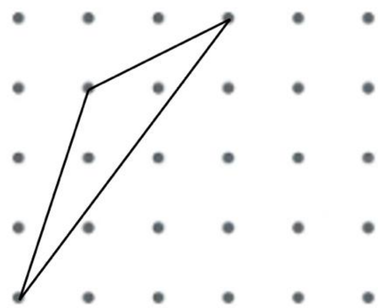
c)



d) Turn d sideways to view it as a parallelogram:

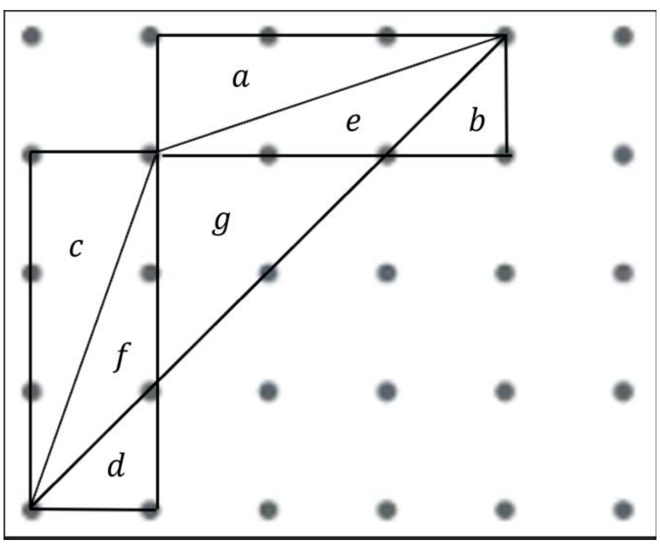


Example 1: Find the area of the triangle below using the subtraction method.



Example 2: Another method rather than surrounding the entire area around the shape is to just surround portions and then use the subtraction method for each of the portions. The shape below is different than the one used in Example 1 and will have a different area. Find the area of triangle *e*-*f*-*g*.

Instead of surrounding the entire area of the triangle, just surround the top portion (a and b) and side (c and d).



Section 4.10 Irrational Square Roots

Looking Back 4.10

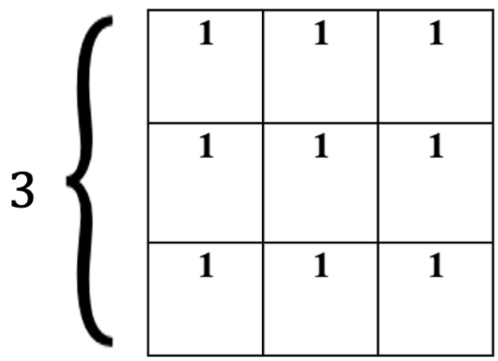
We learned a little bit about roots when we studied exponents in Section 1 of this module. The symbol to indicate a root is a radical sign ($\sqrt{}$) and $\sqrt[n]{b}$ is read: "the nth root of b." The cube root of 27 is written $\sqrt[3]{27}$. Three is the cube root of 27 because 3^3 (read: "three cubed") is $3 \cdot 3 \cdot 3$, which equals 27. If $b = a^3$, then $\sqrt[3]{b} = a$. These are inverse operations.

$$\text{If } \sqrt[3]{b} = a$$
$$(\sqrt[3]{b})^3 = a^3$$
$$\text{Then } b = a^3$$

The Golden Rule of Algebra says that whatever you do to one side of an equation, you must do to the other. So, you can cube the left side of an equation as long as you cube the right side as well. The symbol $\sqrt{1}$ (square root) actually means $+\sqrt[2]{1}$. If there is no number in front of the upper left-hand corner of the root sign, it is a square root. If there is no sign out front, it is the positive square root. We will be working with positive square roots only in this module because they are used in the Pythagorean Theorem that we will learn about in the next module.

We previously learned the positive square root of a number that is a perfect square number is the side length of a square. The number under the radical sign, called a radicand, is the area of the same square. Therefore, $\sqrt{9} = 3$ because $3 \cdot 3 = 9$. If a square has an area of 9, then the length of the side is $\sqrt{9}$ (which equals 3). The area of a square is found using
$A = s^2$ in which A is the area and s is the side length. In this example, $A = 9$ and $s = 3$.

The side length is a whole number for perfect squares:



s	A
side	(s^2)
1	1
2	4
3	9
4	16

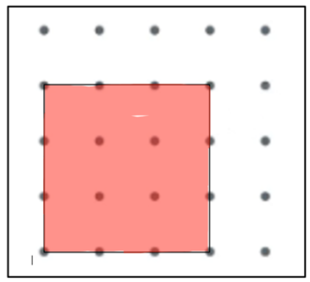
The length of the side of a square can be a decimal number that is irrational; for example, the square root of 2 or ($\sqrt{2}$). This is the side length of a square whose area is 2. We will start with an activity to see what this square root looks like and how we can find the decimal approximation of $\sqrt{2}$; $\sqrt{2}$ is an exact solution, the decimal form is an approximation.

<u>Looking Ahead 4.10</u>

On the centimeter grid paper, the distance between any two dots horizontally or vertically is 1 cm. We will call this 1 unit for our purposes. Connecting two dots diagonally is a little longer than 1 unit (1cm.).

Example 1: Find all the squares that can be drawn on a 4×4 grid (read: "four by four"). These can be in standard position (upright) or non-standard position (titled). Because they are squares, the sides must all be equal. There are eight. We will begin with an activity below to help you find all eight.

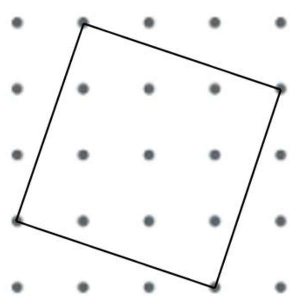
Below is a 3×3 square drawn inside a 4×4 grid. That is a standard square in upright position. It has an area of 9 square units and a side length of 3 units.



Because 9 is a perfect square number, the square root of 9 is a rational number.

In Section 4, we first investigated squares with irrational side lengths. Now, we will explore these more in depth.

Below is the largest non-standard square in a tilted position, which can be drawn on a 4×4 grid. It has an area of 10 square units and a side length of $\sqrt{10}$ units.



Because 10 is not a perfect square number, the square root of 10 is an irrational number. It does not end nor repeat. So, we leave it in exact form. Find six more squares that can be draw in a 4×4 array. Three will be tilted and three will be upright.

$A =$

$A =$

$A =$

$A =$

$A =$

$A =$

In the next section, we will investigate finding the decimal approximations for square roots or the side lengths.

Section 4.11 Approximating Square Roots

Looking Back 4.11

The square root of 9 or ($\sqrt{9}$) simplifies to 3 because 3^2 is $3 \cdot 3 = 9$. This means a square with an area of 9 square units has a side length of 3 units. The exact solution is 3.

The square root of 10 or ($\sqrt{10}$) does not simplify. The exact solution is $\sqrt{10}$ because $\sqrt{10} \cdot \sqrt{10} = \sqrt{10^2} = \sqrt{100} = 10$. This means a square with an area of 10 square units has a side length of $\sqrt{10}$ units. The exact solution is $\sqrt{10}$.

The square root of 20 simplifies to $\sqrt{4 \cdot 5} = \sqrt{4} \cdot \sqrt{5} = 2\sqrt{5}$. We can check this using $(2\sqrt{5}) \cdot (2\sqrt{5})$, which is $(2)(2) \cdot \left(\sqrt{5}\right)\left(\sqrt{5}\right) = 4\sqrt{25} = 4 \cdot 5 = 20$. This means a square with an area of $\sqrt{20}$ has a side length of $2\sqrt{5}$. The exact solution is $2\sqrt{5}$.

Let us investigate why this works. The square root of 5 multiplied by the square root of 5 or $((\sqrt{5}) \cdot (\sqrt{5}))$ is equal to $(\sqrt{5})^2$; that is simply 5 because the square root of anything squared is itself. We could also write this as $(\sqrt{5})^2 = \sqrt{5^2} = \sqrt{25} = 5.$ Both ways result in the same solution. Therefore, as stated above, $(2)(2) \cdot \left(\sqrt{5}\sqrt{5}\right) = 4 \cdot 5 = 20.$

It can get complicated but as you can see, there are several ways to do the mathematics and get the correct solution. Let us review simplifying square roots before reviewing approximating square roots.

Example 1: Simplify $\sqrt{32}$. Find the perfect square factors of 32 first and write the solution in exact form. Use your square root ruler from Section 4.4 and the prime factorization method from Section 4.5.

Example 2: Simplify $\sqrt{15}$. Write the solution in exact form; do not find the decimal approximation.

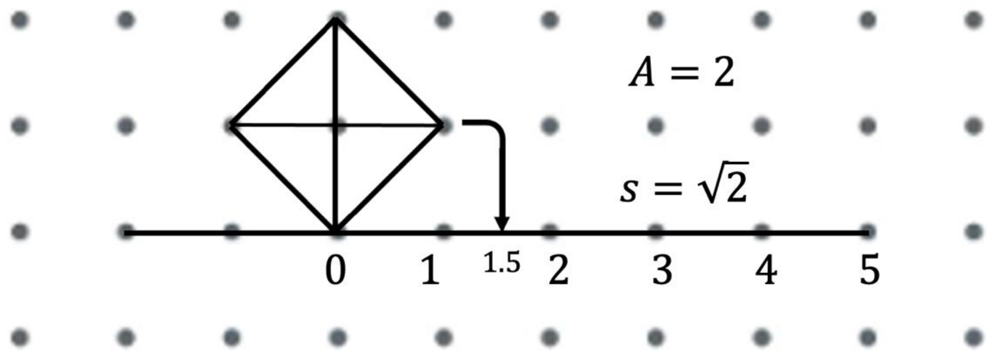
Looking Ahead 4.11

Let us look at the square from the previous section that had an area of 2 and a side length of $\sqrt{2}$. We know that $\sqrt{2}$ is between 1 and 2 because:

$$1^2 = 1 \qquad \sqrt{1} = 1$$
$$?^2 = 2 \qquad \sqrt{2} = ?$$
$$2^2 = 4 \qquad \sqrt{4} = 2$$

The number 2 is closer to 1 than 4, so $\sqrt{2}$ is closer to $\sqrt{1}$ than $\sqrt{4}$. Remember that the distance between two horizontal points or vertical points on the dot grid below is 1, but the distance between two consecutive diagonal points is a little over 1.

Roll the tilted square with an area of 2 onto its side so that it is in an upright position. You will see that one corner is at 0 and the other corner is at a little less than 1.5 on the number line. That means the length of the side of a square with an area of 2 is a little less than 1.5. The decimal approximation for $\sqrt{2}$ is actually1.414213 ..., which is close to 1.5.



If you put "$1.412135623731 \times 1.4142135623731$" in the calculator, the result is "2." Perhaps all the non-zero digits get dropped because the only ones that fit the display of the calculator are 0s and 2.0000 ... etc. is going to be displayed as "2." God could make this calculation (even without a calculator!) but we are unable to multiply two numbers that do not repeat nor end. The decimal approximation for $\sqrt{2}$ never repeats nor ends. It is an irrational number.

Square roots that are not perfect squares are irrational numbers and may be written as decimal approximations.

Example 3: A square with an area of 5 has a side length of $\sqrt{5}$. This is an exact solution. It cannot be simplified any further. However, it can be written as a decimal, which is an approximation, not an exact solution. Simplify $\sqrt{5}$. Write your solution as a decimal approximation.

Example 4: Given the area of a square and the length of a side, write the side length as a decimal approximation to the tenths place. Round your solution to the hundredths place using a calculator.

Area of Square	**Exact Solution**	**Decimal Approximation**	**Calculator**
3	$\sqrt{3}$	1.7	
4	2	2.0	
22	$\sqrt{22}$	4.7	
24	$2\sqrt{6}$	4.9	

In the next section, we will use an estimation method to find a closer decimal approximation.

Section 4.12 Finding Square Roots Using the Estimation Method

Looking Back 4.12

In the previous Practice Problems section, we used a "guess and check" method to find a decimal approximation for an exact square root. This is called the estimation method. Once we determine the two whole numbers on a number line that the square root lies between and decide which one it is closer to, we can estimate as to what we think the decimal would be. When we multiply the "guess" by itself, we are checking to see how close it is to the number under the square root symbol (the radicand). Depending on whether we are too high or low, we can adjust our guess higher or lower. Once we get really close, then we try to change decimals to the hundredths or thousandths place to find a number that when multiplied by itself gets closer and closer to the radicand.

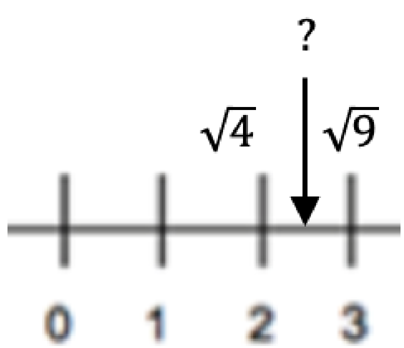
Looking Ahead 4.12

Example 1: Estimate the decimal approximation for $\sqrt{6}$ as close as possible. The square root of 6 or $\sqrt{6}$ is between the two perfect squares 2 and 3 because $2^2 = 4$ and $3^2 = 9$.

$\sqrt{6}$ lies between __________ and __________

__________ $< \sqrt{6} <$ __________

__________ $< \sqrt{6} <$ __________



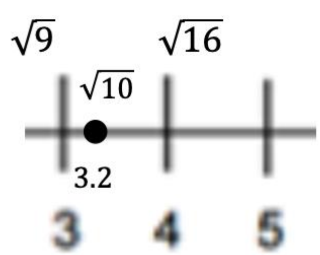
Example 2: Show where $\sqrt{10}$ is on the number line. Use the "guess and check" method to find the decimal approximation that is within 0.01 of the exact length.

$\sqrt{10}$ is between __________ and __________

__________ $< \sqrt{10} <$ __________

__________ $< \sqrt{10} <$ __________

Guess	$(\mathbf{Guess})^2$	Too High/Too Low



Section 4.13 Finding Square Roots Geometrically and Algebraically

Looking Back 4.13

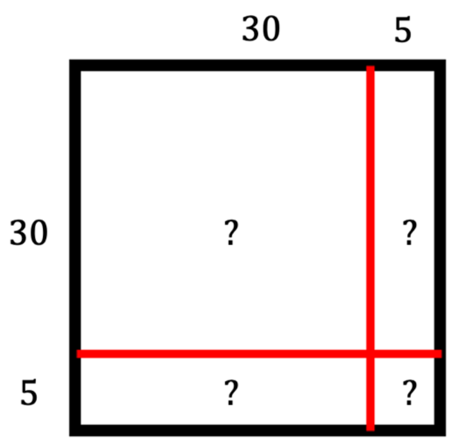
We have seen one way to find square roots. We can use the "guess and check" method or the estimation method. The difficulty with the estimation method is that it could go on and on getting infinitely closer but never exact. It can get lengthy. We have seen throughout mathematics that there are many ways to solve problems and many ways to represent those problems and their solutions. Square roots are the inverse of squaring a number. Because squaring a number uses multiplication, finding a square root must involve division, which is repeated subtraction. We will look at a geometric representation and use the reverse process to derive a division algorithm for finding a square root.

Looking Ahead 4.13

Example 1: Find the value of $(35)^2$ geometrically using a square diagram. The number $(35)^2$ can be written as $(30+5)^2$ when 35 is separated into tens and ones. This is how we do long multiplication and division; we look at the place value.

The square diagram for this is as follows:

1) The __________ comes from 30×30 (top left).
2) The __________ comes from 30×5 (top right).
3) The __________ comes from 5×30 (bottom left).
4) The __________ comes from 5×5 (bottom right).

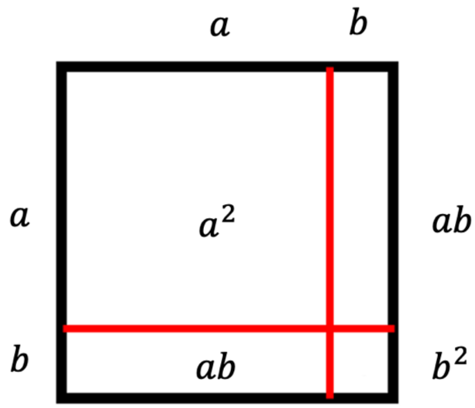


The area is the sum of the areas of the inner quadrilaterals:

__________ + __________ + __________ + __________ = __________

Therefore, $(35)^2 =$ __________

Example 2: Show the algorithm for squaring a number algebraically.



Example 3: Find the square root of 1,225 using the square root division algorithm. Use reverse thinking. Let $a = 30$ and $b = 5$.

$$(a + b)^2$$

Multiply
And
Double

Square It Square It

$$a^2 + 2ab + b^2$$

Example 4: Find the square root of 25 using repeated subtraction.