Module 3 Linear Equations

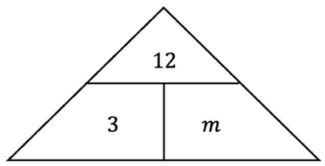
Section 3.1 The Math Triangle

Looking Back 3.1

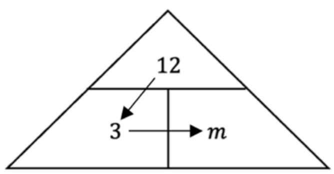
The concept of linear equations is fundamental to algebra. This entire module will be devoted to investigations of linear equations.

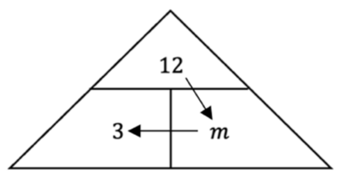
We have been solving equations such as $3m = 12$ by undoing multiplication with division; for example, $\frac{3m}{3} = \frac{12}{3}$ to find that m is 4. Therefore, there is one unique solution to this equation. When we substitute $m = 4$ into the equation, we get $3(4) = 12$ or $12 = 12$; 4 is the unique solution for that equation.

There are three ways to write the equation for $3m = 12$. Using the math triangle makes it easy. Start by drawing a mathematics triangle with three parts: the number or letter that is the product goes in the top of the triangle. The two things being multiplied go in the two halves that make up the bottom of the triangle.



Because multiplication is commutative, $3 \times m = m \times 3$. Using reverse thinking, the top can be divided by the bottom left to get the bottom right, or the top can be divided by the bottom right to get the bottom left.





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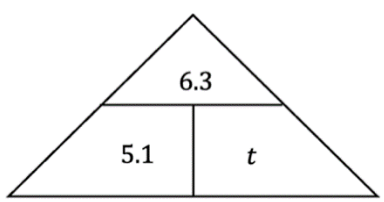
These three parts of the mathematics triangle help us see the three formulas that represent the same thing. This can be very helpful in solving equations when the unknown variable could be in one of three places.

Example 1: Set up three formulas to solve for the unknown variable t in the triangle below. Which is the easiest one to use to solve for t?



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Looking Ahead 3.1

In science, there is a formula to calculate mechanical advantage for an inclined plane, which is: $\text{Mechanical Advantage} = (\text{Length of Slope}) \div \text{Height}$. An inclined plane is a flat surface with a slope used to raise or move heavy objects from one place to another. These are objects are too heavy to lift straight up. An inclined plane produces a mechanical advantage to decrease the amount of force needed to move an object a certain height.

If you are asked to find the mechanical advantage of an inclined plane that has a slope of 8 feet long and a height of 2 feet, you would substitute these values into the formula and solve the equation:

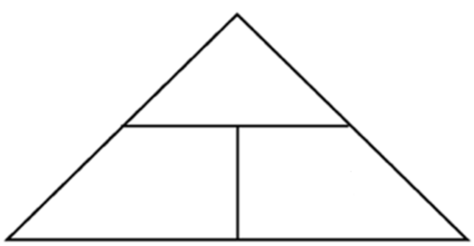
$$\text{Mechanical Advantage} = \frac{8\text{ ft.}}{2\text{ ft.}} = 4$$

What if you were given the inclined plane mechanical advantage and the length of the slope of the inclined plane and were asked to find the height of the inclined plane? How would you write the equation? What if you were given the inclined plane mechanical advantage and the height of the inclined plane and were asked to find the length of the slope of the inclined plane? How would you write the equation?

Using the mathematics triangle can be helpful to finding the formulas for mechanical advantage, length of slope of the inclined plane, and height.

$$\text{Let } IMA = \text{inclined plane mechanical advantage}$$
$$\text{Let } L = \text{length of the inclined plane slope}$$
$$\text{Let } h = \text{height of the inclined plane}$$
$$IMA = \frac{L}{h}$$

Example 2: Put the formula in the math triangle and then write three equations that you derive.



Example 3: If you know the IMA (inclined plane mechanical advantage) is 4 and the L (length of the inclined slope) is 8 feet, which formula would you use to find the height from Example 2?

Section 3.2 More Than One Variable

Looking Back 3.2

We have solved many problems involving one variable. For example, if an occupational therapist worked $4\frac{1}{2}$ hours at a school and was paid $\$202.50$ for her work, we can calculate what she was paid per hour by using an equation: $4.5h = \$202.50$ in which h represents the hourly rate. "Undoing" multiplication with division results in $\frac{4.5h}{4.5} = \frac{\$202.50}{4.5}$, or $h = \$45.00$ per hour.

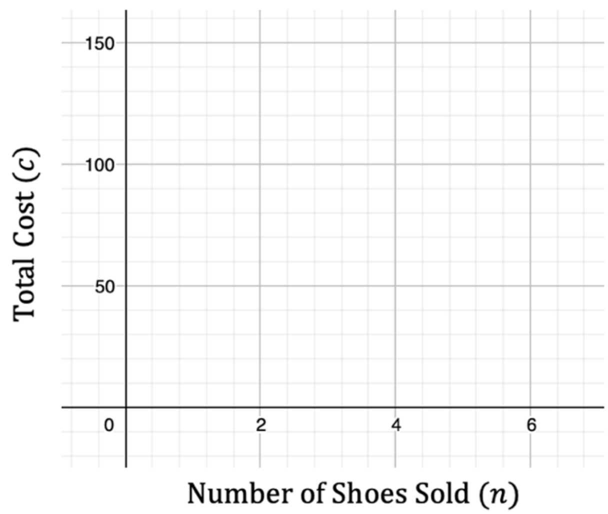
Looking Ahead 3.2

In General Math, we set up equations to solve problems similar to the one described above. We found the expression to represent the total price of corn to be $\$0.35n$ when the cost of each ear was $\$0.35$ and n represented the total number of ears of corn bought. To find the price of 8 ears of corn, we substituted 8 for n and multiplied $\$0.35$ by 8 which equals $\$2.80$, which is the total cost for 8 ears of corn.

In the previous paragraph, the total cost of corn was $\$0.35n$. If the question was "how many ears of corn could be bought with $4.20?" the equation would be $\$4.20 = \$0.35n$ and we would solve for n to find the number of ears of corn. "Undoing" multiplication with division results in $\frac{\$4.20}{\$0.35} = \frac{\$0.35n}{\$0.35}$ or $n = 12$ ears of corn, which means $\$4.20$ could buy one dozen ears of corn.

Example 1: Another problem from General Math was about shoes that cost $\$22.00$ each. We were asked to write an equation to show the total cost for any number of shoes. We let c be total cost and n be number of shoes. The equation we wrote was $c = \$22.00n$. Use this information to complete the table below. Then complete the graph using the table.

Number of Shoes Sold (n)	Total Cost (c)
1	$22.00
2	
3	
4	
5	
n	



These are all examples of linear equations. Linear equations are used to model and solve real-world problems.

Section 3.3 Interpreting Situations

Looking Back 3.3

We have written linear expressions and equations to describe situations mathematically. All of the investigations going forward in this module will be linear as well. That means all the variables will be to the first degree.

Looking Ahead 3.3

Hanna, Evan, and Norton participated in a walk-a-thon to raise money for cancer awareness.

a) Hanna's girl scout troop pledged $75.00 up front and $0.50 for every mile Hanna walked.

b) Evan's troop pledged $3.00 for every mile Evan walked, but he paid his own $10.00 entry fee. (Evan's troop paid him back his $10.00 for the entry fee with the total money collected for the walk.)

c) Norton's troop pledged $4.25 for every mile Norton walked.

Example 1: If $t =$ total money earned and $m =$ number of miles walked, three different equations could be written to represent the amount of money each participant earned for cancer awareness. Write an equation for the total earnings of each participant (Hanna, Evan, and Norton).

Example 2: You could make a table for each of the situations to see which participant would earn the most money after 6 miles, 8 miles, and 10 miles. Complete the table below and see who earns the most money at 6 miles, 8 miles, and 10 miles.

Miles	**Hanna**	**Evan**	**Norton**
m	$\$0.50m + \$75.00$	$\$3.00m - \$10.00$	$\$4.25m$
0	\$75.00		
1	\$75.50	−\$7.00	\$4.25
2	\$76.00	−\$4.00	\$8.50
3		−\$1.00	
4	\$77.00	\$2.00	\$17.00
5	\$77.50	\$5.00	\$21.25
6		\$8.00	\$25.50
7	\$78.50		
8	\$79.00		\$34.00
9	\$79.50	\$17.00	\$38.25
10		\$20.00	\$42.50

Example 3: Use the table above to answer the questions below and interpret the situation.

Q_1: What do the numbers in the first row next to $m = 0$ mean?

Q_2: How many miles does Evan have to walk before he is out of debt?

Q_3: How much money did the three participants earn in total after the sixth mile?

Q_4: When will Evan earn more money than Norton?

Q_5: How much is the amount of money in Hanna's column increasing by per mile? Explain why.

Section 3.4 Generating Tables

Looking Back 3.4

We have looked at a few tables and seen how they relate to equations. The best way to see the connection is to start with an equation and generic table. Connections between tables, graphs, and equations are very important not only in algebra but in real-world situations.

Looking Ahead 3.4

Example 1: The equation $3x + 2y = 10$ has two variables. Substitute 0 and 1 in for x and solve for y.

Example 2: Rearrange the equation $3x + 2y = 10$ by isolating y. (You are solving for y in terms of x.)

Example 3: Now substitute 2 and 3 in for x using the equation you derived in Example 2. Why is this equation easier to use than the equation in Example 1?

Example 4: The forms of an equation:

a) Standard Form:

b) Slope-Intercept Form:

Section 3.5 Graphing Linear Equations

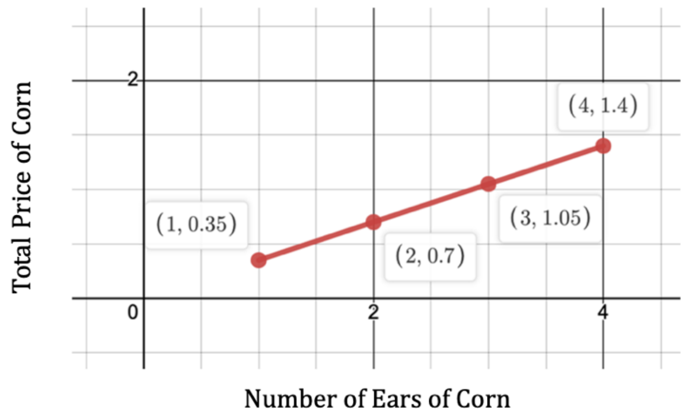
Looking Back 3.5

In General Math, we moved from tabular representations to graphical representations. In this module (of Pre-Algebra), we are going to do the same thing. This will help us transition to the important concepts of Algebra.

Looking Ahead 3.5

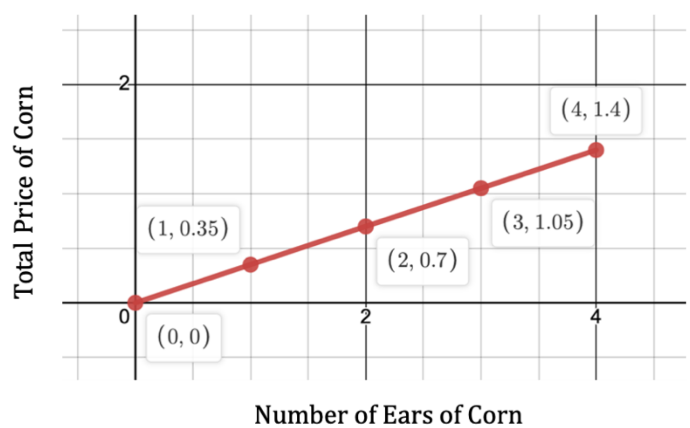
In General Math, we made a table for the number of ears of corn in which we let $x =$ number of ears of corn and $y =$ total price of corn. We said that each ear of corn was \$0.35 so the total price of corn is represented by the expression $\$0.35x$, which was equal to y. Therefore, the equation for total price of corn was $y = \$0.35x$.

Number of Ears of Corn (x)	Total Price of Corn (y)
1	\$0.35
2	\$0.70
3	\$1.05
4	\$1.40



We started the table from 1, not 0. If we bought 0 ears of corn the table and graph would look as below:

Number of Ears of Corn (x)	Total Price of Corn (y)
0	\$0.00
1	\$0.35
2	\$0.70
3	\$1.05
4	\$1.40



This is called a direct variation equation because y varies directly with x and goes through the point $(0, 0)$. A direct variation equation is written in the form $y = kx$ in which k is the constant of variation. In this case, the equation is $y = 0.35x$ and $k = 0.35$.

This constant of variation is also the slope (constant rate of change), \$0.35 for each ear of corn. This is also known as our unit rate in this situation. The y-intercept is 0. Therefore, $y = mx + b$ is $y = mx + 0$ or $y = mx$ for a linear equation that goes through the origin.

Example 1:	Suppose you pay \$0.15 for a plastic bag to carry your corn in.
a)	How does this change the previous equation ($y = 0.35x$)?
b)	How does this change the total cost of the corn?
c)	How does this change the graph for total price of corn and number of ears of corn?
d)	Is this still a direct variation equation?

Number of Ears of Corn	**Total Price of Corn**

Number of Ears of Corn	**Total Price of Corn**

Slope-intercept form of an Equation: $y = mx + b$

Now you have the slope-intercept form of an equation as shown below:

y-intercept where the line crosses (hits) the y-axis

$y = mx + b$

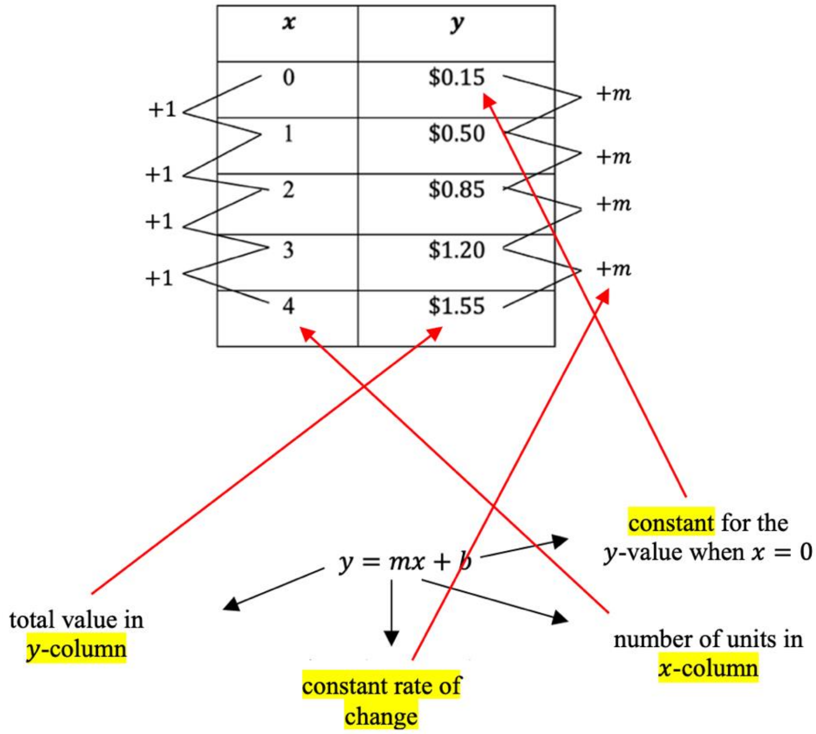
y-coordinate on graph

slope (steepness) of graph

x-coordinate on graph

Here is the table from Example 1 for the price of corn including the cost for the bag of $0.15.

x	y
0	$0.15
1	$0.50
2	$0.85
3	$1.20
4	$1.55



It is not direct variation because y does not vary directly x.

Example 2:	Is the equation $y = 3x - 9$ a direct variation? How do you know whether it is or is not?

Section 3.6 Finding the x and y-intercepts

Looking Back 3.6

In the three equations from the walk-a-thon problems over the previous sections (Hanna, Evan, Norton's money earned and miles walked), you were asked to find the y-intercept of each equation. The slope-intercept form of an equation is $y = mx + b$ in which b (the constant value) represents the y-intercept. If b is being added in the equation, the y-intercept is positive; if b is being subtracted in the equation, the y-intercept is negative.

We were also asked to find the y-intercept from the graph of Hanna's equation, Evan's equation, and Norton's equation. To do this, we simply located the point on the graph where the line of the equation crossed the y-axis. In this section, we will continue to work on locating y-intercepts from a table.

Looking Ahead 3.6

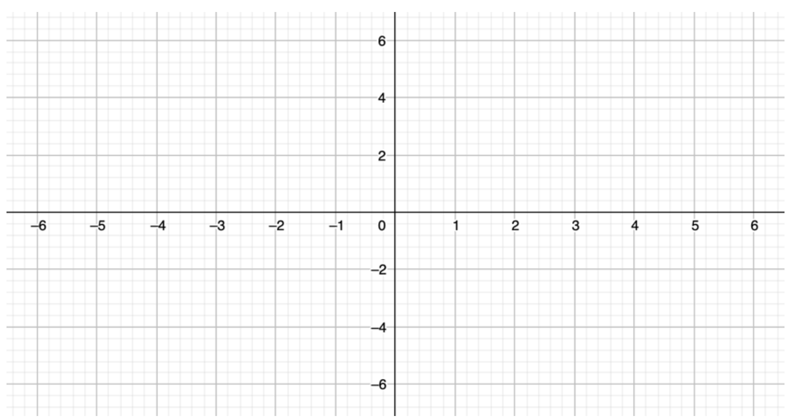
Example 1: If a football player intercepts a pass, he/she catches it. If a spy intercepts a message, he/she retrieves it. Fill in the blanks below for the definition of the y-intercept.

The y-intercept is the ____________________ where the graphed ____________________ crosses the ____________________. It is at that point that x is equal to _______________. It can be written as the ordered pair ____________________ or ____________________.

Example 2: From what you learned in Example 1, use your logic to tell what you would expect the x-intercept of an equation to be. Fill in the blanks for the definition of the x-intercept.

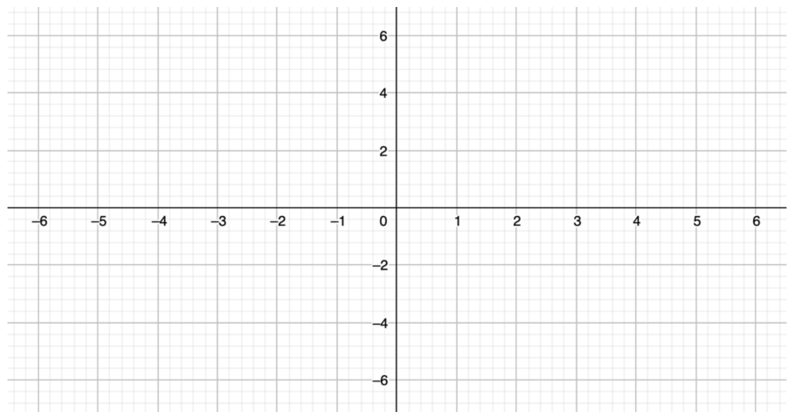
The x-intercept is the ____________________ where the graphed ____________________ crosses the ____________________. It is at that point that y is equal to _______________. It can be written as the ordered pair ____________________.

Example 3: Graph the x-intercept and y-intercept of the equation $y = 2x + 4$ and connect the points to make a line for the linear equation.



Example 4: Find the x-intercept and y-intercept for the equation $y = -5x + 4$. Do not graph this equation.

Example 5: Graph the x-intercept and y-intercept of the equation $y = 2x + 3$ and connect the points to make the line for the linear equation.



Example 6: Find the y-intercept of $y = 2x + 3$ (the equation from Example 5) using a table.

x	y
0	
1	
2	
3	

Example 7: Find the x-intercept of $y = 2x + 3$ (the equation from Example 5 and Example 6) using a table.

x	y
0	
1	
2	
3	

Example 8: Interpolation was used to find the x-intercept in Example 7. Is there an easier way to find the x-intercept without using the table?

The descriptions below give us a summary of x-intercepts and y-intercepts:

x-intercepts

- Crosses the x-axis where $y = 0$
- The coordinate point is $(-\frac{b}{m}, 0)$
- Moves right or left only

y-intercepts

- Crosses the y-axis where $x = 0$
- The coordinate point is $(0, b)$
- Moves up or down only

Section 3.7 Isolating the Variable

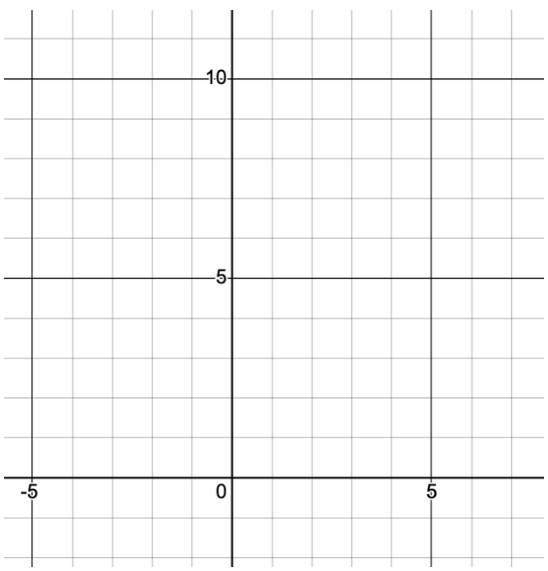
Looking Back 3.7

When a standard form equation, such as $3x + 2y = 6$, is written in slope-intercept form, it is called "solving for y in terms of x." The variable y is isolated on one side of the equation by itself and all the other terms are moved to the other side of the equation. This form allows us to quickly graph the equation as we learned in the previous section.

Example 1:	Solve the equation $3x + 2y = 6$ for x in terms of y.

Looking Ahead 3.7

Example 2:	Plot the y-intercept and slope to graph the equation $y = -\frac{3}{2}x + 3$.



When you solved for y in terms of x, you were able to identify the slope ($m = -\frac{3}{2}$) and the y-intercept ($b = 3$). You could go to the coordinate $(0, 3)$ on the y-axis where the equation intersects (intercepts) the y-axis. The x-coordinate of the y-intercept is always 0.

Draw your first point from the equation at $(0, 3)$. From there, the slope is rise over run, so you could go down 3 for -3 and right 2 for $+2$ to get the slope of $\frac{-3}{2}$; or, knowing the slope is $\frac{3}{-2}$, you could go up 3 from point $(0, 3)$ and left 2 for -2 and plot another point. Then you could connect the two points to make a line. Both are shown to the left. It only takes two points to draw a straight line.

You know the standard form of a linear equation is $Ax + By = C$ in which A, B, and C are constants. These numbers are called parameters as they are fixed and non-changeable for any given line. The variables A and B are coefficients of x and y. They are the numbers in front of the variable. The variable C is a constant only. It is a parameter that is a number. However, x and y are variables that represent the infinite points that lie on the line and make the equation true.

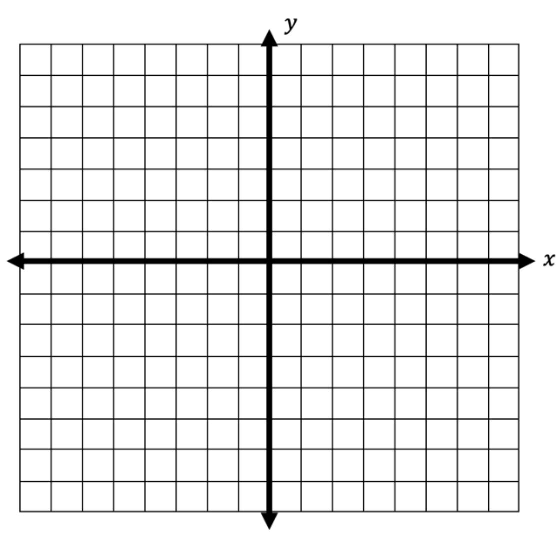
Example 3: Take the standard form of an equation, $Ax + By = C$, and solve for y in terms of x, which gives us the slope-intercept form of an equation. The variable y will be isolated on one side of the equal sign by itself.

Example 4: Solve for x in terms of y in the standard form equation $Ax + By = C$.

Now, the slope is not rise over run but run over rise. Would the graph of this line look different? Will you get the same line using the same the same equation but solving for x?

Example 5: In the equation $x + 2y = 6$, the parameters are $A = 1$, $B = 2$, and $C = 6$. Substitute these values into the equations below and draw them both on the same graph.

$y = -\frac{A}{B}x + \frac{C}{B}$ and $x = -\frac{B}{A}y + \frac{C}{A}$



Section 3.8 Slope as a Ratio

Looking Back 3.8

Hopefully you are beginning to see that there are multiple ways to represent a linear equation, not only as an equation, but in a table and with a graph as well. Many real-world situations, particularly in science, can be modeled by equations, tables, and graphs; we have seen this with the mechanical advantage problems in Section 3.1.

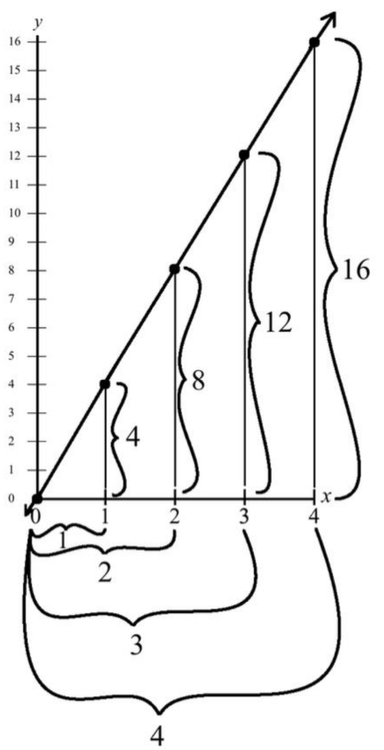
At this point, the y-intercept has been defined as "the point where the graphed line crosses the y-axis," and the x-intercept has been defined as "the point where the graphed line crosses the x-axis." Slope has been defined as a "constant rate of change" in a linear equation. In the real-world (specifically civil engineering), slope is defined as gradient, incline, or pitch. This was found using the equation and table in the Practice Problems of Section 3.5. In this section, we will investigate slope as a ratio and learn how that ratio relates to the graph. In the next section, we will review how to find the slope of a line from its graph.

Looking Ahead 3.8

Let us use the graph of a line to find slope, not just numerically, but in meaning and origination. We will begin with the slope-intercept form of an equation ($y = mx + b$), which is $y = mx$ when $b = 0$. If we want to solve for slope (m), we must write this equation as $m = \frac{y}{x}$. In higher-level mathematics, the ratio $\frac{\Delta y}{\Delta x}$ (read: "delta y divided by delta x") means the change in y as compared to the change in x. The Δ (delta) symbol means "change in."

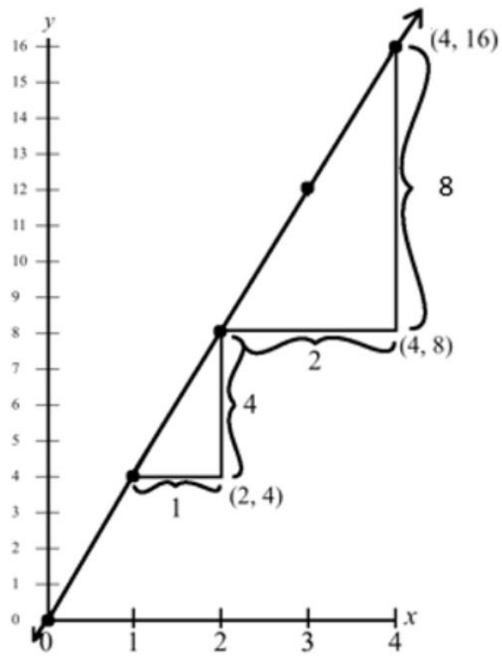
Example 1:	Find the graph of $y = 4x$. (In the graph, recall the triangles under the line are "slope triangles.")

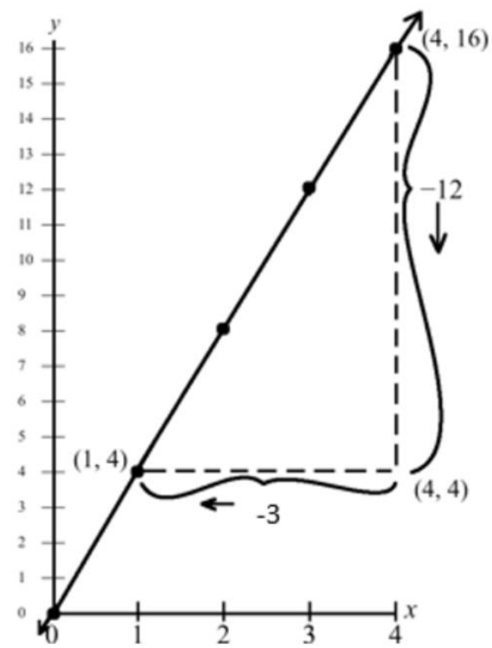
This steepness of the line is represented by the angle the line makes with the horizontal axis (x). Because the slope is the ratio of the y-distance over the x-distance, we can make a table of the slope triangles below.



Slope Triangle	1	2	3	4
Ratio $\frac{y}{x}$				

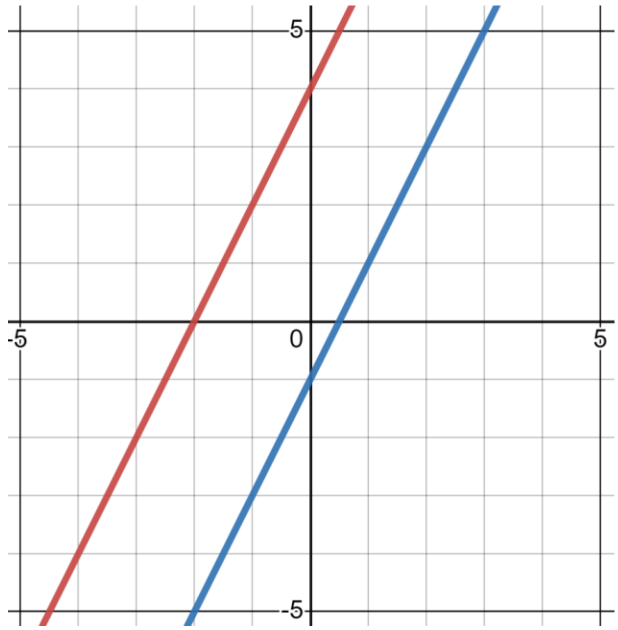
Example 2:	What is the slope of the line from Example 1 if we use different slope triangles.





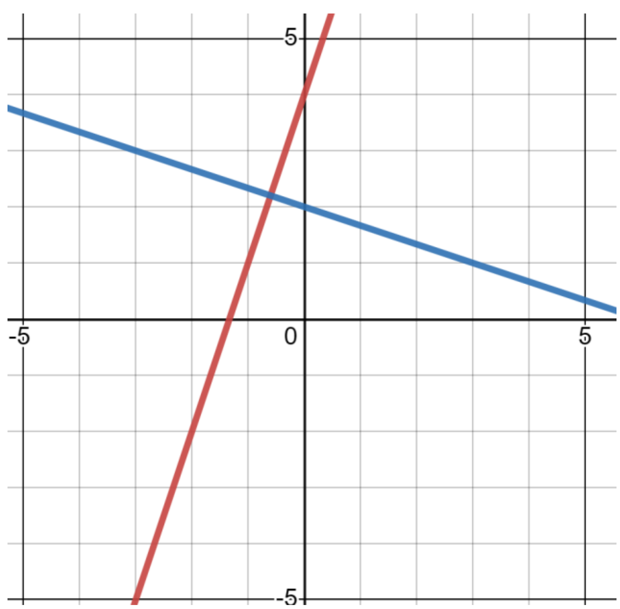
Lines in a plane, which extend in two directions (such as a piece of paper) are parallel if they have the same slope. Parallel lines are equidistant from each other at every point.

Example 3:	Show that the lines $y = 2x + 4$ and $y = 2x - 1$ are parallel.



Lines in a plane, which intersect at 90° (right angles) have an L-shaped corner. They are perpendicular if their slopes are opposite reciprocals.

Example 4:	Show that the lines $y = 3x + 4$ and $y = -\frac{1}{3}x+2$ are perpendicular.



Section 3.9 Direction and Steepness of Slope

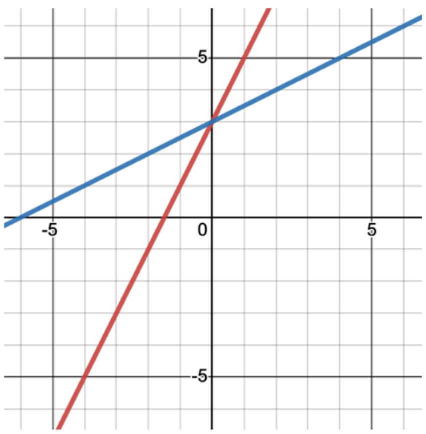
Looking Back 3.9

We have learned a lot about slope. We can locate slope in an equation in slope-intercept form and we can calculate it from a table or from counting the squares on a graph. When none of this is practical, we can use the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$ as long as we know two points on the line. We can find these ordered pairs on the table that is generated from the equation. In this section, we will look specifically at the direction and steepness of the slope of a line.

Looking Ahead 3.9

Example 1: Between the two equations given, which has a steeper slope? How could you find out?

$$y = 2x + 3 \text{ and } y = \frac{1}{2}x + 3$$



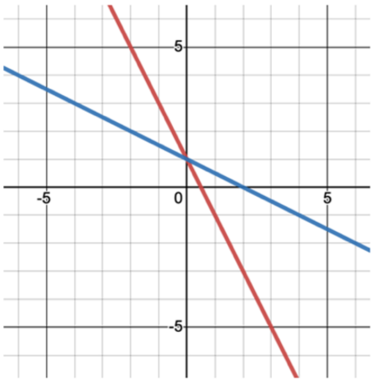
$\boldsymbol{y = 2x + 3}$	
$\boldsymbol{x}$	$\boldsymbol{y}$

$\boldsymbol{y = \frac{1}{2}x + 3}$	
$\boldsymbol{x}$	$\boldsymbol{y}$

The slope is positive for both equations. The lines are increasing (going up) from left to right.

Example 2: Between the two equations given, which has the steeper slope? What do you notice about the direction of each line? Can you conclude anything about slope and the direction of a line?

$$y = -2x + 1 \text{ and } y = -\frac{1}{2}x + 1$$

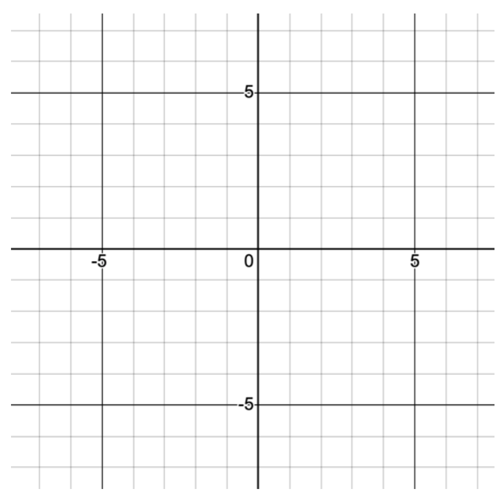


$y = -2x + 1$	
x	y

$y = -\frac{1}{2}x + 1$	
x	y

The slope is negative for both equations. The lines are decreasing (going down) from left to right.

Example 3: Find the slope of $y = 4$ in the equation $y = mx + b$ when $b = 4$ and graph it.



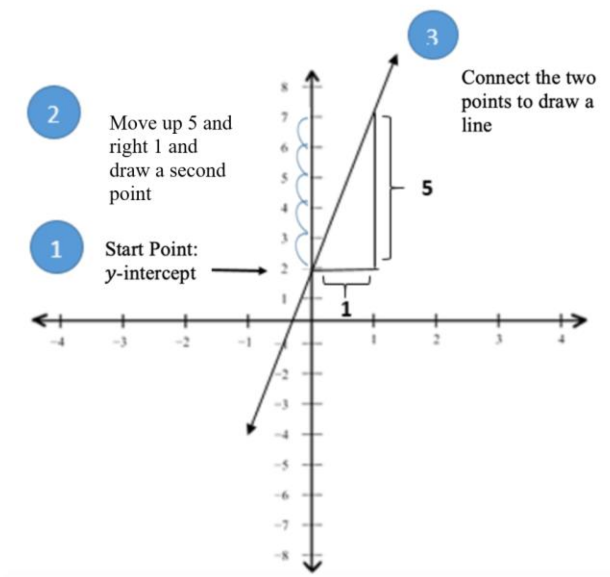
Section 3.10 Comparing Slope and y-intercept

Looking Back 3.10

Now we know how to look at an equation, generate a table, and draw a graph. We can determine if the slope is steep or not and if the slope of the line is positive or negative by looking at whether the line is increasing or decreasing from left to right. In Section 3.7, we drew a graph directly from an equation in slope-intercept form without making a table first (when we practiced isolating the variable). Now that we know so much about slope, we can make a graph directly from slope-intercept form. We will practice that again here.

Looking Ahead 3.10

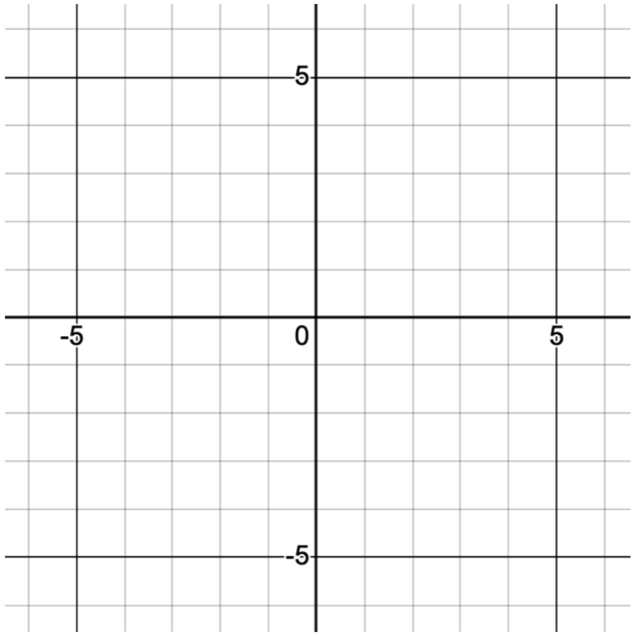
Example 1:	Graph the line $y = 5x + 2$ using the slope and y-intercept.



How does $y = 5x$ change the graph?

How does $y = 5x + 4$ change the graph?

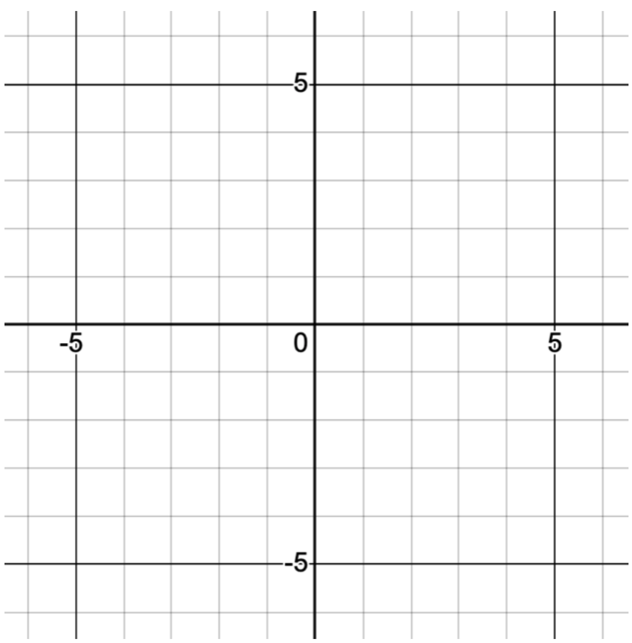
Example 2: The slope of a line is $-\frac{4}{3}$. The y-intercept is 1. Graph the line.



Does changing the slope to $-\frac{2}{3}$ make the line steeper or less step?

Note: A ratio is a fraction. When the ratio is opposite (negative), the negative can either go in the numerator or in the denominator because $-\frac{a}{b}$ is the same as $\frac{-a}{b}$ or $\frac{a}{-b}$.

Example 3: Find the slope of $x = 4$. What is its y-intercept?



Section 3.11 Writing an Equation in Standard Form

Looking Back 3.11

We have been using the slope-intercept form of an equation to graph a line. However, equations are often written in standard form ($Ax + By = C$) in which the x and y are variables that change as we move up and down the line. The parameters (A, B, and C) are constants that do not change as we move up and down the line. They are specific numbers unique to a given line that name the line.

When the standard form of the equation has been rearranged in previous sections of this module, we ended up with the following equation:

$$y = mx + b$$

$$y = -\frac{A}{B}x + \frac{C}{B}$$

The slope of the above equation is $m = -\frac{A}{B}$ and the y-intercept is $\frac{C}{B}$. To find slope (m) and the y-intercept (b), we can write the equation in slope-intercept form each time we solve a problem, or we can memorize that the slope is $-\frac{A}{B}$ and the y-intercept is $\frac{C}{B}$. Notice that b is the y-intercept and B is the coefficient of y.

By now, you may have noticed you are never given only formulas, but also where they come from. The first time you see them, you also see how they are found. Some students like to memorize and remember formulas after that. Some students do not like to memorize formulas and will derive them again each time they need to. Either way, you will know why you are doing what you are doing and where it comes form.

The word "apologetics" means to use reasoned arguments or writing to justify something, particularly in religious doctrine. In the Bible, I Peter 3:15 says: "but sanctify Christ as Lord in your hearts, always be ready to make a defense to everyone who asks you to give an account for the hope that is in you, yet with gentleness and reverence."

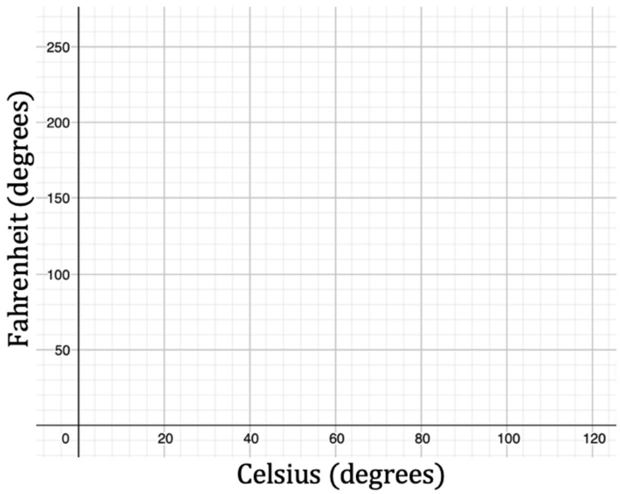
God wants us to be able to defend our faith. He wants us to understand His Word and His World. Part of His world is mathematics and science. Mathematics can be used to defend science; it is the language of science. If you know mathematics, you will be able to discern its use. That is why there are such detailed steps in these modules.

Looking Ahead 3.11

Example 1: $0°$ Celsius $= 32°$ Fahrenheit This is the freezing point.

$100°$ Celsius $= 212°$ Fahrenheit This is the boiling point.

Use this information along with a graph to derive equations and compare Celsius to Fahrenheit and Fahrenheit to Celsius. Let $x =$ Celsius (degrees) and $y =$ Fahrenheit (degrees).
Plot the two points $(0, 32)$ and $(100, 212)$ on the graph to relate the two temperature scales.



a) Find the equation that represents the relationship between Fahrenheit degrees and Celsius degrees.

b) Solve for Celsius degrees in terms of Fahrenheit degrees.

c) How many degrees Celsius is $95°$ Fahrenheit?

d) How many degrees Fahrenheit is $15°$ Celsius?

Example 2: Convert the equations below to standard form.

a) $y = \frac{2}{3}x - 10$

b) $3y = -16x + 3.8$

Section 3.12 Using Slope and Intercept to Find an Equation

Looking Back 3.12

Previously, we have learned to graph an equation using the slope and y-intercept. Now, we will find the equation for a line given one point and the slope.

Looking Ahead 3.12

Example 1: Find the standard form equation for a line that has slope $m = 4$ and contains the point $(2, -3)$. First, put the information in slope-intercept form ($y = mx + b$) and solve for b. Then write it in standard form ($Ax + By = C$).

Example 2: Find the equation of a line with a slope of -6 and x-intercept of -2.

Example 3: Find the equation of a line with a slope of $-\frac{1}{2}$ and the point $(1, 4)$.

Section 3.13 Using Point-Slope Form to Find an Equation

Looking Back 3.13

In the previous section, we found the standard form of an equation given the slope of a line and a point on the line. We used the slope-intercept equation, the slope, and one value for x and y on the line to find the y-intercept. Then we had all the information we needed to write the slope-intercept form of the equation and write it in standard form.

When something is standard, it is common; it is the way we are used to seeing things. In equations, it is standard to put variables in alphabetical order without negative numbers and/or fractions as the leading coefficient. Now, we are going to write an equation for a line when given two points on the line.

Looking Ahead 3.13

Example 1: Use the formula for slope ($m = \frac{y_2 - y_1}{x_2 - x_1}$) and clear the denominator in order to find the point-slope form of an equation.

Example 2: Use the point-slope equation you found in Example 1 and substitution to find the standard form equation of a line that passes through point $(3, 4)$ and has a slope of 2.

Example 3: Given the two points $(6, -4)$ and $(-3, 5)$, find the standard form equation of a line that passes through these points. First, find the slope using the formula for slope ($m = \frac{y_2 - y_1}{x_2 - x_1}$). Then substitute the slope and the point $(6, -4)$ in the point-slope equation. Finally, write that equation in standard form.

Example 4: In the Example 3, we found the slope to be -1. Then we substituted the slope $m = -1$ and the point $(6, -4)$ for x and y in the point-slope form of the equation. We wrote the equation in the standard form and the equation became $x + y = 2$. Repeat the same process using the other point $(-3, 5)$ and find the standard form of the equation. Do you think you will get the same equation as in Example 3?