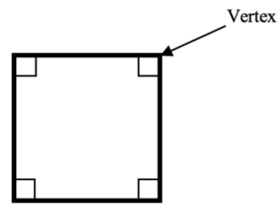
Module 5 The Pythagorean Theorem

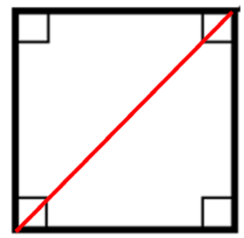
Section 5.1 Acute, Obtuse, and Right Triangles

Looking Back 5.1

A square has 4 right angles. A right angle is 90° and a square symbol at an angle shows that. An angle is created when two sides of a polygon meet at a vertex. Because a square has 4 angles which are 90°, the sum of the angles in a square is 360°. This is true for all quadrilaterals (four-sided polygon).



If you cut a square in half, you get two triangles. That means the sum of the angles in a triangle is 180°. We will prove this when we study Geometry and Trigonometry. Remember, when you cut the squares in half the two angles created belong to right triangles.

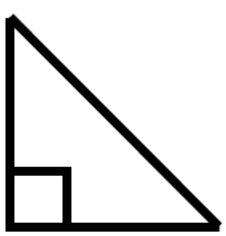


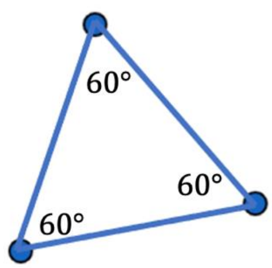
We use a protractor to measure angles. Again, we will use this tool in Geometry and Trigonometry. For now, we will watch a video of how the protractor is used.

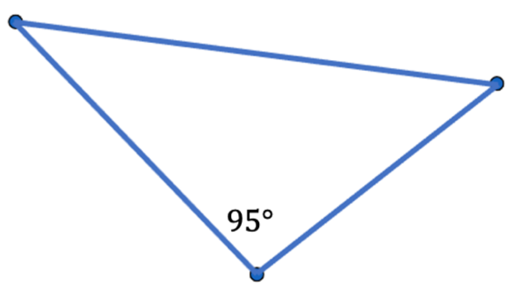
A circle has 360° so a protractor has 180° because it is a semi-circle (half a circle). We will also measure the angles of: an acute triangle, which has all 3 angles less than 90°; an obtuse triangle, which has only one angle greater than 90°. Why do you think a right triangle can only have on right angle? Let us watch the video and see.

Looking Ahead 5.1

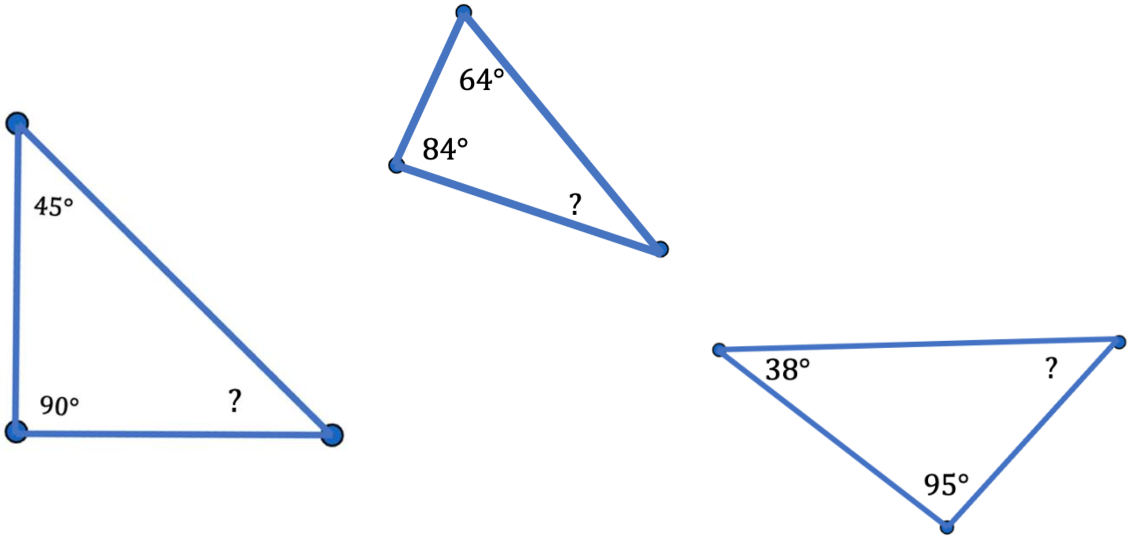
Example 1:	Tell whether each triangle below is a right triangle, acute triangle, or obtuse triangle.







Example 2:	Find the measure of the missing angle in each triangle below.



Example 3:	Find the third angle in each triangle below given the other two angles.

Right Triangle: 48° and 42° and ___________°

Acute Triangle: 36° and 88° and ___________°

Obtuse Tringle: 55° and 23° and ___________°

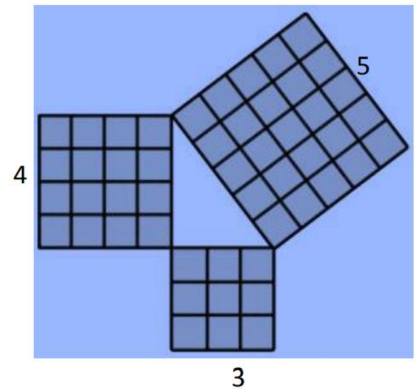
Section 5.2 Right Triangles and the Pythagorean Theorem

Looking Back 5.2

We have been talking about squares in this module. There is a famous theorem called the Pythagorean Theorem that involves square sides of right triangles. We will learn about the Pythagorean Theorem for the rest of this module.

A right triangle has two legs that meet to form a square (90° angle). The long (or slanted) side of a triangle is called the hypotenuse. If we square the length of the legs of a triangle and add the product together, the sum is the square of the length of the hypotenuse.

If we square the length of the legs of the right triangle shown to the right, we end up with 3^2 (9) and 4^2 (16) respectively. If we square the hypotenuse of the right triangle, we get 5^2 (25 square units). If we add the area (square units) of the squares formed on each leg, their sum is the area of the square of the hypotenuse. This is called a 3-4-5 triangle. It gets its name from its side lengths (3, 4, and 5 units). The smaller two areas, 9 and 16, add together to give the largest area, 25. This is how the mathematician Pythagoras arrived at the Pythagorean Theorem.



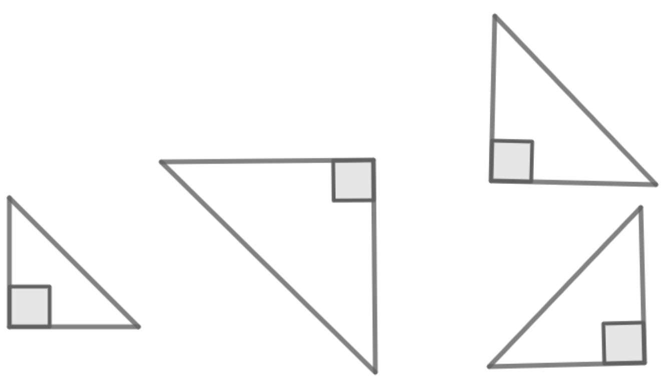
Let us use a Pythagorean Board and marbles to see how this works. Watch the video now which demonstrates the Pythagorean Theorem.

The Pythagorean Theorem states that the sum of the squares of the two legs of a right triangle is equal to the square of the hypotenuse of the triangle. The sum of the area of the squares formed on the legs of a right triangle is equal to the area of the square formed on the hypotenuse.

We usually use the variables a and b to represent the legs and c to represent the hypotenuse for a right triangle. This gives us the formula for the Pythagorean Theorem: $a^2 + b^2 = c^2$.

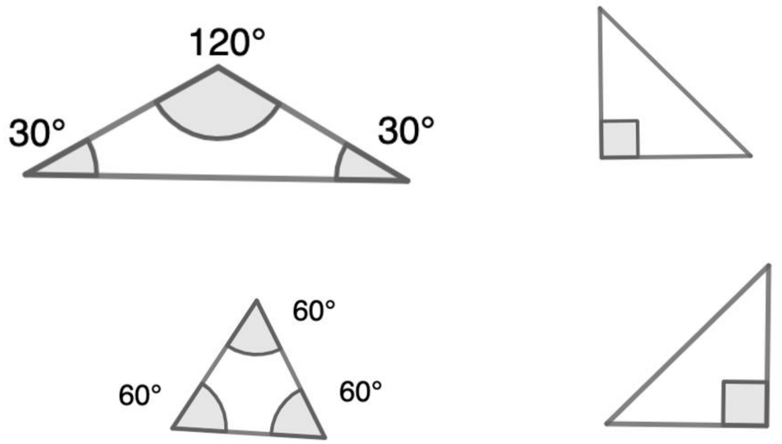
Looking Ahead 5.2

Example 1: Identify the hypotenuse and legs in each triangle below. Mark the legs a and b and the hypotenuse c.



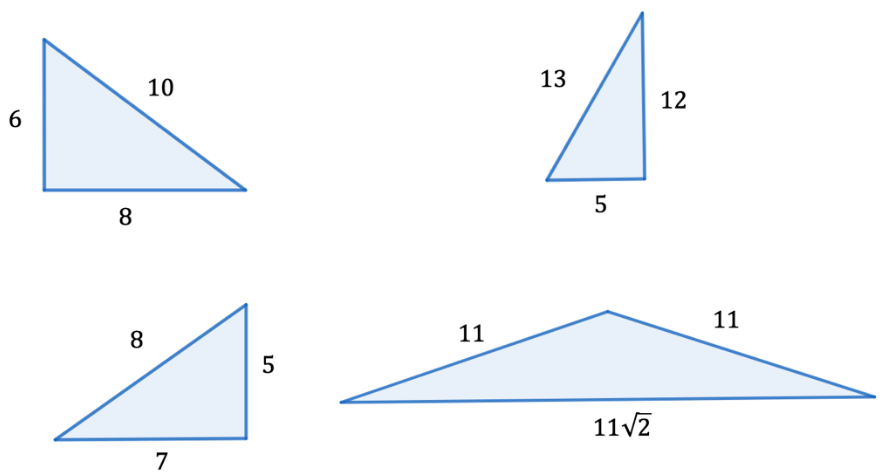
A right triangle has a square symbol where the legs meet to form a right angle. This means the angle is 90°.

Example 2:	Identify which triangles below are right triangles.



If the square of the longest side of a triangle is greater than or less than the sum of the squares of the shorter two sides of a triangle, then it is not a right triangle.

Example 3:	Identify which triangles below are right triangles. Use the Pythagorean Theorem.



Section 5.3 Pythagorean Theorem: Finding the Length of the Hypotenuse

Looking Back 5.3

We know the Pythagorean Theorem and we know how to identify the legs and the hypotenuse of a right triangle. In this section, we are going to use the Pythagorean Theorem to find the length of the hypotenuse of right triangles when given the lengths of the legs. In addition to the theorem, we will need to use the square and square root skills we learned in the previous module as well as our equation-solving skills to calculate the hypotenuse of a triangle.

The Pythagorean Theorem is named after the mathematician and philosopher (philosopher means one who loves wisdom) Pythagoras, but it may have been discovered by the Chinese before his time.

Pythagoras had his own school where he taught philosophy, music, and mathematics. His students were both men and women. For his second-year students, Pythagoras would step out from behind a curtain and teach to them in a white robe, golden sandals, and golden crown. For his first-year students, Pythagoras would teach from behind the curtain. Of Pythagoras' students, many were men who went on to compete in the Greek Olympics (women were not yet allowed to compete in the Olympics), which shows an emphasis he placed on physical strength as well as mental capability.

As far as his beliefs, Pythagoras believed we (humans) are one altogether being; he believed in immortal gods such as Zeus of Greek mythology; he believed that we live like separate branches of a tree connected to a divine source of power (which is the root in his metaphor); he believed if we do not harm ourselves or others, we do not harm the tree.

Pythagoras taught his students that all that is good in the soul stems from the root- kindness, and it is wise to follow the path of goodness.

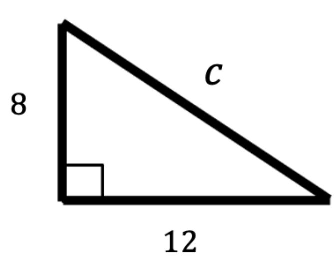
In John 15:4-5, Jesus speaks to this belief. He says: "Abide in Me and I in you. As the branch cannot bear fruit of itself, unless it abides in the vine, so neither can you, unless you abide in Me. I am the vine, you are the branches; he who abides in Me, and I in Him, he bears much fruit; for apart from Me you can do nothing."

Furthermore, Galatians 5:22-23 says: "But the fruit of the spirit is love, joy, peace, patience, kindness, goodness, faithfulness, gentleness, self-control; against such things there is no law."

Looking Ahead 5.3

Example 1: Use the Pythagorean Theorem to solve for c. Given $a = 12$ and $b = 5$, find c. Only find the principal square root. Given there are two solutions, we only want the positive solution, which is the length of the hypotenuse.

Example 2: Given the triangle below, find the hypotenuse. Also find the decimal approximation of the hypotenuse and round to the nearest hundredths place.



Section 5.4 Pythagorean Theorem: Finding the Length of the Leg

Looking Back 5.4

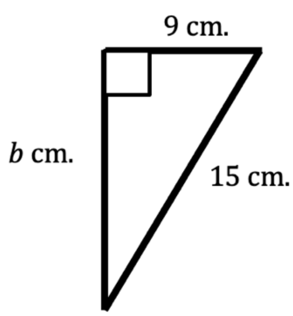
We have used the Pythagorean Theorem to find the measurement of the hypotenuse of a triangle, but now we will use the theorem to find the measurement of one of the legs of a triangle when the measurements of the hypotenuse and other leg are given.

The hypotenuse of a right triangle is easiest to find because it is direct: all the numbers for the legs are on one side of an equation and the variable is on the other side. However, finding the length of a missing leg involves another step to get all the numbers on one side of the equation and the variable on the other side.

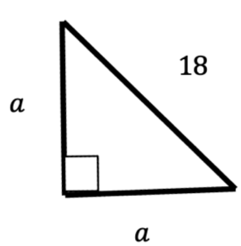
Looking Ahead 5.4

Example 1: If the hypotenuse of a right triangle is 16 inches and one leg is 12 inches, find the measure of the other leg (in inches). Round your solution to the nearest hundredths place.

Example 2: Find the length of the missing leg of the right triangle below.



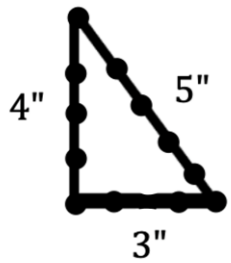
Example 3: In an isosceles triangle, the lengths of the legs are the same. If you have an isosceles right triangle, and the length of the hypotenuse is 18, what is the length of each of the two legs? Notice, leg b is also leg a because $a = b$. Combine the like terms and undo multiplication with division to get your final solution.



Section 5.5 Finding the Length of Any Missing Side of a Right Triangle

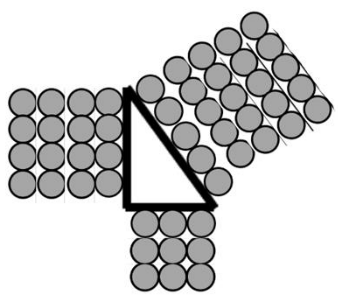
Looking Back 5.5

We have now learned a little bit about Pythagoras and the Pythagorean Theorem, which is attributed to Pythagoras. We have seen a Pythagorean Board that uses marbles to illustrate the Pythagorean Theorem. We have also learned about a triangle whose legs were 3” and 4” and found that the slant line connecting the endpoints of the two legs measure 5” long.



This is called a 3-4-5 triangle (in Geometry and Trigonometry, we will learn how the Egyptians folded a rope of one unit onto itself to construct this triangle and how they found square corners for land boundaries).

Let us surround each side of the triangle with quarters that have a 1” diameter and whose sides are touching and form squares (as we did with the marbles on the Pythagorean Board in Section 5.2).



We will find that the smallest square (3” side) has an area of 9 square quarters. The next largest square (4” side) has an area of 16 square quarters. If we add those together, we get 25 square quarters, and that is the same number of quarters that make up the largest square (5” side).

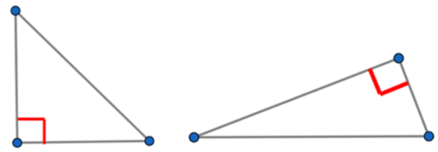
The Pythagorean Theorem states that if you have a right triangle with a 90° (square) angle and you draw a square on each of the sides of the triangle, then the sum of the area of the two smaller squares, formed on each leg, is equal to the area of the largest square, formed on the hypotenuse side of the triangle.

The square on the shortest leg of the triangle has a side length of 3” (equal to the shortest leg of the triangle). The area of that square is $3 \times 3 = 3^2 = 9$ square inches. The square on the next largest leg of the triangle (4”) has a side length of 4” (equal to the longest leg of the triangle). The area of that square is $4 \times 4 = 4^2 = 16$ square inches. The square on the side of the hypotenuse of the triangle (5”) has a side length of 5”. Its area is $5 \times 5 = 5^2 = 25$ square inches. Therefore:

$$3^2 + 4^2 = 5^2$$
$$9 + 16 = 25$$
$$25 = 25$$

This theorem only applies to right triangles. Let us review the definition of right triangles.

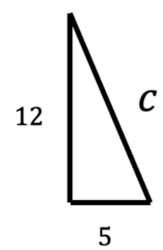
Looking Ahead 5.5



Shown are two right triangles. They each have a 90° angle as shown by the small square in the corner where the two legs meet. That is how you can identify the legs; they are on either side of the 90° angle. The hypotenuse and longest side of the triangle is the side of the right triangle that is across form the right angle.

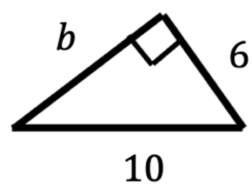
You can use the Pythagorean Theorem to find any missing side of a triangle (legs or hypotenuse) given you know the other two sides (as we have previously seen in the last two sections). We will use all that we have learned here.

Example 1:	Find the unknown length of the hypotenuse in the right triangle below.



$a = 5 \qquad b = 12 \qquad c = ?$

Example 2:	Find the length of the unknown leg in the right triangle below.



The Pythagorean Theorem is used to find unknown sides of triangles, not unknown angles. We will learn how to do that in Geometry and Trigonometry. The Pythagorean Theorem only works for right triangles, not obtuse or acute angle (which are also called oblique triangles).

Section 5.6 Pythagorean Triples

Looking Back 5.6

We can now find the missing side length of a right triangle given the measurements of two legs or one leg and the hypotenuse. We are also on our way to becoming pretty clever with squares and square roots. Now, let us look at some more right triangles with side lengths that are called Pythagorean triples.

Looking Ahead 5.6

As was stated previously in this module, we can never assume a triangle is a right triangle. It must have a little square where the legs meet to mark the right angle.

Another way to prove an angle is right is by using the Pythagorean Theorem. That is what we are going to do in this section.

Example 1: If $a = 3$, $b = 4$, and $c = 6$, is the triangle a right triangle? Put the numbers in the Pythagorean Theorem equation to see if this works.

Example 2: If $a = 3$, $b = 4$, and $c = 5$, is the triangle a right triangle?

If 3, 4, and 5 is a right triangle, then any multiples of 3, 4, and 5 would also be right triangles.

$$3 \times 2 \quad 4 \times 2 \quad 5 \times 2 \quad = 6, 8, 10$$

This is also a right triangle.

$$3 \times 3 \quad 4 \times 3 \quad 5 \times 3 \quad = 9, 12, 15$$

This is also a right triangle.

Example 3: Name two more multiples of $3 - 4 - 5$ triangles that would be right triangles.

Example 4: Below is a chart of the sides of right triangles. The Pythagorean Theorem works for these triangles because they are right triangles. The three numbers for the letters a, b, and c are called Pythagorean triples. Add Pythagorean triples next to each row of the chart below for a total of four Pythagorean triples.

a	b	c
3	4	5
5	12	13
8	15	17
7	24	25

Example 5: Using the Pythagorean chart for Pythagorean triples (with multiples in mind), tell which of the groups of three numbers below are sides of right triangles.

a) $10, 24, 26$

b) $10, 17, 18$

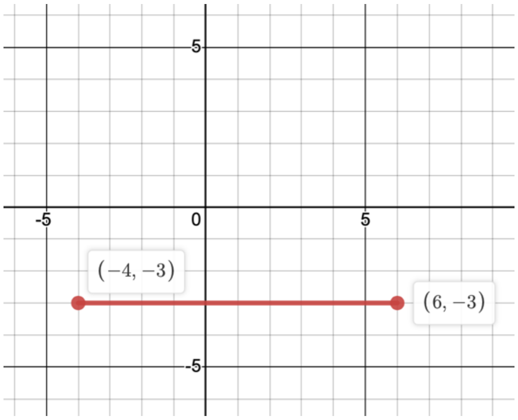
c) $24, 45, 51$

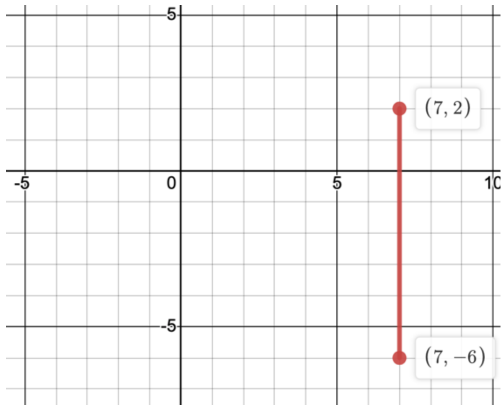
d) $4, 7.5, 8.5$

Section 5.7 The Pythagorean Theorem and the Distance Formula

Looking Back 5.7

We have worked quite a bit on the coordinate plane; therefore, we know how to graph points. We also know how to use the Pythagorean Theorem to solve problems. In this section, we are going to combine the two to find the distance of a line on a coordinate plane. If the line is vertical, simply count the number of spaces going from top to bottom. If the line is horizontal, simply count the spaces going from left to right. Let each space between grid lines be 1 unit.





Above, in the graph to the left, the line segment is horizontal. Count the spaces or subtract the x-coordinates of the ordered pairs of the endpoints. Above, in the graph to the right, the line segment is vertical. Count the spaces or subtract the y coordinates of the ordered pairs of the endpoints.

Because the lines above are horizontal and vertical, like the legs of an upright right triangle, it is no problem to count spaces and get the length of the lines. However, what do we do if the line is a diagonal on the graph such as the hypotenuse of an upright right triangle? In this case, we have to try something different.

Looking Ahead 5.7

To find the distance of a line that is diagonal on a graph, use the Pythagorean Theorem.

Example 1:	Find the distance of a line that goes between the points $(7, 2)$ and $(-3, -5)$.

Step 1: Plot the points on a graph and draw the diagonal line between them to connect them.

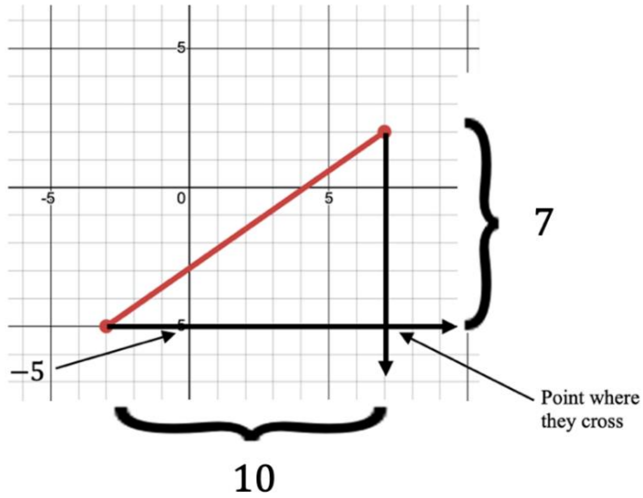
Step 2: Draw a horizontal line through the lowermost point on the graph.

Step 3: Draw a vertical line through the uppermost point on the graph.

Step 4: Let the two points be two vertices of a right triangle. Let the point where the horizontal and vertical line meet be the third vertex of the right triangle.

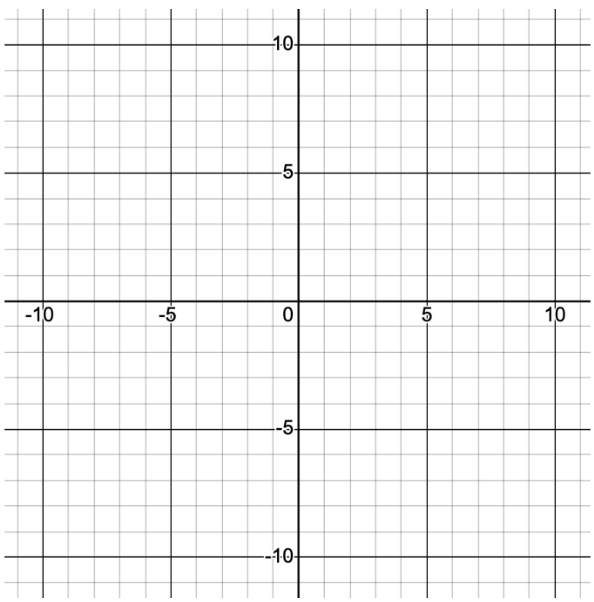
Step 5: Count the horizontal spaces along the horizontal line between two points. This is the length of one leg.

Step 6: Count the vertical spaces along the vertical line between two points. This is the length of the other leg.



Step 7: Use the Pythagorean Theorem to find the diagonal length, which is the hypotenuse.

Example 2:	Find the distance of a line that runs between the points $(-8, 4)$ and $(6, -8)$.



Step 1: Find the horizontal distance between the points $(-8, 4)$ and $(6, -8)$ using $|x_2 - x_1|$. Let this be leg a.

Step 2: Find the vertical distance between the points $(-8, 4)$ and $(6, -8)$ using $|y_2 - y_1|$. Let this be leg b.

Step 3: Substitute these distances in the formula: $a^2 + b^2 = c^2$ and solve for c.

Substituting the horizontal and vertical leg lengths in the Pythagorean Theorem equation results in the formula:

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = c^2$$

Squaring both sides results in:

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{c^2}$$

This gives us the distance of the formula for the diagonal line:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

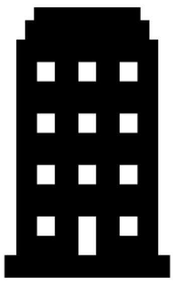
Section 5.8 Pythagorean Theorem Word Problems

Looking Back 5.8

We have explored many aspects of squares and radicals, especially square roots, most of which center around the Pythagorean Theorem. In this section, we want to see a few of the many different uses of the Pythagorean Theorem to solve word problems.

Looking Ahead 5.8

Example 1: There is a fire in an old building downtown. The firemen arrive and set up their ladder. The window the firemen will be entering is 35 feet up in the building. The ladder they are using is 50 feet long. How far should the firemen set the ladder from the base of the building to get to the window? Draw a sketch of the situation and write in the measurements to help solve the problem. This is a right triangle, and the missing length can be found using the Pythagorean Theorem.



Example 2: A boat gets caught in a storm and the mast breaks. The base of the sail is 10 feet from the base of the mast and the diagonal of the sail from the base to the top of the mast is 17 feet. How tall is the mast? Make a sketch of the situation to help solve the problem.

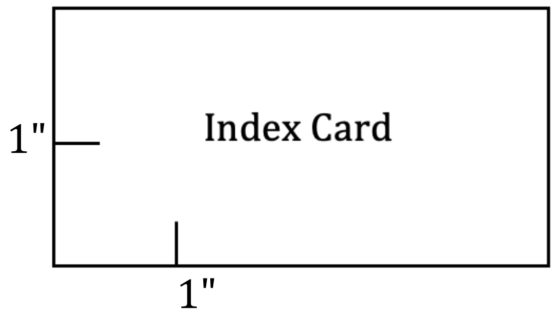
Section 5.9 The Wheel of Theodorus

Looking Back 5.9

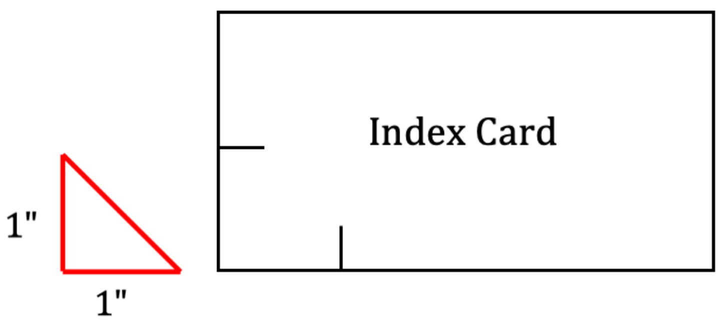
A fun way to use the Pythagorean Theorem is to make the Wheel of Theodorus. Theodorus of Cyrene was a Greek mathematician of antiquity; you will read more about his mysterious history in the Practice Problems for this section. All that we have learned about the Pythagorean Theorem will allow us to do this. When we are done making the Wheel, we will make a picture out of it. This will make sense as we move through the lesson.

Looking Ahead 5.9

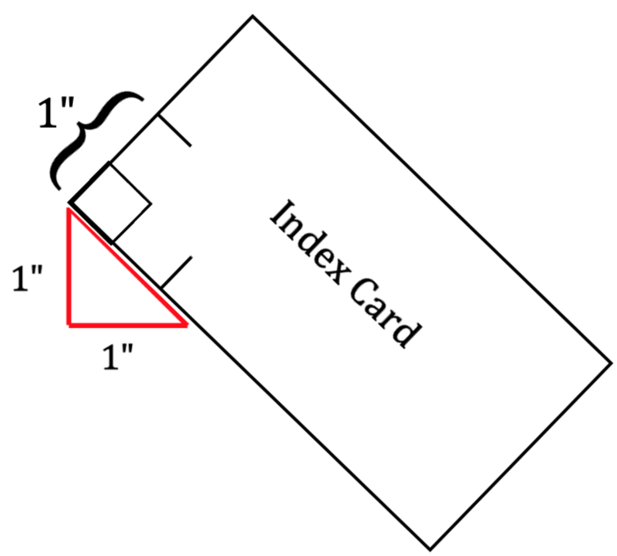
If you would like to make the Wheel of Theodorus, start with an index card in the upright position. Make tick marks one inch up from the bottom left corner and one inch to the right of the bottom left corner.



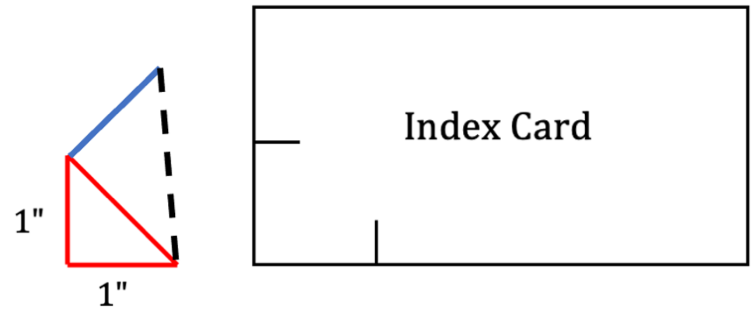
1. From the bottom left corner of the index card, draw a 1" line up to the tick mark. Then remove the card and connect the endpoints of the two 1" legs to make a right triangle.



2. Put the index card along the hypotenuse aligned with the uppermost point. The square makes a 90° angle. Draw a line 1" out from the uppermost point of the triangle along the index car.



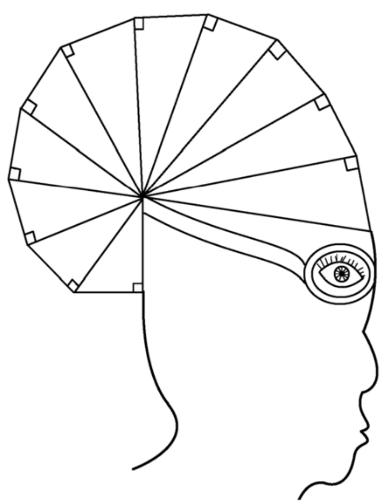
3. Remove the card and connect the new upper endpoint back to the original endpoint of the lower leg of the original triangle. This is shown by a dotted line.



4. Keep moving to the right in this way in circular fashion. Put the index card along the dashed line and draw another line out to the 1" tick mark. You can make the original triangle 2 by 2 inches if you like (or by more inches if you want a bigger design). What does the Wheel look like to you: the top of a lollipop, the tail of a Python, a snail? Draw a design around your Wheel of Theodorus and create a work of art that is pleasing to your eye.

As you can see by my design below, the Wheel looks like a swimmer's cap to me. It may look like an Egyptian prince or something else to you.

You will be making your own design in todays practice problems and learning more about mysterious Theodorus.



Section 5.10 Deriving the Pythagorean Theorem using Algebra and Geometry

Looking Back 5.10

We have seen marbles and quarters used to build a Pythagorean Board so that we can understand how the Pythagorean Theorem works. This is not a proof; we could keep changing the side lengths and adding quarters looking for a sum that does not work. We would have to test many possibilities or all possibilities to make sure that a counterexample does not exist. However, because the Pythagorean Theorem has been around for a few thousand years, it has been proven many times and in many ways. It can be demonstrated using algebra and geometry.

Looking Ahead 5.10

Let us look at some of the non-standard squares you discovered earlier in this module using 3×3 grids and 4×4 grids on the grid dot paper. Rather than counting the sides by their units to determine the numerical area, let us assign the side lengths letters and find the area in terms of variables. Then we will have a general formula for area. From the geometric models, we will derive formulas to determine areas for the large square.

Fill in the blanks using the examples with diagrams below them:

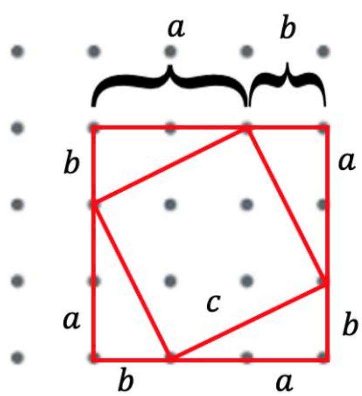
Each side length of the larger square is two segments. Let one segment length along a side be called "____________"

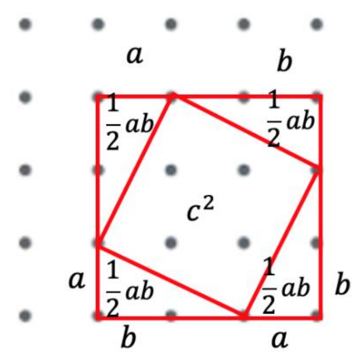
and the second segment length along the same side be called "____________." Each side length is ________________

or ________________. The area of the larger outside square is ____________________ or ____________________.

Example 1: Use the distributive property to find the area of the outside square using $(a + b)(a + b)$. This is equal to c^2.	

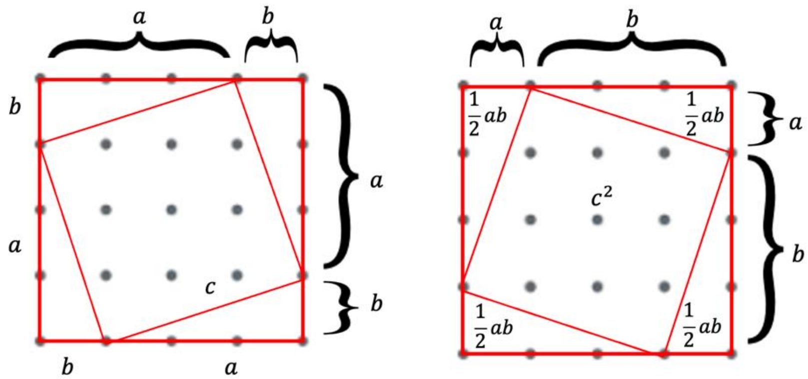
3×3 Grid





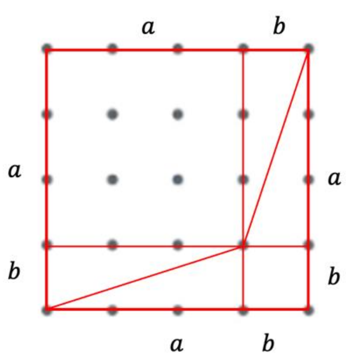
Example 2: Use the chop strategy to find the area of the inner square in terms of the area of the outer square minus the area of the triangles on all four corners of the square.

4×4 Grid



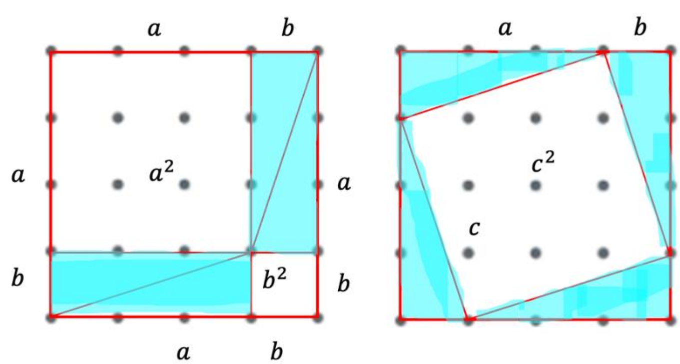
Another method for finding the area of the large outside square is to use the chop strategy and add the area of the small inner squares plus the area of the four surrounding triangles.

Example 3: Find the area of the large inner square and the area of the small inner square as well as the area of the surrounding four triangles in terms of a and b. Show that the sum of the inner areas is equal to the area of the entire square.



This model is the same size as the first model. Both models have four triangles.

The model on the left and the model on the right have equal areas. Therefore, geometrically and algebraically, $a^2 + b^2 = c^2$, which is the Pythagorean Theorem.



Section 5.11 The Pythagorean Theorem and the Golden Rectangle

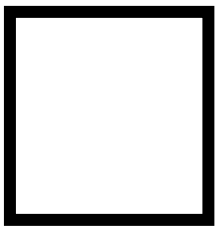
Looking Back 5.11

In General Mathematics we learned about the Golden ratio: it is the ratio of an arm length from the shoulder to the tip of the finger to the length from the elbow to the tip of the finger, and the length of a playing card to the width of a playing card. The symbol used for the Golden ratio is φ (phi). Phi (φ), much like pi (π), does not end or repeat: The Golden ratio is 1.61803399…

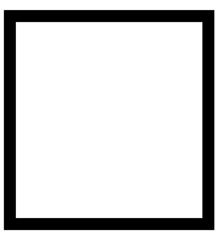
The discovery of the Golden ratio is attributed to Pythagoras. The Ancient Greeks used it in their art (for example, the statue of King David) and architecture (for example, the Parthenon). The Golden rectangle is most pleasing to the eye. Therefore, in this section, we will construct a golden rectangle and see how we can use the Pythagorean Theorem to calculate the Golden ratio.

Looking Ahead 5.11

1. Draw a square that is 1" long by 1" wide. Label it $ABCD$.

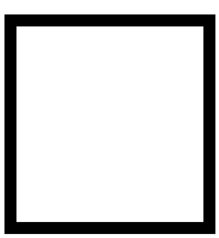


2. Bisect the bottom side of the square (cut it in half).

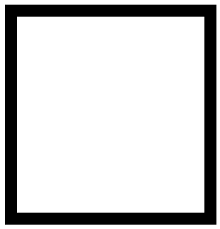


Label the midpoint E. The distance of DE is $\frac{1}{2}^{"}$ and the distance of EC is $\frac{1}{2}^{"}$.

3. Draw a dashed line segment connecting E and B. Extend side DC with a dashed line.



4. Putting your compass on the point E, open it to the length from E to B and draw an arc so that it intersects the line extended from point C. If you do not have a compass, use a piece of string or yarn: Hold one end still at point E and put the other end tight at B. The string can extend past your hand at both ends but hold it with your left hand at E and your right hand at B. Move your right hand down (keep the string tight and straight) gradually until it crosses the dotted line extended from point C and make a mark there. Label the mark point F.



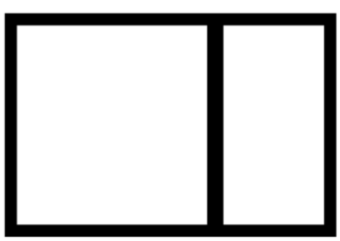
5. Extend side AB with a dashed line.

6. Draw a line from F straight up to the extended line from point B or perpendicular to the opposite side extended from point B. Stop at the point of intersection and label it point G.

Rectangle $AGFD$ is a golden rectangle. It consists of a large rectangle and a smaller one inside, rectangle $BGFC$. Label side AB with an a and side BG with a b. Since the rectangles are similar, we can set up a proportion:

$$\frac{a}{b} = \frac{a+b}{a}$$

Example 1: Let us find the lengths of our original square and rectangle and use those numbers in the Golden ratio algorithm.



1. The sides of the square are 1".

2. The distance of DE and EC are each $\frac{1}{2}$".

3. The legs of the right triangle ECB are the segment EC and the segment CB. The distance EC is $\frac{1}{2}$". The distance CB is 1".

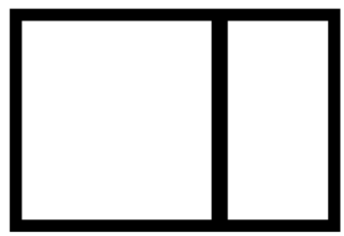
4. Let c be the dashed line, EB.

Use the Pythagorean Theorem to solve Problem 5.

5. The distance EF is the same as the distance EB. What is this distance?

6. The distance EF is $\frac{\sqrt{5}}{2}$". The distance EC is $\frac{1}{2}$". What is the distance of the distance CF?

$$EF - EC = CF \ \therefore CF = \frac{\sqrt{5}}{2} - \frac{1}{2} = \frac{\sqrt{5}-1}{2}$$



7. The length of a is 1".

8. The length of b is $\frac{\sqrt{5}-1}{2}$; $b = \frac{\sqrt{5}-1}{2}$ because that is the length of the extended side found in Problem 6.

9. Using the proportion $\frac{a}{b} = \frac{a+b}{a}$, substitute these values in for $\frac{a+b}{b}$ to solve for $\frac{a}{b}$.

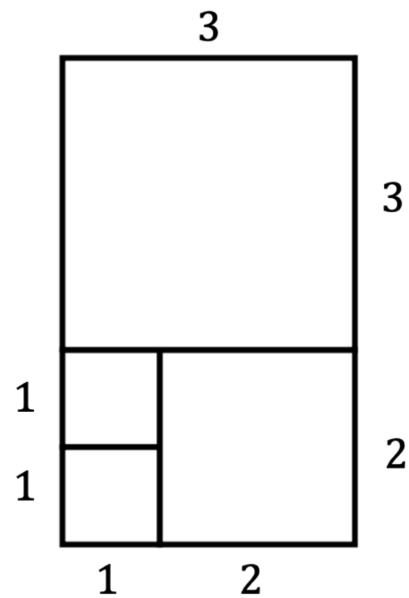
And there we have it, the Golden Ratio!
Another unique property of the Golden ratio is that $1 + \varphi = \varphi^2$.

(This is the phi symbol as in the Pyramids of Giza in Egypt; if you have some free time, look them up!)

<u>Section 5.12 The Pythagorean Theorem and the Fibonacci Numbers</u>

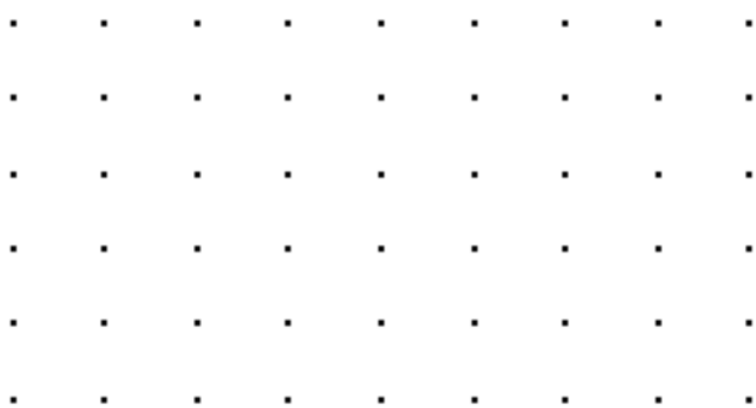
<u>Looking Back 5.12</u>

We can use unit squares to build a Golden rectangle. We begin by putting one unit cube next to another to get a side length of two units. Next to that, we put a new square with a side length of two units. That makes a rectangle that is three units long and two units wide. Lastly, put a three-by-three square next to that along the side that is one plus two units long.



If we continue this process, we continue to build a Golden rectangle. We are always building onto the new length. The next rectangle has a side length of $3 + 2 = 5$. The length of the next rectangle will be $5 + 3 = 8$.

If we take the ratio of the new side length to the old (previous) side length, we get the following:

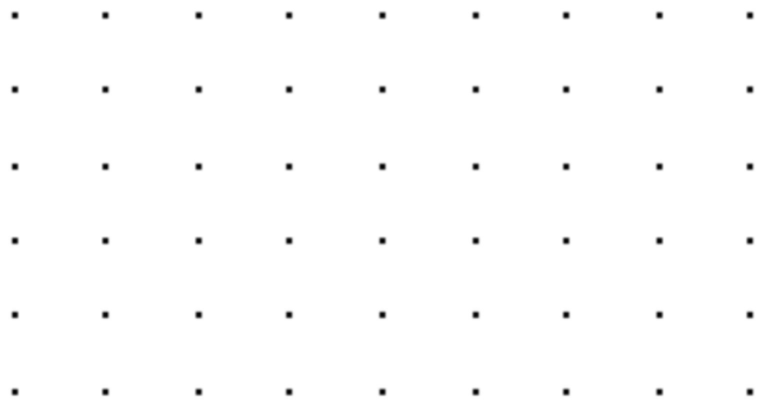


$\frac{2}{1} = 2$ $\frac{3}{2} = 1.5$ $\frac{5}{3} = 1.66667$ $\frac{8}{5} = 1.6$

We can see that it is approaching the Golden ratio! The ratio of the longer side of the length to the shorter segment of the previous length gets closer and closer to the Golden ratio as the rectangle gets larger and larger.

If we look at the side lengths, we see that they are Fibonacci numbers: $1, 1, 2, 3, 5, 8$ ... in which a number is added to the previous number to get the new number. The ratio of a Fibonacci number $F(n)$ and the previous Fibonacci number $F(n-1)$ approaches the Golden ratio. Both are found in nature for they are the handiwork of the Master Designer, the one true God.

If we connect these squares in a spiral moving along an arc from one corner to the next, we get what is called the Golden spiral or logarithmic spiral.



This spiral abounds in God's creation. If you cut a snail shell or a nautilus shell in half, you can see this spiral. It is called the Golden spiral. It is found in the horns of some goats and rams, in the cochlea of the inner ear (yes, you are golden!), and in many spider webs. The Milky Way has several spiral arms, each being a logarithmic spiral of 12 degrees; they follow a similar Fibonacci pattern as do hurricanes and facial features.

Looking Ahead 5.12

We have investigated how the Pythagorean Theorem relates to the Golden ratio. Now, let us explore the relationship between the Pythagorean Theorem and the Fibonacci Sequence.

We have found Pythagorean triples in a table previously. The ones on the left are primitive and the ones on the right are multiples of a primitive.

Primitive	**Multiples of a Primitive**
$3, 4, 5$	$6, 8, 10\ (2 \times (3, 4, 5))$
$5, 12, 13$	$9, 12, 15\ (3 \times (3, 4, 5))$
$8, 15, 17$	$12, 16, 20\ (4 \times (3, 4, 5))$
$7, 24, 25$	$15, 20, 25\ (5 \times (3, 4, 5))$
$20, 21, 29$	$10, 24, 26\ (2 \times (5, 12, 13))$
	$16, 30, 34\ (2 \times (8, 15, 17))$
	$18, 24, 36\ (6 \times (3, 4, 5)$

You can generate a Pythagorean triangle from four Fibonacci numbers. Now, let us look at the Algebra behind Fibonacci numbers.

1	a	Let the first number be a
1	b	Let the second number be b
2	$a + b$	
3	$(a + b) + b = a + 2b$	
5	$(a + 2b) + (a + b) = 2a + 3b$	
8	$(2a + 3b) + (a + 2b) = 3a + 5b$	
		The next numbers will always be the sum of the previous two.

Do you see a pattern?

The coefficients of a are on the left; the coefficients of b are on the right.

Coefficients of a		**Coefficients of b**
	b	1
1	$a+b$	1
1	$a+2b$	2
2	$2a+3b$	3
3	$3a+5b$	5

These are Fibonacci numbers so the next coefficients will be 5 for a and 8 for b for the expression $5a+8b$.

Now, let us use these letters and the Fibonacci numbers to generate a Pythagorean triangle.

1 a	2 b	3 $a+b$	5 $a+2b$

Follow these steps:

1. Multiply the two inner numbers: $2 \times 3 = 6$

2. Double the result: $2 \times 6 = 12$
(This is one side of the Pythagorean triangle)

3. Multiply the two outer numbers: $1 \times 5 = 5$

4. Add together the squares of the inner two numbers: $2^2 = 4$ and $3^2 = 9$ $4 + 9 = 13$
(This is the third side of the Pythagorean triangle)

This process gives us the Pythagorean triple $5, 12, 13$. You can use this process with any two numbers a and b, not just Fibonacci numbers.

This particular example has two Fibonacci numbers: 5 and 13. The Pythagorean triple $3, 4, 5$ has two Fibonacci numbers. It is still an unsolved problem if there are more Pythagorean (right-angled) triangles that contain just two Fibonacci numbers as lengths of the sides.

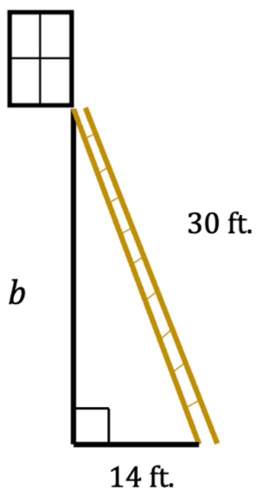
Section 5.13 More Pythagorean Theorem Problems

Looking Back 5.13

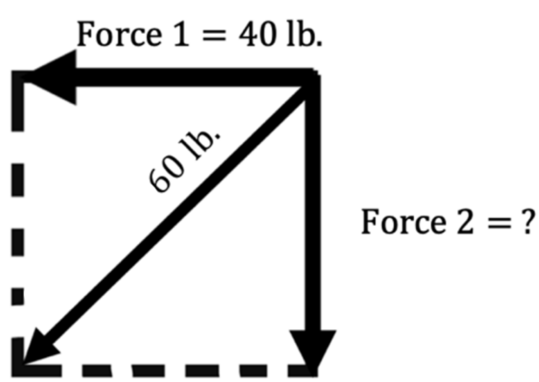
We have investigated how the Pythagorean Theorem relates to the Golden ratio and the Fibonacci sequence. We have also explored proofs of the Pythagorean Theorem and its uses. To finish this module, we are going to solve real-world problems using the Pythagorean Theorem. Any problem that involves a right triangle and has unknown sides, lengths, or distances may be an opportunity to use the Pythagorean Theorem.

Looking Ahead 5.13

Example 1: The escape route from a building is blocked due to a fire. A ladder is placed up to a second story window. The ladder is a 30-foot ladder and is placed 14 feet from the building at an angle. How high up is the window from the ground (b)?



Example 2: Two forces pull at right angles to each other. The resultant force is a diagonal in between the two forces. If one force is 40 lbs. and the resultant force is 60 lbs., what is the poundage of the opposite force?



Example 3:	Determine if the forces below are pulling at a right angle toward each other.

a) 25 lbs. and 15 lbs.; the resultant force is 35 lbs.

b) 4.5 lbs. and 2.4 lbs.; the resultant force is 5.1 lbs.