Module 8 Plane and Solid Figures

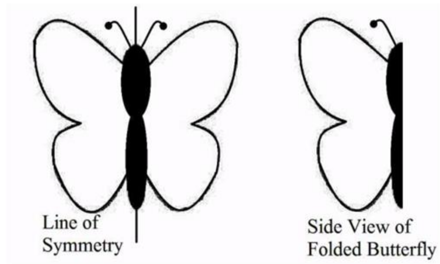
Section 8.1 Translation Symmetry

Looking Back 8.1

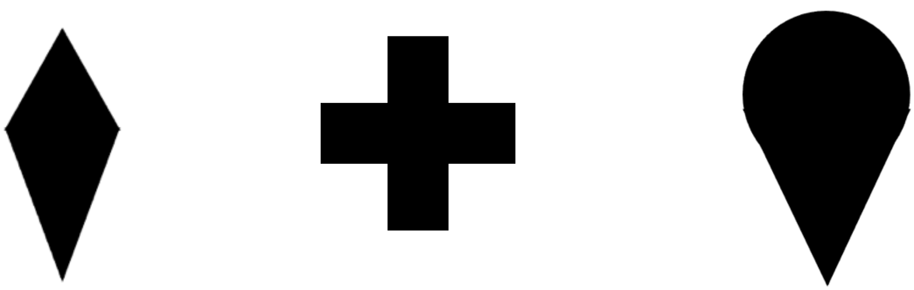
This module will focus on geometry as an introduction to the course you will be taking in two years after studying Algebra 1 next year. We have studied some geometry previously: we have represented algebra problems geometrically and we have solved geometry problems algebraically. The Pythagorean Theorem in Module 5 applies to both geometry and algebra.

"Geo" means earth and "metry" means measure. Circumference is one topic studied in geometry that can be used to measure the earth. We will study circumference later in this module. However, to begin we will investigate transformations.

Symmetry is balanced design that God has placed in nature. The body of a butterfly is the line of symmetry down the middle of the wings of the butterfly. If one wing is folded onto the other wing over the line symmetry, the one half looks like the other when opened. Each side matches the other. You are symmetrical as well. The right side of your body has the same features as the left side of your body. You are evidence of God's design in the universe.



Example 1: Draw the line or lines of symmetry in the figures below so the one part can be folded over the line of symmetry onto the other part; the parts should match up with no gaps or overlaps.

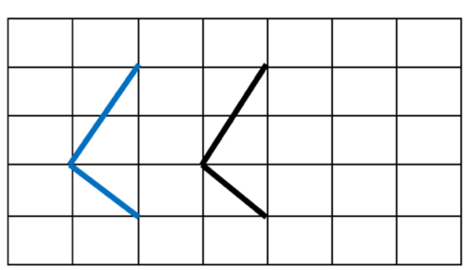


Looking Ahead 8.1

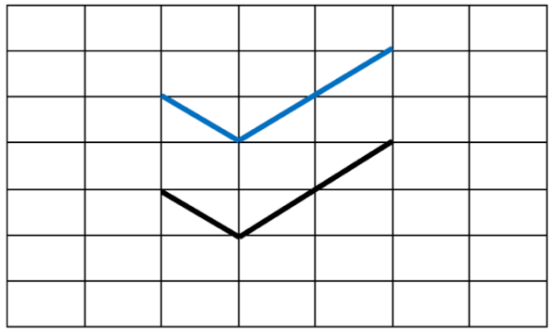
There are three types of symmetry in nature that we are going to study in this module. The three are: translations; reflections; rotations. We will begin with translations.

With translations, we can slide a pattern horizontally, vertically, or diagonally at some angle on a plane. A plane is something like this piece of paper that extends infinitely in two directions.

If you translate a design horizontally, you can move it right or left. Each point slides along a vector. The blue design below has been translated two units right. These vectors have a size of two units and a direction of East.

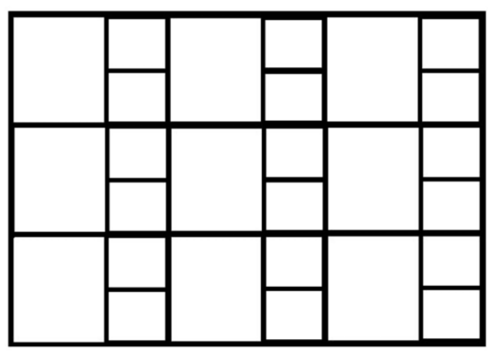


If you translate a design vertically, you can move it up or down. The blue design below has been translated two units down. What is the size and direction of each vector below?



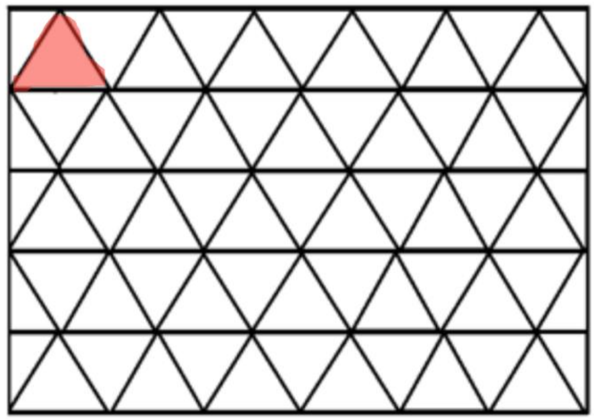
Example 2: A basic element of design is drawn below. When you slide it to the right, so that the left endpoint moves to the right most endpoint, you can cover a row. You can then move to a second row so the top left endpoint moves down to the bottom left endpoint, and slide the design across that row from left to right.

You can keep doing this until the page is covered and below is the design you will see:

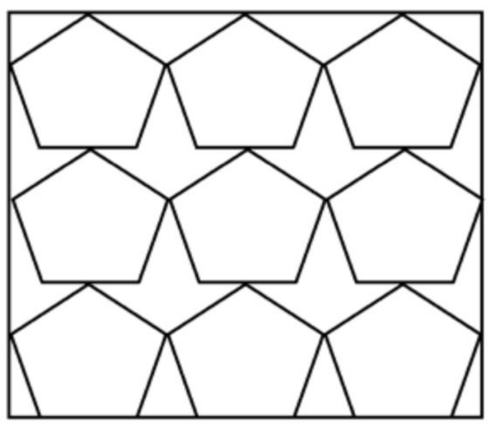


When each point of a figure or shape slides and maps onto a new image, you have a translation.
A basic design element is often used for art mosaics and wallpaper or tile designs. When you cover a plane with no gaps or overlaps the design becomes a tiling.

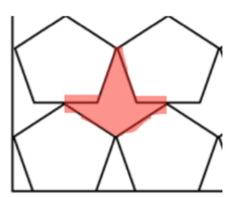
Example 3: Slide the colored triangle along the diagonal lines. Shade in all the triangles that result from the translation.



Example 4: Look at the pentagons below. Are they translated vertically, horizontally, or diagonally?



The pentagon is a basic design element that makes a border design when translated. The triangles that together look like an arrow, one of which is highlighted in red to the right, is also a basic design element that will make the same border design when translated.

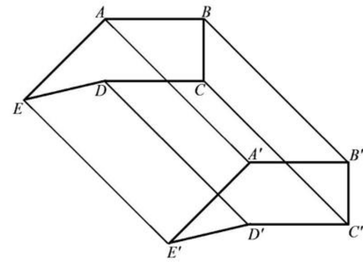


Section 8.2 Reflection Symmetry

Looking Back 8.2

In the previous section, we learned about translation symmetry. We looked at border designs on wallpaper and found the design element that had been translated. A translation slides a figure matching each point of the pre-image to its image.

Look at the shape to the right. Lines that extend from each pre-image point to each image point can be drawn to show the movement. For example, point A to point A', point B to point B', etc.



The extended lines from the pre-image to the image must be the same length so the image is not distorted or changed in any way from the pre-image.

Notice that the pre-image Pentagon $ABCDE$ is moved to the image Pentagon $A'B'C'D'E'$ and is still a pentagon. The size and shape have not changed.

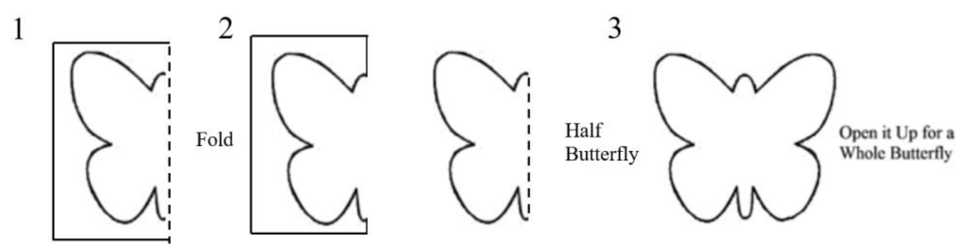
In this section, we will investigate reflection symmetry. The image also preserves the size and shape of the pre-image when reflected.

Looking Ahead 8.2

Activity 1

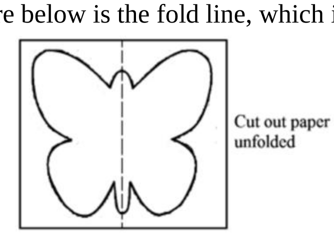
Remember the butterfly in the previous section? Let us make one.

1. Fold a paper in half and draw half of a butterfly on the folded half of the paper.
2. Cut the butterfly out.
3. You should have a whole butterfly when you open up the fold.

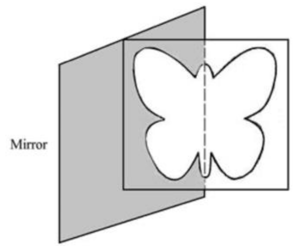


The open butterfly is symmetrical.

If you open the leftover folded paper with the cutout, you also get a symmetrical design. The fold is the line of symmetry. The dashed line in the picture below is the fold line, which is the line of symmetry.

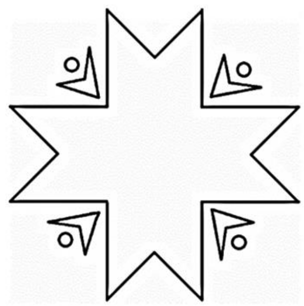


If you fold the paper in half again and put it up to a mirror on the line of symmetry (which is the dashed line where the fold is, also shown below), in the mirror, you will see the other half of the butterfly design. The line of symmetry is the line of reflection. The other half of the butterfly appears in the line of reflection. This is why it is called a mirror image. A mirror image is the same size and shape as the original design.

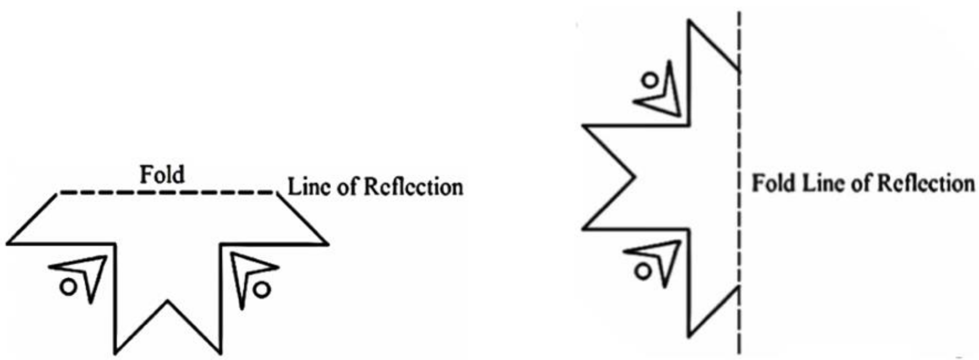


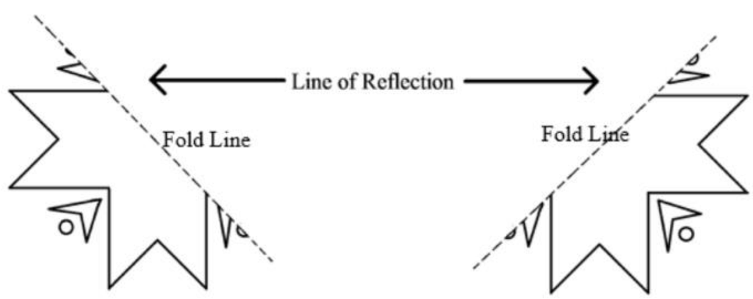
Activity 2

1. Copy the design below on tracing paper.
2. Draw lines that you think are lines of symmetry or reflection.
3. Fold along different lines to see when the design folds onto itself with no gaps or overlaps in the design.



Can you fold different ways to show all the lines of symmetry? (These are the lines of symmetry or reflection.)





When a design is folded onto itself so that all points of the design match up, the design is said to be superimposed onto itself. You will practice this in the Practice Problems section.

Section 8.3 Rotation Symmetry

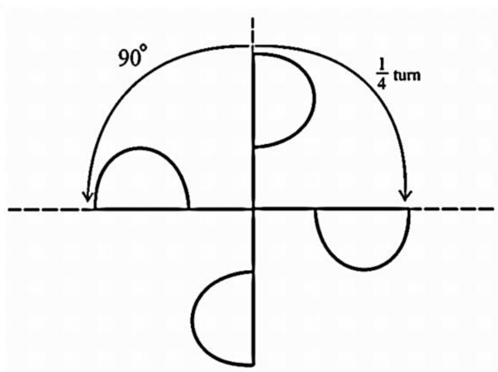
Looking Back 8.3

We have learned that slides along a given line of translation move a pre-image to its image without distorting the image. We have also learned that reflections in a given line of symmetry move a pre-image to its image without distorting the image. In this section, we are going to learn about rotation symmetry. Images with rotation symmetry are turned around a given point of rotation and move a pre-image to its image without distorting the image. If an image retains its original size and shape it is not distorted.

Looking Ahead 8.3

Turning a figure around a given point of rotation and moving the image without distortion is called a rotation. The rotation can be clockwise or counterclockwise.

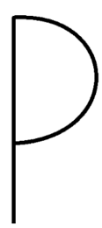
In Problem 9 of the previous Practice Problems section a question was asked: Does the design (below) have reflection symmetry?



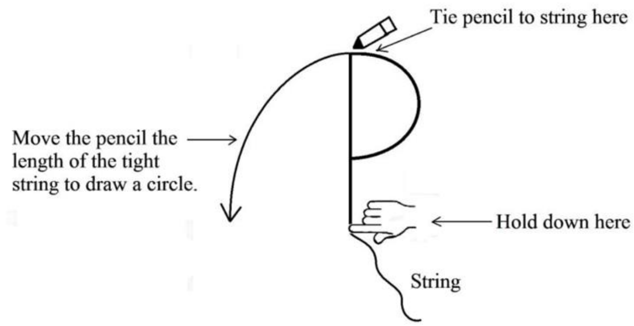
If the top reflects in the x-axis, the top does not superimpose onto the bottom image. Because the pre-image does not line up with the image, the image does not have reflection symmetry.

Now, trace the image on a piece of tracing paper. Let the point where the x and y axes meet be the point of rotation. Line up the tracing paper copy with the pre-image and rotate it one-quarter of a turn clockwise. Notice that the tracing lines up with the image. Next, move the image 90° counterclockwise. It still lines up with the pre-image. This means it has rotation symmetry. It can be turned around a given point of rotation without distorting the image.

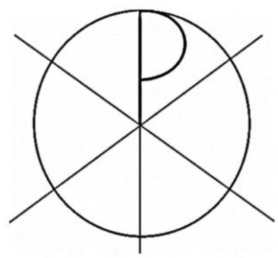
Use the basic design element P to begin. Let the bottom end of the pole of the P be the center of rotation. Follow the steps to create a design using the basic design element rotated either 60° clockwise or 60° counterclockwise.



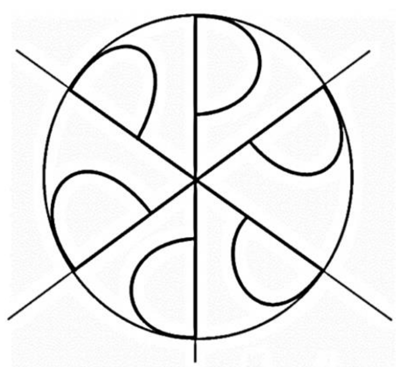
1. Draw a capitol P on a piece of paper. Use a ruler to make the pole of the P straight. This will be the axis of rotation.
2. Hold a string open the length of the pole of the P. This will be the axis of rotation from the point of rotation at the bottom point of the pole of the P. Tie a pencil to the upper point of the pole of the P.
3. Put your right finger on the string at the bottom point of the P at the axis of rotation.
4. Holding the string down at the bottom point of the pole of the P with your right finger, use the pencil tied to the upper point to draw a counterclockwise circle that has a radius the length of the pole of the P.



5. There are $360°$ in a circle; the rotations from tick mark to tick mark will be $60°$ each: $360°/60° = 6$ rotations. Use a protractor so the bottom is lined up with the pole of the P. Make a tick mark at $60°$ on the circle counterclockwise from the pole.



6. Use a ruler to draw a straight line from the tick mark to the opposite side of the circle.
7. Line up the base of the protractor with the new line and again mark $60°$ on the circle, then draw a line from the tick mark to the opposite side. Repeat this process one more time.
8. Put tracing paper over the original paper. Copy the P, circle, and three lines onto the tracing paper.
9. Rotate the tracing paper around the center point $60°$ clockwise and copy the P onto the tracing paper.
10. Keep moving the tracing paper around the circle, stopping when the lines are lined up to copy the P onto the tracing paper until a full circle has been made (as shown below).

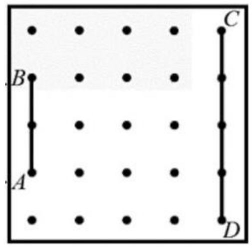


There are other ways to do this if you do not have string available. You could open a compass the length between the center point of rotation at the bottom of the P and the point being rotated to make each tick mark. You could also use a protractor to draw the circle and measure out each rotation.

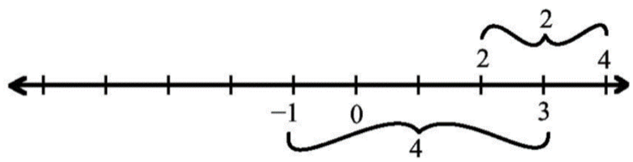
Section 8.4 The Distance Formula

Looking Back 8.4

In Module 5, we learned how to find the length of horizontal and vertical lines on a coordinate plane: You simply count the number of units between points or ordered pairs.



A mathematical way to find the distance between points on the number line is to subtract the two points. The distance from 2 to 4 is $4 - 2 = 2$. The distance between -1 and 3 is $3 - (-1) = 3 + 1 = 4$. The space between the points is the distance between them.



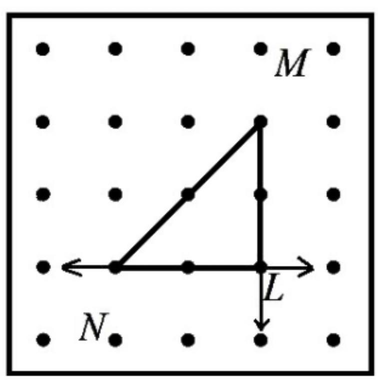
This is what we did with slope. We call slope the change in y over the change in x or $\frac{\Delta y}{\Delta x}$ or $\frac{\text{rise}}{\text{run}}$. The change in y is the distance between the two y points $|y_2 - y_1|$. The change in x is the distance between the two x points $|x_2 - x_1|$. That is where the slope formula comes from:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Just as we derived the formula for slope by looking at patterns, we derived the formula for the distance of a diagonal line when given two points in Module 5. Let us revisit this now.

Looking Ahead 8.4

Example 1: Suppose you are given the two points M and N. Let N be the point (x_1, y_1) and M be the point (x_2, y_2) Find the coordinates for L in terms of x and y.



Example 2: Find the distances of LN, LM, and NM.

The distance of LN is the horizontal distance between $L(x_2, y_1)$ and $N(x_1, y_1)$. It is $x_2 - x_1$. The distance of LM is the vertical distance between $L(x_2, y_1)$ and $M(x_2, y_2)$. It is $y_2 - y_1$.

We can use the Pythagorean Theorem (which we learned in Module 5) because we are looking for a missing side length in a right triangle and we already know the distance of the other two sides (the legs).

If we call the length of LN side a, the length of LM side b, and the length of NM side c, then we have the equation below from the Pythagorean Theorem ($a^2 + b^2 = c^2$).

$$(LN)^2 + (LM)^2 = (NM)^2$$

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = (NM)^2 \qquad \text{Substitute}$$

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(NM)^2} \qquad \text{Take the square root of both sides}$$

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = NM$$

Therefore, the distance between any two points (x_1, y_1) and (x_2, y_2) is as follows:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This is true for lines in which d is the distance of the diagonal line on the dot grid or the hypotenuse of the right triangle created.

Example 3: Use the distance formula to find the distance between points $(3, -4)$ and $(1, 6)$. Let $(3, -4)$ be the point (x_1, y_1) and let $(1, 6)$ be the point (x_2, y_2).

Example 4: Use the distance formula to find the distance between the points $(3, -4)$ and $(1, 6)$, but in this example let $(1, 6)$ be the point (x_1, y_1) and $(3, -4)$ be the point (x_2, y_2).

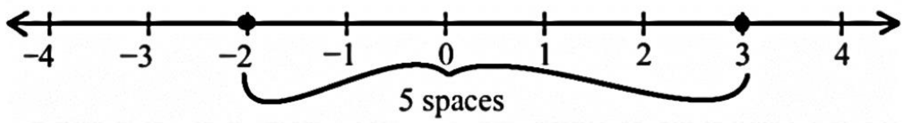
Section 8.5 The Midpoint Formula

Looking Back 8.5

Geometry involves shapes and measures. When shapes are represented on the coordinate grid, there is a frame of reference for length of sides, dimensions, areas, perimeters, and many measurements. Once formulas are developed, these measurements can be calculated using numbers given. By first analyzing these shapes on the coordinate grid, formulas can be developed using variable representations.

Looking Ahead 8.5

Find the distance between -2 and 3.



Half of the space between -2 and 3 is:

$$\frac{1}{2}(3-(-2))$$
$$\frac{1}{2}(5)$$
$$\frac{5}{2}$$
$$2.5$$

Or

$$\frac{3-(-2)}{2}$$
$$\frac{5}{2}$$
$$2.5$$

Now, add this to -2:

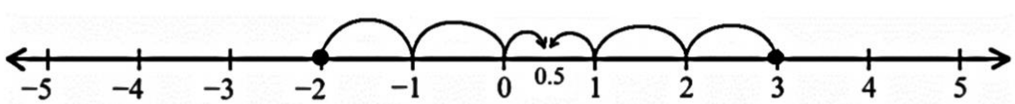
$$-2+2.5=0.5$$

Or

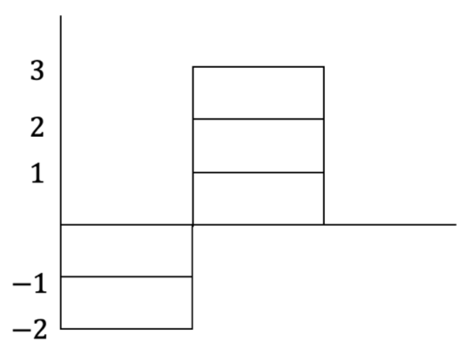
Subtract this from 3:

$$3-2.5=0.5$$

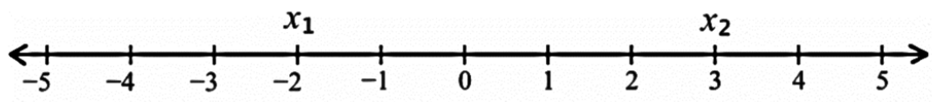
Find the midpoint between -2 and 3.



The midpoint is in the middle of the two points. It is the average of the two points.

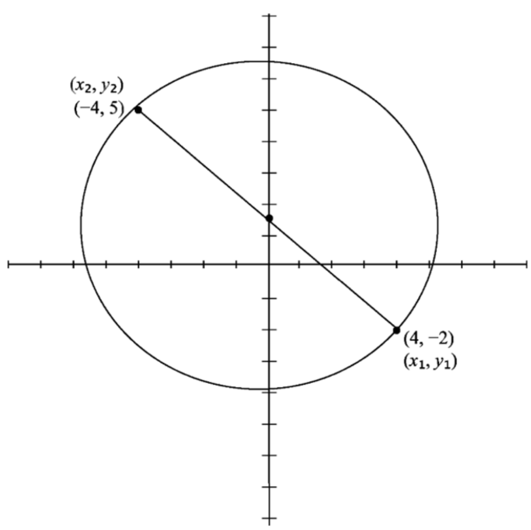


Example 1: Find the midpoint algebraically in terms of x_1 and x_2.



The midpoint formula for a line segment with endpoints (x_1, y_1) and (x_2, y_2) is $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$. The midpoint is the average of the two points on the number line. It is also the average between the horizontal and vertical points on the coordinate grid.

Example 2: In Problem 6 of the previous Practice Problems, you found the distance of the diameter of a circle with endpoints $(4, -2)$ and $(-4, 5)$. Now find the midpoint of the diameter of the circle below.

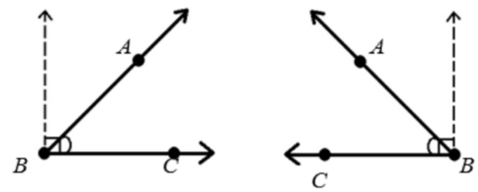


<u>Section 8.6 Angle Relationships</u>

<u>Looking Back 8.6</u>

We have learned that angles can be classified as right if they are 90°. In a right triangle, the rays that meet at the vertex form a square angle. Angles are obtuse if they are greater than 90° and acute if they are less than 90°.

When two rays meet at a common endpoint, they form an angle. The common endpoint is the vertex of the angle. The angle below may be called $\angle B$, $\angle ABC$, or $\angle CBA$.

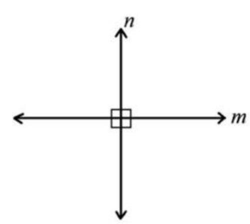


Example 1: Fill in the blanks.

We have learned that angles that are 90° or form a square angle are called _____________________ angles. If the angles are greater than 90°, they are called _____________________ angles. If they are less than 90°, they are called _____________________ angles.

In the diagram above, the angle is acute because it is smaller than a right angle, which is shown with the dashed lines; that means $\angle ABC$ is less than 90°.

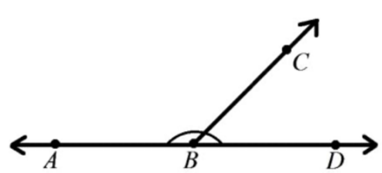
As you can see in the diagram below, the rays of a(n) ______________________ angle are perpendicular. Perpendicular lines meet at a point that forms an L shape, a T shape, or a cross +.



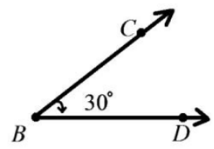
Lines m and n are perpendicular ($\perp$).

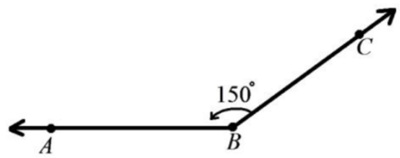
Example 2: Fill in the blanks.

A straight line has an angle of __________°. This is the semi-circle, which is the shape of a protractor. If you draw a ray coming out of the straight line or straight angle, you get two angles that are adjacent (next to each other) because they share a common side (the ray) and a common vertex.



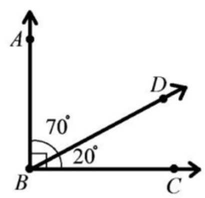
Angle ABC and $\angle DBC$ are _____________________ (next to each other). The common ray between the two angles is ______________. Also, $\angle ABC$ and $\angle DBC$ are supplementary because they have a sum of __________°.

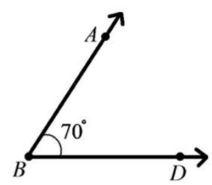


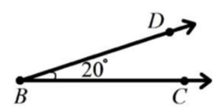


Even if they are separated, they are still called supplementary angles. The common vertex of the two angles is point __________.

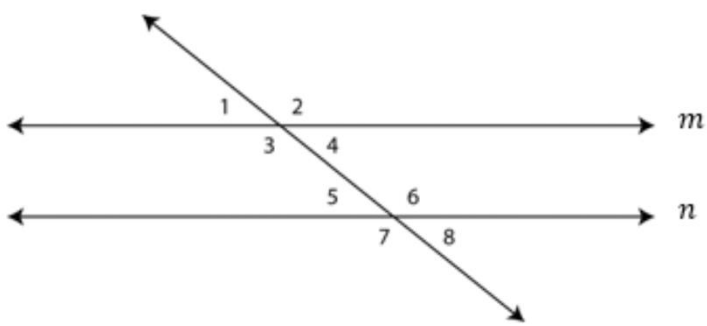
If you put a ray in between the two rays that are the sides of a right angle, you get two angles again. They are complementary (they are also adjacent and connected by a common side and common vertex). They are called complementary because they add up to 90°. That means $\angle ABD$ and $\angle DBC$ are complementary because they have a sum of 90°. We cannot label any angle as $\angle B$ because there are two angles at that vertex, not just one angle. That would be confusing!







Example 3: Parallel Lines: Lines m and n are parallel because they never cross when extended infinitely.
A transversal is a line that intersects two other lines. The transversal below cuts through two parallel lines m and n. All the angles are numbered. Find the pairs that supplementary.

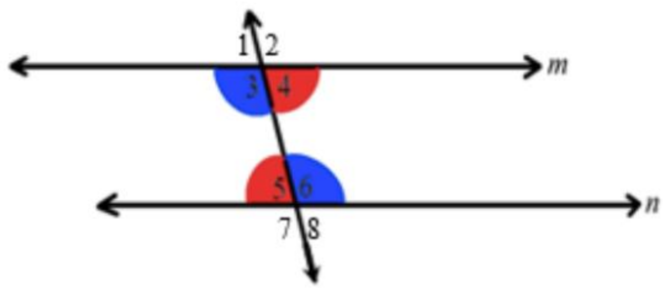


Find all the pairs of angles that are supplementary:

_______________ _______________ _______________ _______________

_______________ _______________ _______________ _______________

Example 4: Alternate Interior Angles are on opposite sides of the transversal between the two parallel lines.

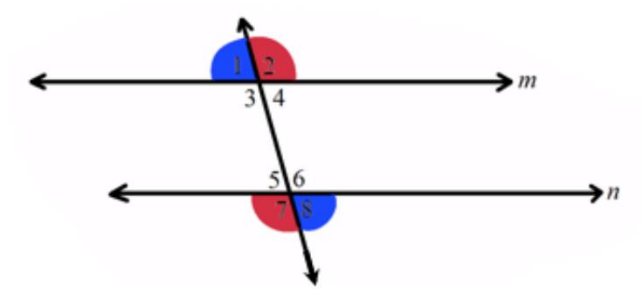


There are two pairs of Alternate Interior Angles in the diagram above; can you name them?

_______________ _______________

What do you notice about the measures of the Alternate Interior Angles?

Example 5: Alternate Exterior Angles lie above and below the parallel lines and are on opposite sides of the transversal.



There are two pairs of Alternate Exterior Angles in the diagram above; can you name them?

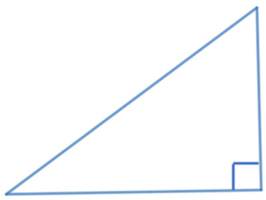
_______________ _______________

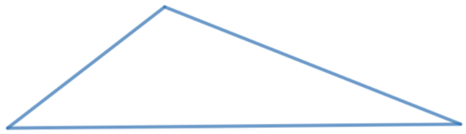
What do you notice about the measures of the Alternate Exterior Angles?

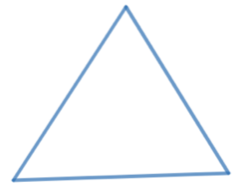
<u>Section 8.7 Classifying Triangles and Quadrilaterals</u>

<u>Looking Back 8.7</u>

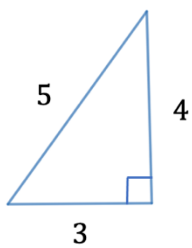
Just as there are three types of angles: right, obtuse, and acute, there are three types of triangles that are identified by their angles. A right triangle has one right angle, and an obtuse triangle has one obtuse angle. In an acute triangle, all three angles are acute.


Right

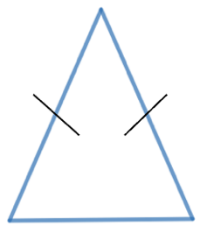

Obtuse

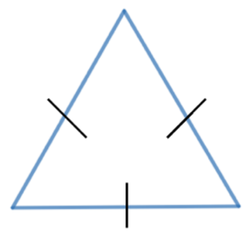

Acute

Triangles can also be named by their sides. Scalene triangles have no equal sides, isosceles triangles have two equal sides, and equilateral triangles have three equal sides.



Scalene


Isosceles


Equilateral

Polygons are named by their sides, which is shown in the table below. The prefix in the name tells the number of sides the polygon has and the suffix “gon” means many-sided.

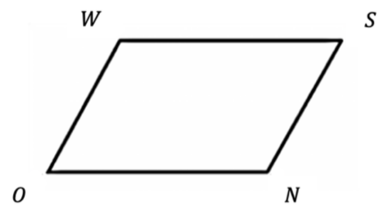
Number of Sides	Name	Prefix Root Word
3	<u>Triangle</u>	(<u>Three</u> angles)
4	<u>Quadrilateral</u>	(<u>Four</u> sides)
5	<u>Pentagon</u>	(Five)
6	<u>Hexagon</u>	(Six)
7	<u>Heptagon</u>	(Seven)
8	<u>Octagon</u>	(Eight)
9	<u>Nonagon</u>	(Nine)
10	<u>Decagon</u>	(Ten)

<u>Looking Ahead 8.7</u>

A quadrilateral is a four-sided figure.

Example 1:	Fill in the blanks for a parallelogram.

A parallelogram is a __________________________.



In the parallelogram $SNOW$, $SN \parallel OW$ (read: "segment S-N is parallel to segment O-W") and $SW \parallel NO$ (read: "segment S-W is parallel to segment N-O").

Parallelograms have the following three special properties:

a) Opposite sides are ___________________________.

b) Opposite sides are ___________________________.

c) Opposite angles are ___________________________.

That means that same-sided angles are supplementary.

Example 2: Fill in the blanks for a rectangle.

A rectangle is a quadrilateral (and parallelogram) in which:

a) Opposite sides are ___________________________.

b) Opposite sides are ___________________________.

c) All angles are ___________________________.

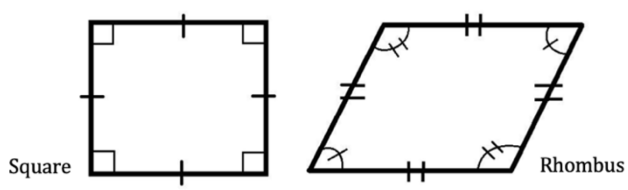
Example 3: Fill in the blanks for a square and a rhombus.

A quadrilateral (or parallelogram) with all sides equal, opposite sides parallel, and all four angles equal is a

___________________________. The angles are all ________________.

A rhombus has opposite sides ___________________________, _____________________, and four sides equal, but

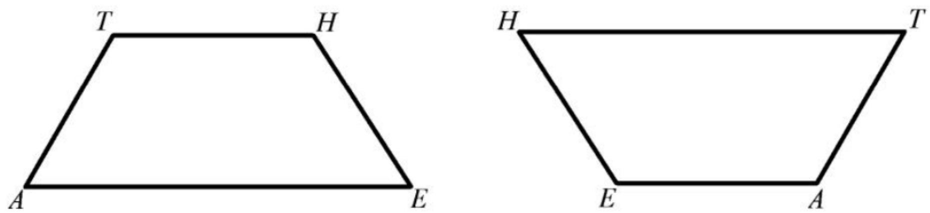
is different from a square. Tell how.



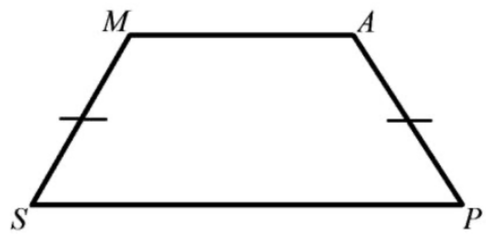
A square is a special rectangle whose sides are equal and a special rhombus whose angles are equal.

Example 4: Fill in the blanks for a trapezoid.

A trapezoid is a quadrilateral with only _________ ___________ of parallel sides. The other pair are ____________ parallel.

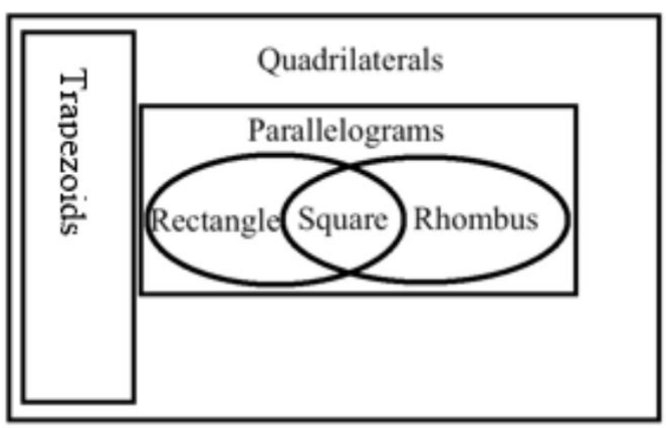


Above, $TH \parallel AE$ but TA is not parallel to HE. Because the sides are not necessarily equal, the opposite angles are not equal. The parallel sides are called bases and the non-parallel sides are called legs.



Opposite sides can be equal in a trapezoid.

If the legs are equal, then the trapezoid is called an isosceles trapezoid. The base angles are also equal.
Below is the traditional Venn Diagram for Quadrilaterals:



There is some debate as to whether the parallelogram is a special kind of trapezoid in which both pairs of opposite sides are parallel. This would change the definition of a trapezoid to: "a quadrilateral with at least one pair of parallel sides."

Using this definition, the flowchart would look as below:

Quadrilateral

Trapezoid

Parallelogram

Isosceles Trapezoid

Rhombus

Rectangle

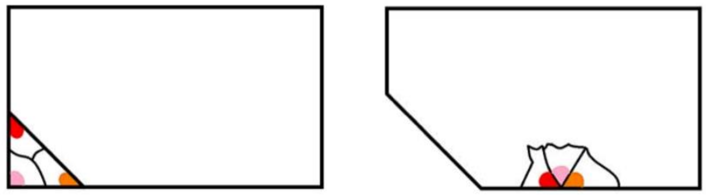
Square

<u>Section 8.8 Angle Sums of Polygons</u>

<u>Looking Back 8.8</u>

The angles of a triangle add up to 180°. We have learned this previously in our studies. Now, let us complete this activity to prove it.

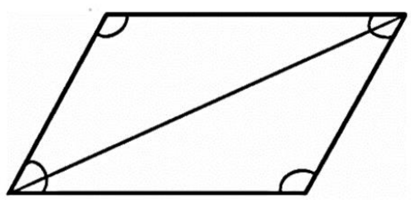
If you take an index card and draw a triangle in the corner and cut it in three pieces, you can line up all the angles on the side of the index card to see how they fit along the line with no gaps or overlaps. Make sure you color each angle of the triangle a different color initially, before cutting the triangle into three pieces.



Example 1:	Use the diagrams above and below to fill in the blanks.

The index card is a rectangle and has ____________________ _______________ angles. The sum of the angles in a rectangle is _______________, the same as that of a circle.

A parallelogram is a quadrilateral with _______________ angles and can be sliced into two triangles. The six angles of the two triangles fit into the four angles of a quadrilateral. Because each of the angles of the triangle total _______________, then the two triangles in the quadrilateral total _______________.



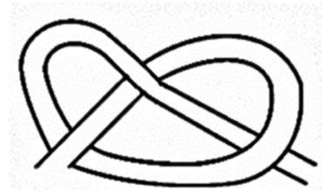
<u>Looking Ahead 8.8</u>

Because we know the sum of the angles of a triangle, we can break polygons up into triangles to find the sum of the angles of the polygon. We can divide the sum of the angles by the number of angles to find the degree of each angle in a polygon. This only works for regular polygons in which all the angles are equal and all the sides are equal. So, our investigation will be of regular polygons only.

Let us start by making a paper pentagon:

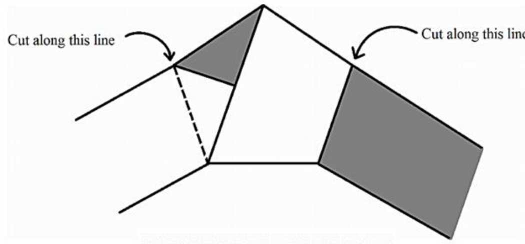
1. Lay a ruler along the longer side of a sheet of paper with the straightedge towards the paper. Make a line on the paper along the straightedge of the ruler. Cut along the line so you have a long strip of paper with a constant width.

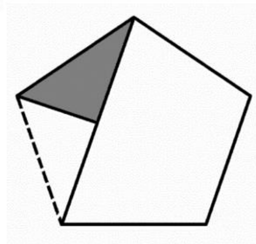
2. Tie an overhead knot in the long strip of paper as if you are tying your shoes. It looks like a pretzel (to the right).



3. Pull each end slowly to tighten the pretzel and then flatten the creases.

4. Cut off the lengths hanging out from the sides of the pentagon.



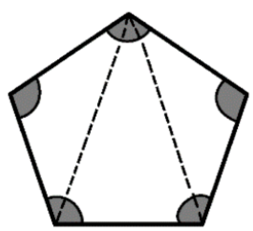


5. Use a ruler to measure the edges. They should each be approximately equal.

6. Use a protractor to measure each angle. They should each be approximately equal.

Example 2: Use the diagram below to fill in the blanks to learn how to calculate the angles of a regular pentagon.

Draw lines from a vertex to the other vertices. Notice two of the lines are sides of the pentagon.



You can divide a regular pentagon into _______________ triangles.

Each triangle has a sum of _______________.

Multiply the number of triangles by 180°: ____________________

The sum of the interior angles of a pentagon is _______________.

Divide the sum of the angles by the number of angles to find the measure of each angle:

_______________ ÷ 5 = _______________

The triangles come from one vertex and the diagonals in the polygon connect to all the non-adjacent vertices of the polygon. The two on either side of the angle are sides of the polygon.
A four-sided figure can be sliced into two triangles. A five-sided figure can be sliced into three triangles. How many triangles can an eight-sided polygon be sliced into? Do you see a pattern?

Example 3: Fill in the blanks to find a formula for the measure of each angle in a regular polygon.

If n is the number of sides in a regular polygon, and the sum of the angles of one triangle is _______________, then

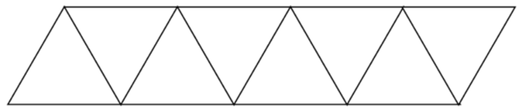
the sum of the triangles in an n-sided polygon is ___________ $\cdot\ 180°$. The degree of each interior angle in an

n-sided polygon is _______________.

Section 8.9 Regular Tessellations and More

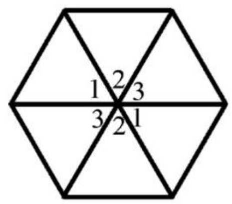
Looking Back 8.9

Earlier in this module, we learned about tiling. A tiling or tessellation is a pattern of shapes or polygons that completely cover a flat surface or plane with no gaps or overlaps. Regular triangles will tessellate (cover) a plane without gaps or overlaps. You can cover a plane with triangles only (as shown below). However, regular pentagons do not tessellate a plane. If you try to cover a plane in regular pentagons only, there will be spaces between them that are not pentagons. (See Example 3 and 4 of Section 8.1.) A regular tessellation uses one regular polygon to cover a plane. The angle surrounding each vertex is congruent. Let us see if we can figure out what other regular polygons will tessellate the plane and why.



Looking Ahead 8.9

Any given point on a plane can be surrounded by triangles. The triangles can be rotated around any vertex. There are six triangles at each vertex; they form a hexagon.



Angles 1, 2, and 3 above are on a straight line. Their angles sum up to $180°$. As you know from the previous section, each angle of a regular triangle measures $60°$; therefore, three angles of a regular triangle measure $60° + 60° + 60° = 180°$.

By vertical angles, which you know from Section 8.6, the three angles below the horizontal line are also each $60°$ and sum up to $60° + 60° + 60° = 180°$.

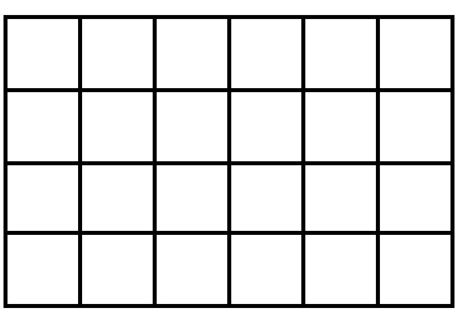
Therefore, there are $360°$ surrounding a vertex.

A tessellation is named by the polygons surrounding a vertex. Each polygon is named by the number of its sides. The shape above is called a 3.3.3.3.3.3 because the vertex is surrounded by six triangles.

Now, do you know why a regular pentagon will not tessellate (cover) a plane? The interior angles of a regular pentagon are $108°$, which you might remember from your table in the previous section. If you put three pentagons around a point, then you have $108° \times 3 = 324°$; that is $36°$ less than $360°$ so there is a gap. If you add a fourth pentagon, you have $108° \times 4 = 432°$; that is $72°$ over $360°$ so there is an overlap.

There are two other regular polygons that tessellate a plane. Look at the table from the previous practice problems section to see if you can figure out which two, they are. Ask yourself, "Which regular polygon has interior angles that are multiples of $360°$?" This is the only way the shapes can fit perfectly around a vertex with no gaps or overlaps.

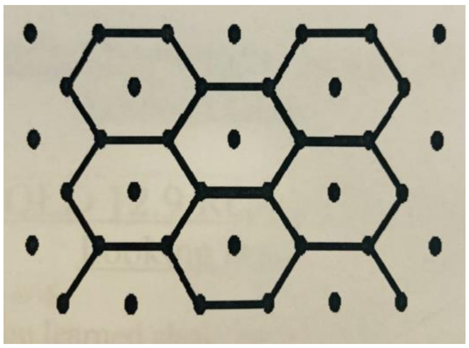
Example 1:	What is the name of the regular tessellation below?



This pattern nicely covers a paper. It is often used for floors and ceilings because of its simplicity.

Example 2:	Can you tessellate a plane with the basic design element below? Is it a regular tessellation?

Example 3:	What is the name of the third regular tessellation, which is made out of hexagons?

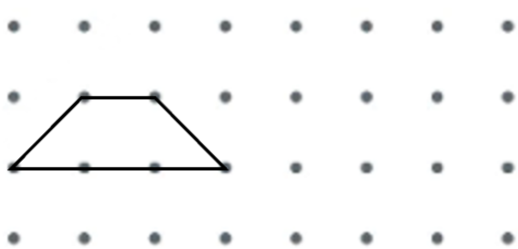


Notice, a normal dot grid has horizontal and vertical distances of one unit and the diagonal distance between two dots is $\sqrt{2}$ units. This does not work for the regular hexagon (which is shown above) so an isometric grid is used.

Hexagons can be found in nature and are particularly durable structures. Bridges built from triangles and hexagons are strong structures because of God's intelligent design. Bees build their homes in hexagonal patterns because of the instincts God gave them.
From your table, you can see the interior angles of a regular hexagon are $120°$ each. There are three hexagons that surround a point: $120° \times 3 = 360°$.

The three regular polygons that tessellate a plane are triangles, squares, and hexagons. These designs are all around you. You can find them in architecture, such as floor and ceiling tiles, and mosaics all over Europe. You can also find them many places in nature, such as beehives.

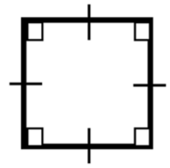
Example 4: Use the dot grid below to tessellate trapezoids. These are not regular. The top is 1 unit long and the bottom is 3 units long. You know the diagonals are $\sqrt{2}$ from your work in Module 5. (These will tessellate the plane, but you may have to do some translations and rotations.)

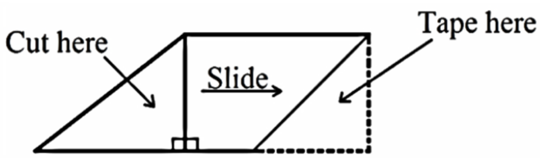


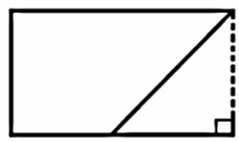
Section 8.10 Surface Area of Polygons and Solids

Looking Back 8.10

The area of a polygon is the number of square units that cover its surface. If we break a square or rectangle into rows one unit wide and columns one unit long, we can count the squares or multiply the length by the width to find the number of square units. If the length is measured in inches, the units are inches and the squares formed by the intersection of the length and width are square inches.



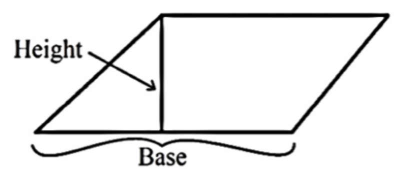




The area of a square is length times width but because it is the same, we can say "side by side." The area of a square is Area $=$ side $\times$ side ($A = S^2$). The area of a rectangle is length $\times$ width. The length is usually the longer side and the width is the shorter side. The area of a rectangle is Area $=$ length $\times$ width ($A = l \times w$).

Any parallelogram has an area of base $\times$ height. Locating the base and height are key to finding the correct area. If you draw a line from the top vertex of a parallelogram perpendicular to the base on the opposite side, you will get a triangle. If you cut off the triangle and translate it to the other side of the parallelogram, you will get a rectangle.

The base of the parallelogram is the length of the bottom side. When you slide it from left to right, the length of the base stays the same size. The height is the vertical line, which is the original line you drew and cut along to form a triangle (represented by the dashed line above).



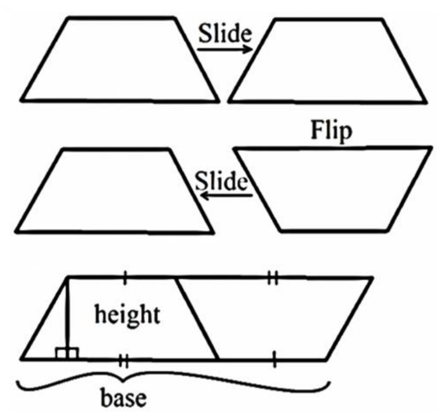
The slanted side is not the actual height; it is longer than the height.

The area of a parallelogram is Area $=$ Base $\times$ Height ($A = b \cdot h$) in which the height is the perpendicular line from the base to the opposite side.

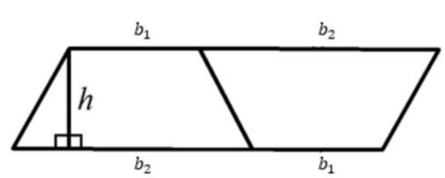
If you start with a trapezoid and slide a copy of it to the right, you will have two trapezoids. However, if you flip the trapezoid over and slide it back next to the trapezoid so the slanted sides are just overlapping, you will have a parallelogram. Knowing the area of a parallelogram will help you find the area of the trapezoid.

We can see that the height is once again the perpendicular line from a vertex to the opposite side and the base is actually $b_1 + b_2$. So, the area of the parallelogram is $(b_1 + b_2) \cdot h$, but we want only the area of half the parallelogram, which is one trapezoid. Therefore, the area of a trapezoid is $A = \frac{1}{2}(b_1 + b_2) \cdot h$.

These are area formulas for two-dimensional figures (plane figures). They can be used to find the surface area of the faces and bases of three-dimensional figures (solid figures).



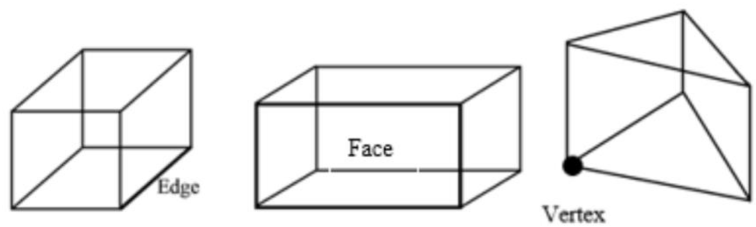
$A = b \times h$



Let us go over the formulas before we move on:
The area of a square is Area $=$ side $\times$ side ($A = S^2$).
The area of a rectangle is Area $=$ length $\times$ width ($A = l \times w$).
The area of a parallelogram is Area $=$ Base $\times$ Height ($A = b \cdot h$) in which the height is the perpendicular line from the base to the opposite side.
The area of a trapezoid is $A = \frac{1}{2}(b_1 + b_2) \cdot h$.

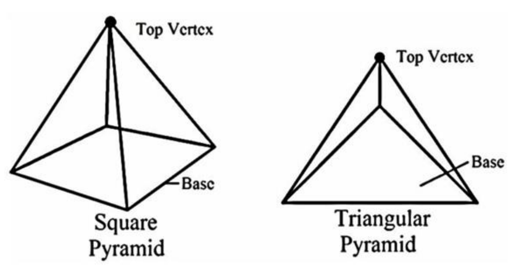
<u>Looking Ahead 8.10</u>

Prisms have all faces made out of polygons. They are named by their bases. A square pyramid is a hexahedron. All six faces are squares including the base. A rectangular prism must have at least four rectangular faces that are not square with the two ends being squares that are not bases, but are faces. A triangular prism has five faces, three of which are squares and two of which are triangles. The vertex is where the two edges meet. An edge is in between two vertices; it connects them. A face is each of the sides, which are made up of polygons.

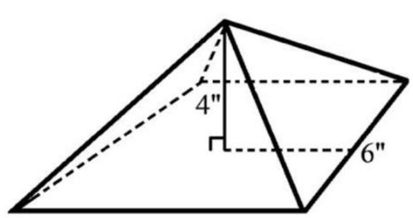


A pyramid is different than a prism. A pyramid comes to a point at the top. All the faces that are not the base meet in one point, which is the top vertex. This vertex is opposite the base, which is on the bottom.

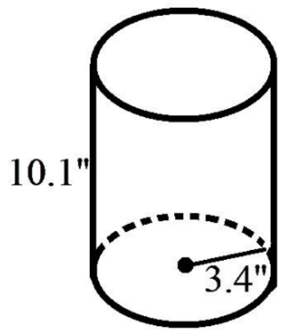
A square pyramid has a square base, a triangular pyramid has a triangular base. When you find the surface area, you are looking for the area on the surface of each of the faces (including the base) and then adding them all up for a total area of all the faces of the solid (the base is a face).



Example 1:	Find the surface area of the square pyramid.



Example 2: A cylinder is not a prism or a pyramid because a circle is not a polygon. Find the surface area of the cylinder.

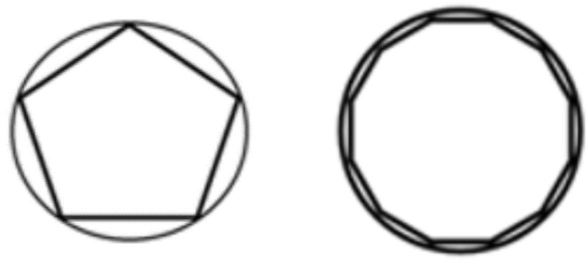


Section 8.11 Area and Circumference of a Circle

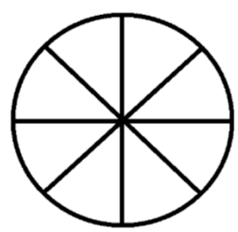
Looking Back 8.11

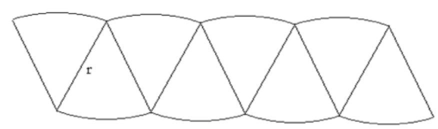
To find the areas of newly introduced polygons we have been breaking them down into smaller polygons whose areas we know how to find. In the previous section, we divided a polygon into triangles to find the sum of the interior angles of the polygon.

In the figure on the left below, a pentagon is inscribed in a circle and the circle is circumscribed about the pentagon. On the right below, what can you say about the dodecagon in relationship to the circle? What can you say about the circle in relationship to the dodecagon?



Let us cut our circle into eight equal pieces. Now, let us see if we can put these pieces together to make a shape whose area formula is known. Once we have a formula, we can use the names of the parts to find the formula for the area of a circle.



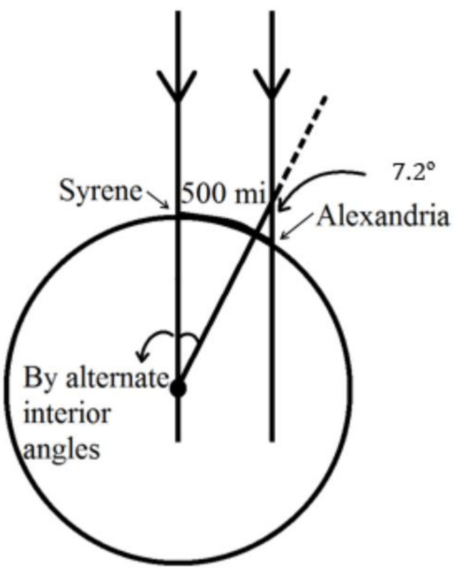


Example 1:	Find the area for a circle in terms of the radius.

Eratosthenes was a head librarian, mathematician, and astronomer in Alexandria, Egypt around 200 BC, which was then considered the center of learning for the world. Fun fact, Eratosthenes thought the earth was round 1,700 years before Christopher Columbus sailed to America. How he knew it is described below.

Eratosthenes knew the summer solstice, June 21st, was the longest day of sunlight in the year. He also knew that on that day in Syene, Egypt the sun would be directly overhead at noon, making the buildings have no shadow, or a 0° angle to their shadow. So, that same day in Alexandria, Eratosthenes held a pole perpendicular to the ground from the sun. He measured a 7.2° angle east to the shadow cast from the pole. To find the circumference of the earth, Eratosthenes used the sun's rays as parallel lines, and the line from his pole to the earth's center as a transversal to form alternate interior angles.

Example 2:	Use Eratosthenes' Method to find the circumference of the earth.

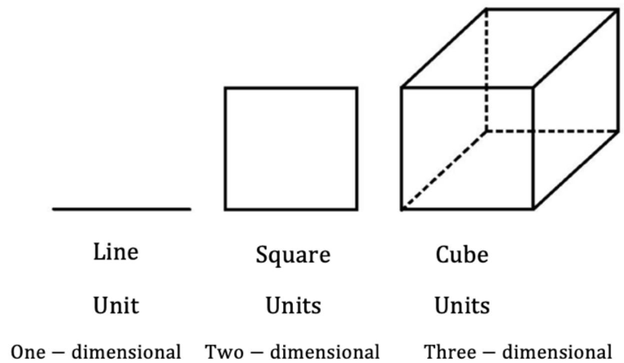


He was only about 100 miles off of the actual distance of 24,900. This was over more than 2,000 years ago before the birth of Christ.

Section 8.12 Finding the Volume of Solids

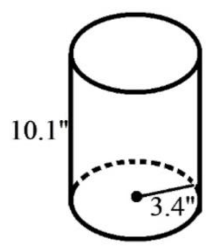
Looking Back 8.12

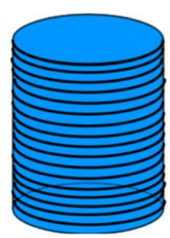
Volume is the inside of a three-dimensional space. The amount of water that will fill a glass and the amount of sand that can be poured into a sandbox are two examples of volume. When we find surface area, we are finding how many two-dimensional square units cover a space, but when we find volume, we are finding how many three-dimensional cubes fill up a space. If water were cut into $1 \times 1 \times 1$ (one by one by one) cubes, how many cubic units would be in the glass? If sand were sculpted into 1 unit × 1 unit × 1 unit cubes, how many cubic units would fill a sandbox?



Looking Ahead 8.12

Example 1: If the diagram below shows a soft drink can, how much soda can it hold in cubic units? We are looking for the volume of the can… We are counting the square units on the circle on the bottom of the can and stacking the circles up 10.1" high.





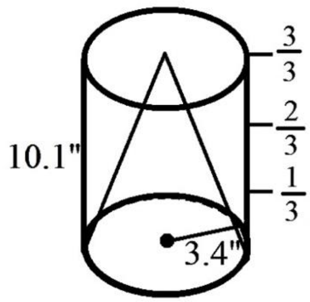
$$V = Bh$$
$$V = \pi r^2 h$$

Cylinder

The B is the area of the base of the cylinder, which is a circle.

A cone is like a cylinder but comes to a point at the top. If we put the cone inside a cylinder with the same base and same height, it would look as shown below Example 2.

Example 2:	Find the volume of the cone.

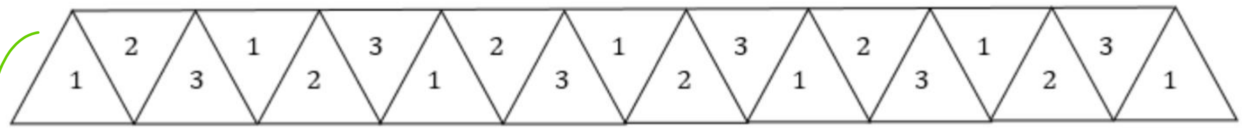


$$V = \frac{1}{3}Bh$$
$$V = \frac{1}{3}\pi r^2 h$$

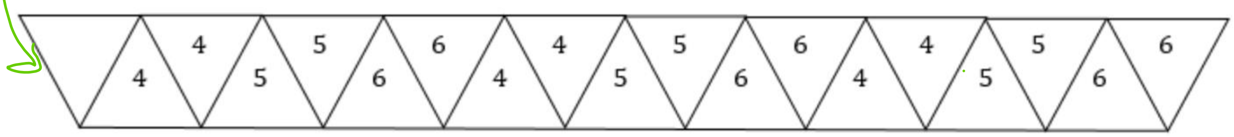
Cone

Section 8.13 Art Project: Hexaflexagons

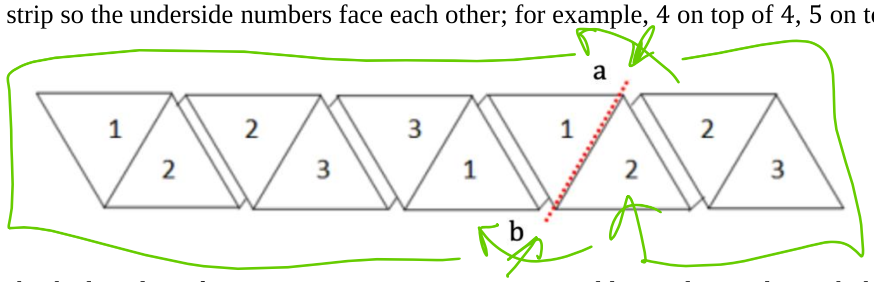
Step 1: Using the templates on the next page, make a strip as below and number it as below:



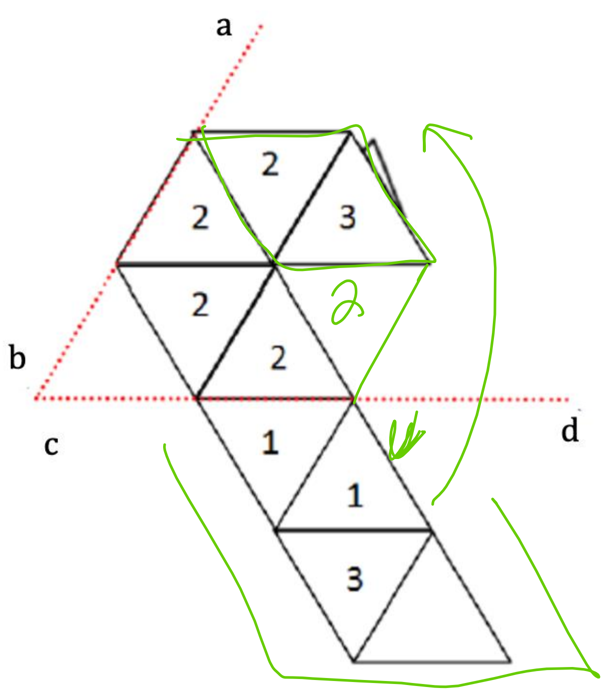
Step 2: Number the underside of the strip as below:



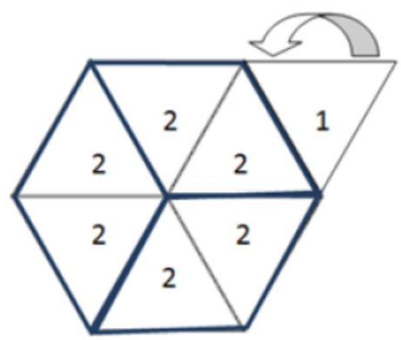
Step 3: Fold each strip so the underside numbers face each other; for example, 4 on top of 4, 5 on top of 5, 6 on top of 6.



Step 4: Fold to the back along line ab, then fold back along cd so it looks like the shape shown in Step 5.

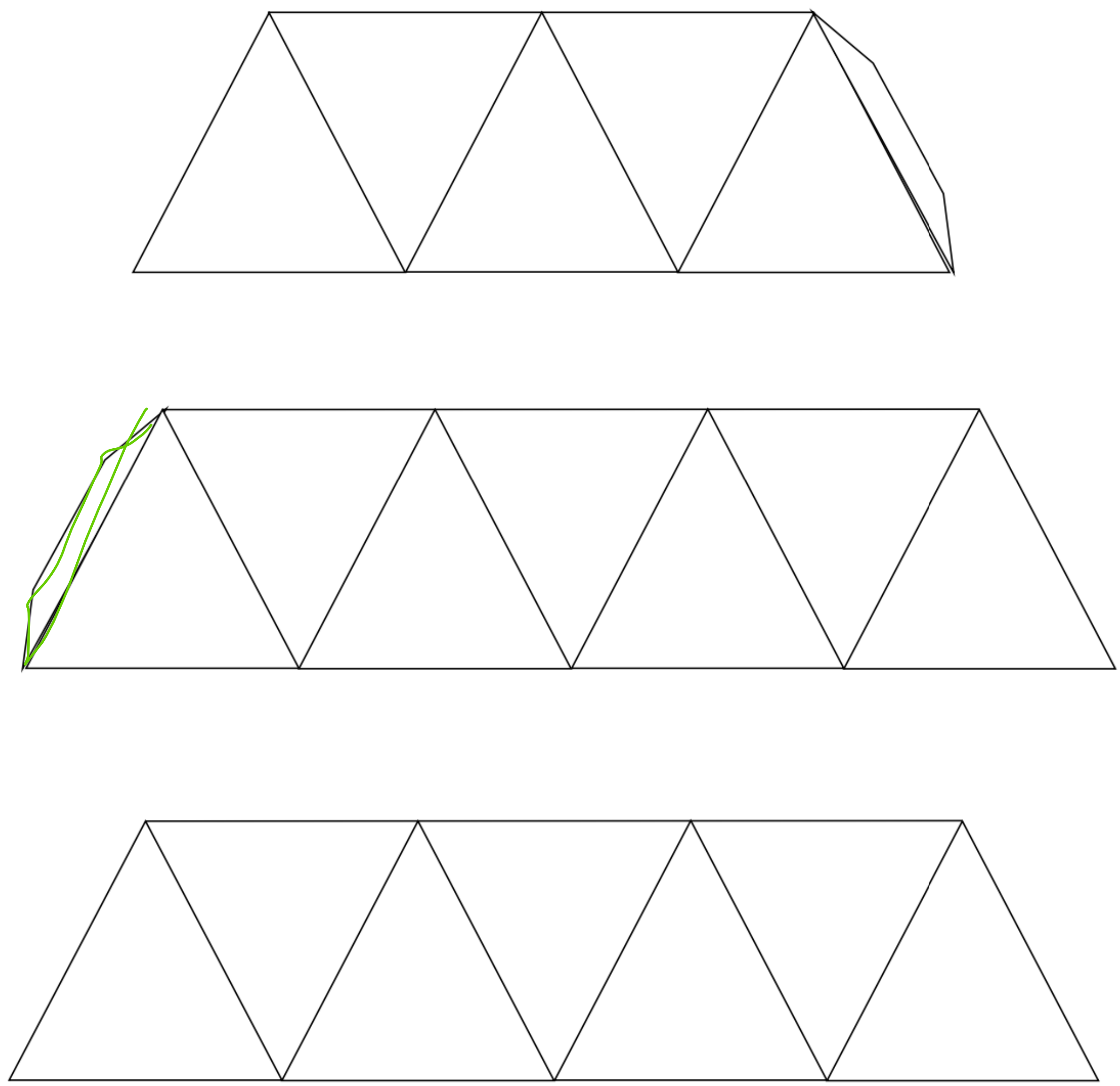


Step 5: Fold triangle 1 underneath the hexagon and paste it to the back.



Step 6: Flex the paper hexagon and move it around to reveal more surfaces. Write your favorite verses on each section to inspire you throughout the day.

Hexaflexagon Template:



Just like the pattern you followed to make the hexaflexagon, God has designed patterns in the universe. In Exodus 25:40, God says: "Make everything according to the pattern I have shown you." God made predictable patterns: 24 hours in a day, 7 days in a week, 4 seasons in each year, the sun to shine during the day, the moon to glow at night, and birth followed by infancy, childhood, youth, adulthood, and old age, so on. This predictability in God's world provides us comfort, control, and safety. After Monday comes Tuesday, and after Saturday comes Sunday, the day of rest. Look for more patterns in God's universe!