Module 5: Polynomials and Factoring

Section 5.1 Defining and Combining Polynomials

Looking Back 5.1

We have been introduced to polynomials and methods to multiply polynomials. In this module, we will take an in-depth look at polynomials and methods to factor polynomials. "Poly," from the Greek language, means *many*, and "nomial," also from the Greek language, means *term*.

Remember, a monomial is one term as "mono," from the Latin language, means *one*; a binomial is two terms as "bi," from the Latin language, means *two*; a trinomial is three terms as "tri," from the Latin language, means *three*.

The variable x has a coefficient of 1. The variable x in $-2x$ has a coefficient of -2. The variable x is sometimes called an "intermediate" because it does not have any fixed value. In fact, in a polynomial expression, the variable x takes on many values. Look at the trinomial below:

$$-5x^2 + 3x - 4$$

The first term is $-5x^2$; it has a coefficient of -5 and a degree of 2.
The second term is $3x$; it has a coefficient of 3 and a degree of 1.
The third term is -4; it has a coefficient of -4 and a degree of 0; $-4x^0$ is the same as -4 because any variable to the 0 power is 1 ($x^0 = 1$); $(-4)(1) = -4$.

The lead coefficient is the number in front of the monomial term with the highest degree. It comes first when an equation is written in standard form with the exponents of each term written from greatest to least. Terms of a polynomial are separated by a plus or minus sign.

Only the polynomial 0 is not taken to the 0 power; 0 could be considered to have no terms at all. It is called the *zero polynomial*, but its degree is not 0. It is left undefined except in Euclidean Geometry.

A polynomial is written in descending order or descending degrees, the largest exponent being first:

$$-3x^3 + 4x^2 - 6x + 2$$

If there are more than one variable in a polynomial, the terms are put in alphabetical order within the monomial. In this case, we will use the x. That means y may increase as x decreases:

$$4x^3y^2 - 7x^2y^2 + 6xyz$$

The degree of a polynomial is the degree of the largest monomial term. It is found by looking at the exponent in each term. If there is more than one variable in the term, find the sum of the exponents of all of the variables in the term.

Example 1: Find the degree of the polynomial below. Underline the coefficients of each term and name the lead coefficient.

$$-2x^3 + 6x^2 - 6x + 1$$

Example 2: Find the degree of the polynomial below. Underline the coefficients of each term and name the lead coefficient.

$$4x^3y^2 - 6x^2y^2 + 7xyz$$

Example 3: Find the degree of the polynomial below. Underline the coefficients of each term and name the lead coefficient.

$$-x^3y + x^2y^2 + 11xy^3$$

This is called a *homogenous polynomial* because the degrees of all three terms are 4.

Looking Ahead 5.1

Polynomials are many terms with each term being made up of numbers, variables, and exponents. The exponents may be only whole numbers $(0, 1, 2, 3, \ldots)$. The exponents may not be negative or fractions. The fraction $\frac{2}{x}$ is not a term of a polynomial because it can be written "$2x^{-1}$." You can have equations with negative exponents such as $y = \frac{2}{x}$, but these are not considered polynomial equations.

Polynomials with four terms have a specific name, but the term polynomial is used for anything with four or more terms.

A second-degree polynomial with only one variable is called *quadratic*, $2x^2 + 1$. A third-degree polynomial with only one variable is called a *cubic*, $x^3 - 4$. You have seen the graphs of the parent functions of these equations, $y = x^2$and $y = x^3$.

Example 4: Sketch the graph of $y = x^2$ and $y = x^3$.

To simplify a polynomial, combine the like terms. To evaluate a polynomial, substitute values in for the variables and perform the operations to get a numerical solution. Terms of a polynomial are separated by a plus or minus sign.

Section 5.2 Using Polynomial Blocks

Looking Back 5.2

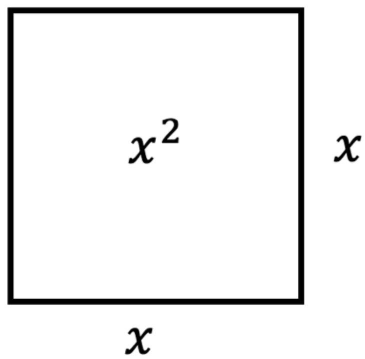
We have learned that like terms can be combined. If the terms are constants (numbers only), they can be added or subtracted. This is called combining the constant terms. With variables, xs can be added to or subtracted from xs, and ys can be added to or subtracted from ys, but xs and ys cannot be combined. In the previous module, we solved these types of problems algebraically and geometrically. In this module, we will investigate polynomials geometrically in more depth. What does x look like? What does y look like? What does xy look like?

Looking Ahead 5.2

The variable x is any unknown length. Our polynomial blocks use this length for the length of x.

If the line above is the length, then the length below is $x + x$ or $2x$.

What does $x \cdot x$ look like? It is a length of x x times, or a length of x with a width of x.



x are linear units (one-dimensional)

x^2 are square units (two-dimensional)

The number 1 is represented by a unit square that looks like this:

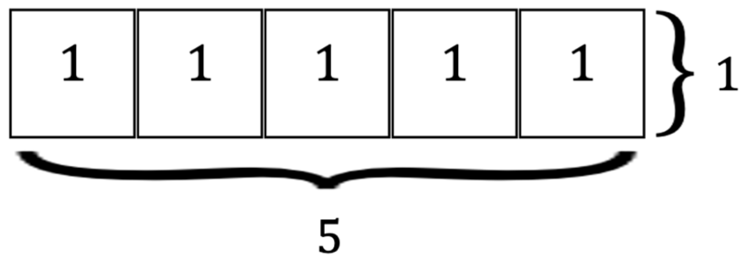
$$\boxed{1}$$

The unit square is 1-unit-long by 1-unit-wide. Therefore, it has an area of 1 square unit ($A = l \times w$). The length and width are both the same. If x is the side length, then $A = x^2$.

The number 2 is two 1-unit squares.

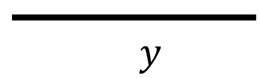
$$\boxed{1} + \boxed{1} = \boxed{1}\boxed{1}$$

You can use a square Post-it® note to represent an x. The side length of the Post-it® note is x.
The number 5 is five 1-unit squares put together.

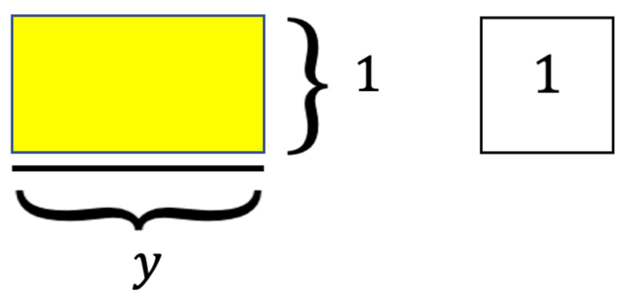


This is a length of five squares (each 1-unit long), and a width equal to the side of the 1-unit square. The area is $A = l \times w$ for a rectangle. For these blocks, $A = 5 \times 1 = 5$ square units.

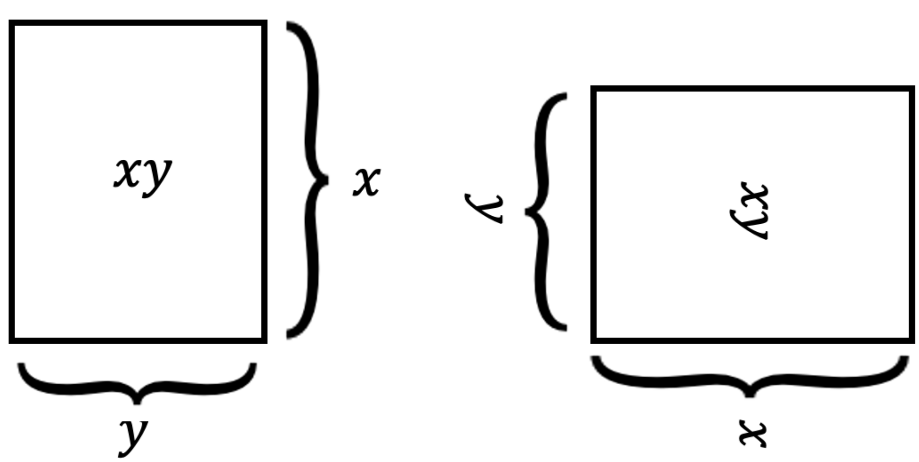
What does y look like? The variable y is another unknown length different than the length of x. It is non-commeasurable with x. There are no integer units of x that fit onto y. This means y cannot be divided by x without a remainder of space. Our polynomial blocks use this length for the unknown length of the variable y.



If the width is 1, then the area of this polynomial block is y because $y \cdot 1 = y$.

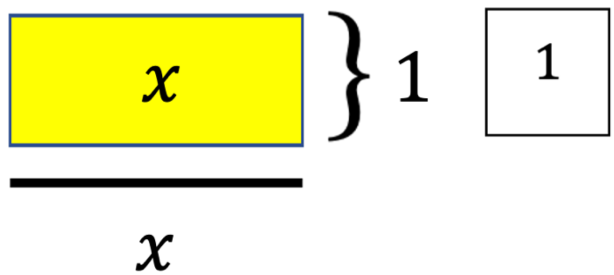


So, xy has a length of x and a width of y and looks as follows:

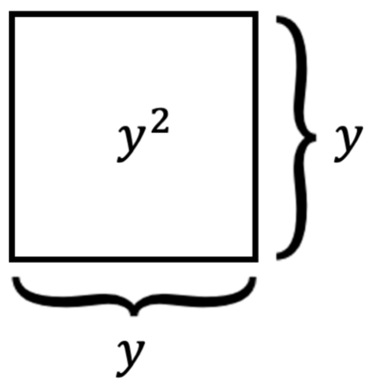


Now, you have seen five of the polynomial blocks: the 1, 5, x^2, y, and xy. The other two polynomial blocks we want to look at are x and y^2.

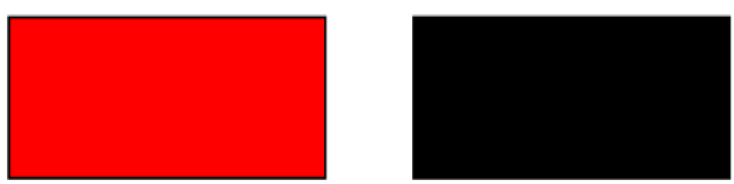
This is the x block with a length of x and a width of 1.



The y^2 block is the same as $y \cdot y$, so the length and width are both equal to y (hence, we get a square, not a rectangle).



Example 1: If the black blocks are positive and the yellow blocks are negative, then when the same size and shape are added together, they become zero.



If the red blocks are negative and the black blocks are positive, then the blocks cancel each other out.

You can copy our template or you can make your own blocks. The important thing is that x-blocks are always the same length, y-blocks are always the same length, and 1-unit blocks are always the same length and width. The x-blocks, y-blocks, and unit blocks are all non-commensurate with each other. The width of x and the width of y must be the length or width of one unit-square.

Section 5.3 Factoring Out a Common Factor

Looking Back 5.3

In the previous module, we learned how to factor out a common monomial from a polynomial and how to use the Distributive Property to write it in the factored form. In the binomial $2x^2 + 8x$, the greatest common factor is $2x$. To factor is to divide. Dividing the binomial $2x^2 + 8x$ by $2x$ is equal to the calculations shown below:

$$\frac{2x^2 + 8x}{2x} = \frac{2x^2}{2x} + \frac{8x}{2x} = x + 4$$

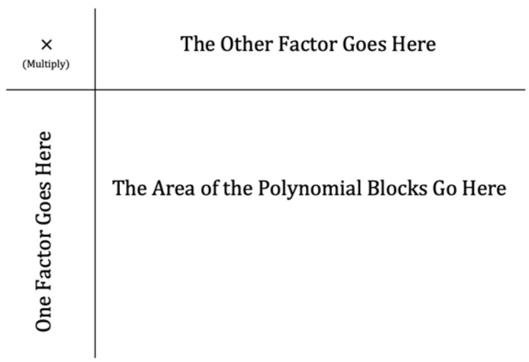
If $\frac{2x^2+8x}{2x} = x + 4$, then using reverse thinking, $x + 4 \cdot 2x = 2x^2 + 8x$. Using algebra, it looks as this: $2x(x + 4) = 2x^2 + 8x$. To check it, use the Distributive Property and multiplication:

$$2x(x + 4) = 2x(x) + 2x(4) = 2x^2 + 8x$$

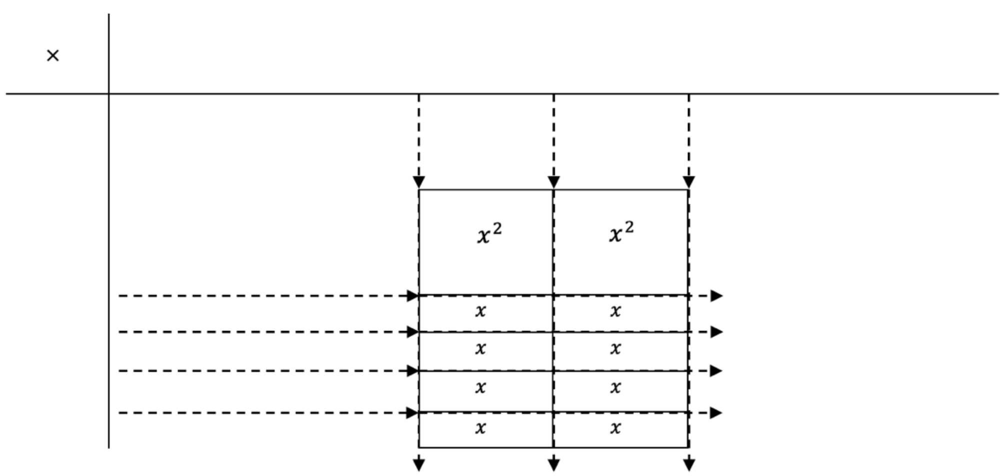
The Distributive Property can be used to multiply through (expand) a polynomial: $2x(x + 4) = 2x^2 + 8x$; or, the Distributive Property can be used to divide out (factor) a polynomial: $2x^2 + 8x = 2x(x + 4)$.

Looking Ahead 5.3

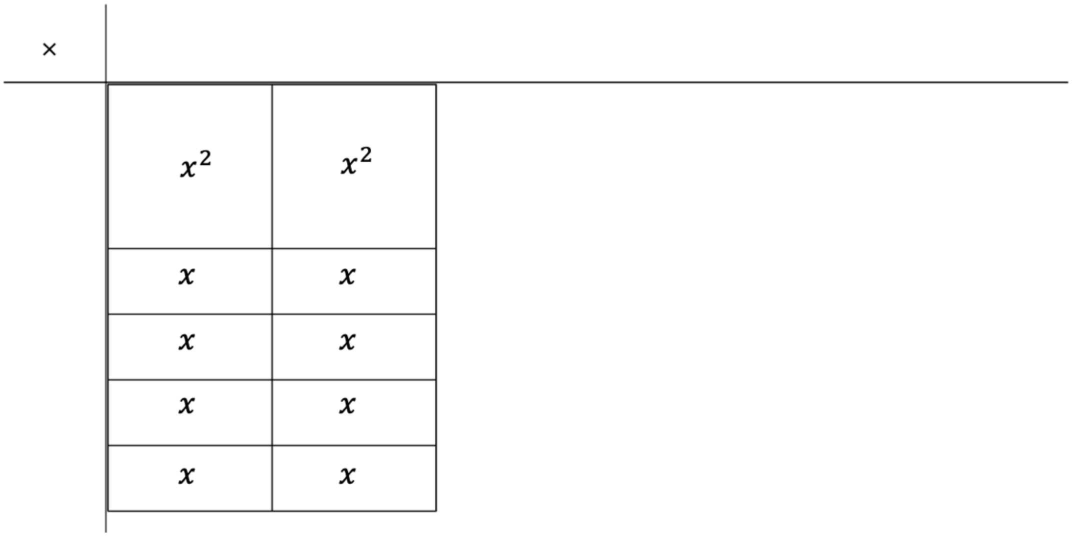
The polynomial blocks can be used to show the factoring process. If you know the common monomial factor, then the other factor can be found using division. Think of the area of a rectangle ($A = lw$). We will use a t-chart to find the other factor. It will look as below:



For this to work, we must use the line-up rule, which means that the edges of all the shapes must line up from top to bottom and from left to right. Below is the geometric representation of combining like terms:

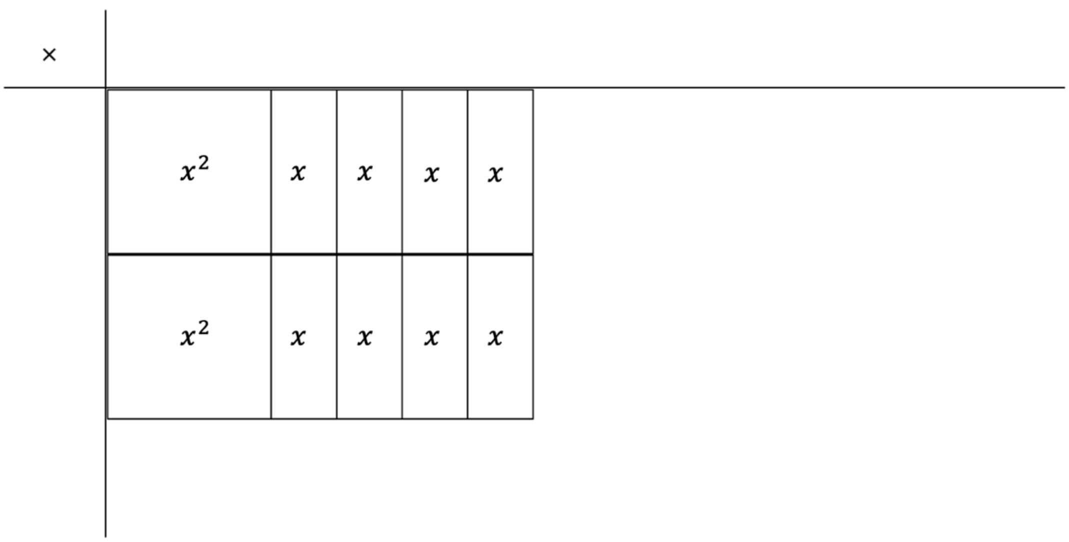


Example 1: Use the polynomial blocks to demonstrate that $2x^2 + 8x = 2x(x + 4)$ and that $2x$ is the greatest common factor. Use the t-frame to factor the polynomial.



The area is: $x^2 + x^2 + x + x + x + x + x + x + x + x = 2x^2 + 8x$

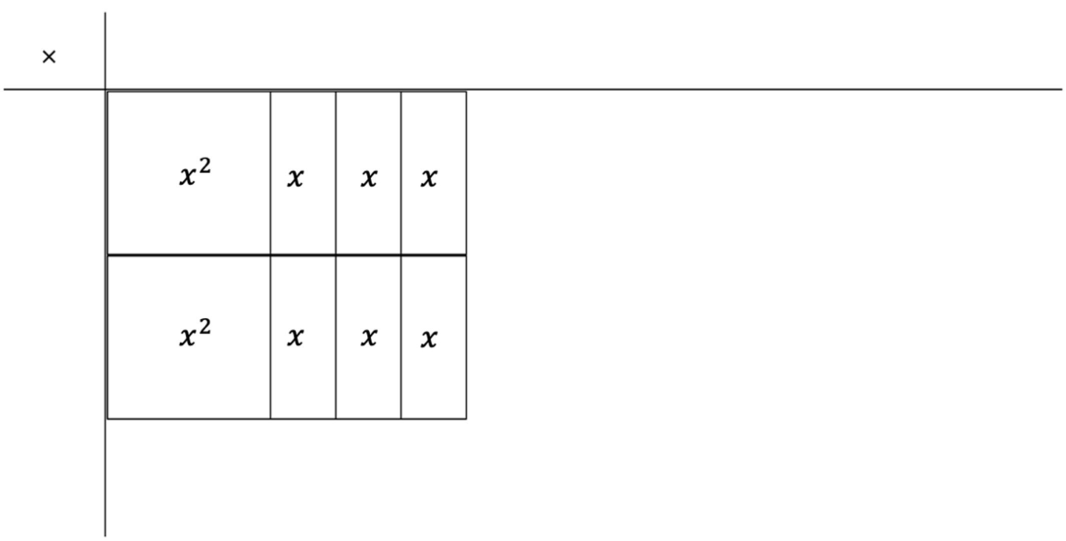
Example 2: Factor the polynomial in Example 1, but this time switch the factors to opposite positions.



Example 3: The area of a polynomial is $x^2 + 6x + 9$. Put these in the t-frame and use the line-up rule to find the side lengths of the square.

$\times$

Example 4: Find the factors (side lengths) and area (polynomial) using the t-frame below.



Example 5: Using the t-frame and polynomial blocks, factor out the common factor of $3x^2 + 6x$. Sketch or trace your solution below.

$\times$

Example 6: You can use an array and multiply for negative numbers. Find the missing boxes for the arrays and find the area.

$(x - 2)(x - 2)$

$(x - 4)(x + 2)$

Section 5.4 Multiplying Binomials

Looking Back 5.4

The two polynomials multiplied in the previous section were the length and width of a rectangle. The area was the binomial that resulted from multiplying the monomial $2x$ by the binomial $x + 4$. In reverse, dividing the area by the length resulted in the width: $\frac{2x^2+8x}{x+4} = 2x$. And, dividing the area by the width resulted in the length:

$$\frac{2x^2+8x}{2x} = x + 4.$$

The product of the length and width is the area: $2x(x + 4) = 2x^2 + 8x$; $2x(x + 4)$ is called the factored form. It is the product $2x^2 + 8x$ written as two factors. One factor is $2x$ and the other factor is $x + 4$.

In the previous module, we used the Distributive Property to multiply polynomials:

$$2x(x + 4)$$
$$2x(x) + 2x(4)$$
$$2x^2 + 8x$$

The monomial gets multiplied by the first term of the binomial, then the monomial gets multiplied by the second term of the binomial.

Now, when two binomials are multiplied, the Distributive Property can be used, or a t-frame and polynomial blocks can be used to find the area of the polynomial:

$$(a + b)(c + d) = ac + ad + bc + bd$$

Looking Ahead 5.4

Example 1: Multiply $(x - 3)$ by $(x + 2)$ using the Distributive Property.

Example 2: Multiply $(x - 3)$ by $(x + 2)$ using FOIL as an acronym for the Distributive Property. This method only works for multiplying binomials.

Example 3: Multiply $(2x + 4)$ by $(x + 3)$ using a t-frame and polynomial blocks. Sketch your solution below.

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Example 4: Multiply $(2x + 4)$ by $(x - 3)$ using the Distributive Property and see if you get the same solution as for Example 3.

Example 5: Multiply $(2x - 1)$ by $(x + 5)$ using a two-by-two array.

Example 6: Multiply $(2x - 1)$ by $(x + 5)$ using the Distributive Property to see if you get the same solution as for Example 5.

Section 5.5 Binomial Squares

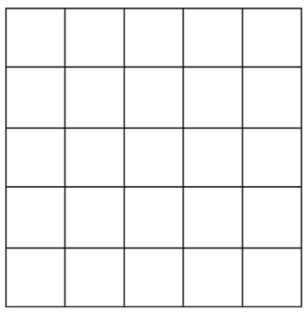
Looking Back 5.5

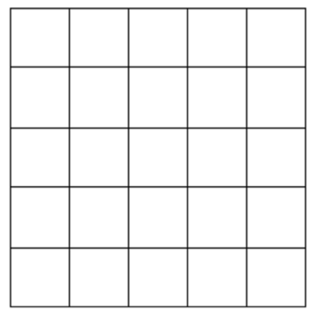
A polynomial that has exactly two terms is called a binomial. To square a term is to multiply it by itself; $x^2 = x \cdot x$; $y^2 = y \cdot y$; $(x+y)^2$ is the same as $(x+y)(x+y)$. When we multiply or expand $(x+y)(x+y)$, called a binomial square, we get the following:

$$(x+y)(x+y) =$$
$$(x)(x) + (x)(y) + (x)(y) + (y)(y) =$$
$$x^2 + 2xy + y^2$$

The square of a binomial results in a perfect trinomial square. The middle term of the trinomial solution is the sign of the binomial square. The last term of the trinomial solution is always positive. Let us see why.

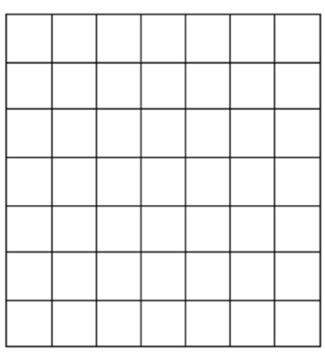
Example 1: Demonstrate 5^2 is equal to $(3+2)^2$ using an area diagram.

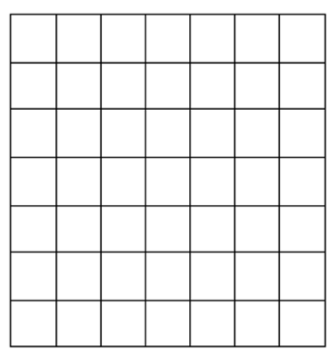




Negative numbers can be represented by the color red and positive numbers can be represented by the color black.

Example 2: Demonstrate 5^2 is equal to $(6-1)^2$ using an area diagram with positive and negative numbers.





<u>Looking Ahead 5.5</u>

The t-frame can be used to demonstrate binomial squares.

Example 3: Demonstrate $(y+2)^2$ which is equal to y^2+4y+4 is equal to $(y+2)(y+2)$ using a t-frame and polynomial blocks. Sketch your solution below.

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Example 4: Demonstrate $(x-1)^2$ which is equal to x^2-2x+1 is equal to $(x-1)(x-1)$ using a t-frame and polynomial blocks. Sketch your solution below.

×	

Hopefully, you are starting to see a pattern:

1. $(x+y)^2 = x^2 + 2xy + y^2$
2. $(y+2)^2 = y^2 + 4y + 4$
3. $(x-1)^2 = x^2 - 2x + 1$

1) The first term of the trinomial is the ____________________ of the ________________ _______________ of the binomial.

2) The middle term of the trinomial comes from multiplying both _______________________ ____________ of the binomial and _________________________ it. This is because the terms are the same. You are adding or subtracting the same term twice.

3) The third term of the trinomial is the ____________________ of the _________________ _____________ of the binomial.

This is more simply stated below:

1. Square the first term
2. Multiply both terms and double the product
3. Square the last term.
4. Put them all together to get the final trinomial.

Example 5:	Use the steps above for $(m-4)^2$ and check your solution using the Distributive Property.

Example 6:	Use the steps above for $(y+7)^2$ and check your solution using the Distributive Property.

Section 5.6 Difference of Squares

Looking Back 5.6

There is a pattern for the sum of a binomial square or the difference of a binomial square. Remember, sum means to add, and difference means to subtract. In the previous section, we found the patterns for finding the product of a binomial square. Think of it as two terms squared or $(\text{two terms})^2$.

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

The binomial square that is a sum has a positive sign in front of the middle term of the trinomial. The binomial square that is a difference has a negative sign in front of the middle term of the trinomial. The last term is always positive because a positive number multiplied by itself is positive, and a negative number multiplied by itself is also positive.

A binomial square that is a difference is not the same thing as the "difference of squares."

Looking Ahead 5.6

In the previous Practice Problems section, one problem used the expression $t^2 + s^2$. This was not a binomial square. That would have to be written "$(t+s)^2$" for $(t+s)(t+s)$ or "$(t^2+s^2)^2$" for $(t^2+s^2)(t^2+s^2)$. It would have to be written as one quantity, using parenthesis, then squared. In a "difference of squares," there is a subtraction sign and each term is squared, not the entire quantity. You are not squaring a binomial quantity. You are squaring each term. Therefore, $t^2 - s^2$ would be a difference of squares. There is a special way to factor $t^2 - s^2$. Notice that $t^2 + s^2$ could be called a "sum of squares," but there is no way to factor that.

In summary…

$$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$$
$$(a-b)^2 = (a-b)(a-b) = a^2 - 2ab + b^2$$
$$(a+b)(a-b) = a^2 - ab + ab - b^2 = a^2 - b^2$$

And now, $(a^2 - b^2) = (a+b)(a-b)$ or $(a-b)(a+b)$

In a binomial square in which the terms are added, the middle term (the sum) is positive because two positive numbers are being added.

In a binomial square in which the terms are subtracted, the middle term (the sum) is negative because two negative numbers are being added.

In the difference of squares, the middle term becomes 0 because the same terms are added but with opposite signs. You are left with the square of the first term of the binomial, and the square of the last term of the binomial.

Example 1:	Use the shortcut to find the solution to $(x+7)^2$.

Example 2:	Expand and multiply $(x+7)^2$ and see if you get the same solution you did in Example 1.

Example 3:	Use the short cut to multiply the square $(t-4)^2$.

Example 4:	Expand and multiply $(t-4)^2$ and see if you get the same solution as you did in Example 3.

Example 5:	Multiply $(x + 6)$ by $(x - 6)$.

Example 6:	Multiply $(y - 0.8)$ by $(y + 0.8)$.

Section 5.7 Factoring Special Cases of Polynomials

Looking Back 5.7

In the previous two sections, we saw that when we expand a binomial square, we have two binomials with the same terms and same sign. When we multiply these, we get a product that is a trinomial and there is a pattern in each product. We also learned that if we multiply two binomials with the same terms and opposite signs, the product is a binomial and there is a pattern in that product as well that results in a difference of squares.

Now, we are going to work in reverse. Instead of multiplying binomials to get the product, we are going to divide binomials to get the factors. In algebraic terms this is called *factoring*.

Looking Ahead 5.7

When you have a binomial that looks like this: $a^2 + 2ab + b^2$, you will notice the first and last terms are perfect squares and the middle term is double the product of the first and last term. It is called a perfect trinomial square and can be factored as the square of the sum of the bases of the first and last term, which is $(a + b)^2$. The factored form is called the sum of a binomial square.

Because $a^2 + 2ab + b^2 = (a + b)(a + b)$; $(a + b)$ and $(a + b)$ are the two factors. Because they are the same, these represent a factor that is repeated: $a + b$. The repeated factors can be written: $(a + b)^2$.

When you have the same binomial but there is a negative middle term (a minus sign instead of a plus sign), then it looks like this: $a^2 - 2ab + b^2$. This is also called a perfect trinomial square. It can be written as the square of the differences of the bases of the first and last terms, $(a - b)^2$. That is the same as $(a - b)(a - b)$. The factored form is called the difference of a binomial square. This is not the difference of squares, which is $a^2 - b^2$.

Both factors, again, are the same. The common and repeated factor is $a - b$. The repeated factors can be written as $(a - b)^2$.

Finally, the binomial $a^2 - b^2$ is called a difference of squares. The middle term became 0 because it was the sum of additive inverses (or opposites), which is 0. The factors are $(a - b)(a + b)$.

The product of the outer terms is positive. The product of the inner terms is negative. Because they are the same terms with opposite signs, when added together, become 0.

These are the special trinomials that when factored become:

Polynomial	Factored Form
$a^2 + 2ab + b^2$	$(a + b)(a + b) = (a + b)^2$
$a^2 - 2ab + b^2$	$(a - b)(a - b) = (a - b)^2$
$a^2 - b^2$	$(a + b)(a - b)$

Example 1: Let us investigate a sum of squares (below). Does that create a special pattern? Can that be factored?

$$a^2 + b^2 = (?)(?)$$

Example 2:	Factor $x^2 - 4x + 4$.

Example 3:	Factor $y^2 + 6y + 9$.

Example 4:	Factor $t^2 - w^2$.

Example 5:	Factor $a^2 - 8ab + 16b^2$.

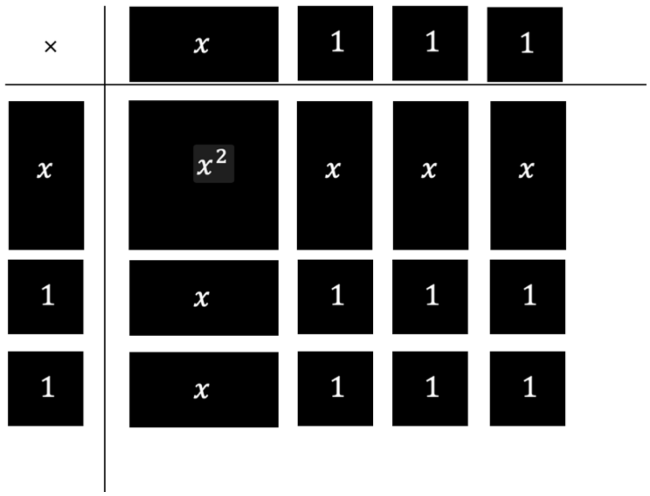
Example 6:	Multiply $(y - 0.8)$ by $(y + 0.8)$.

Section 5.8 Factoring Trinomials Using Rectangular Arrays

Looking Back 5.8

$$x^2 + 5x + 6 = (x + 3)(x + 2)$$

When we used the t-frame and polynomial blocks, we put the factors on the outside of the t-frame and the area on the inside. The area is the polynomial.



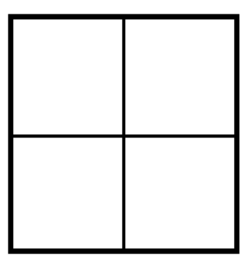
Notice the upper-left corner of the inside of the t-frame is the product of the first terms, the lower-right corner of the inside of the t-frame is the product of the last terms, and the top-right and lower-left are the two terms that combine to result in the middle term. This sum comes after multiplying the outer terms and multiplying the inner terms.

Knowing the trinomial is like starting with the area of the polynomial. An area model can be used to work in reverse to find the length of the sides that will be the factors of the polynomial.

Looking Ahead 5.8

Example 1: Factor $x^2 + 5x + 6$.

Make a rectangle with four sections inside for the trinomial $x^2 + 5x + 6$. Put the first term of the triangle in the upper-left corner and the last term of the trinomial in the lower-right corner.



There are two rectangles left open. The one on the upper-right comes from multiplying the outer terms of the two binomial factors. The rectangle on the lower-left comes from multiplying the inner terms of the binomial factors.

Now approach this as a puzzle and think backwards!

Start with the first term, x^2, which is ___________ × ___________.

Now, the last term, 6, can be ___________ × ___________ or ___________ × ___________.

Try ________ × ________ first. Next, try ________ × ________.

This seems like a guess-and-check method. Actually, the sum of ________ and ________ is ________ (________ + ________ = ________). That is not the coefficient of the middle term of the trinomial.

However, ________ + ________ = ________ (________ + ________ = ________) is the coefficient of the middle term of the trinomial.

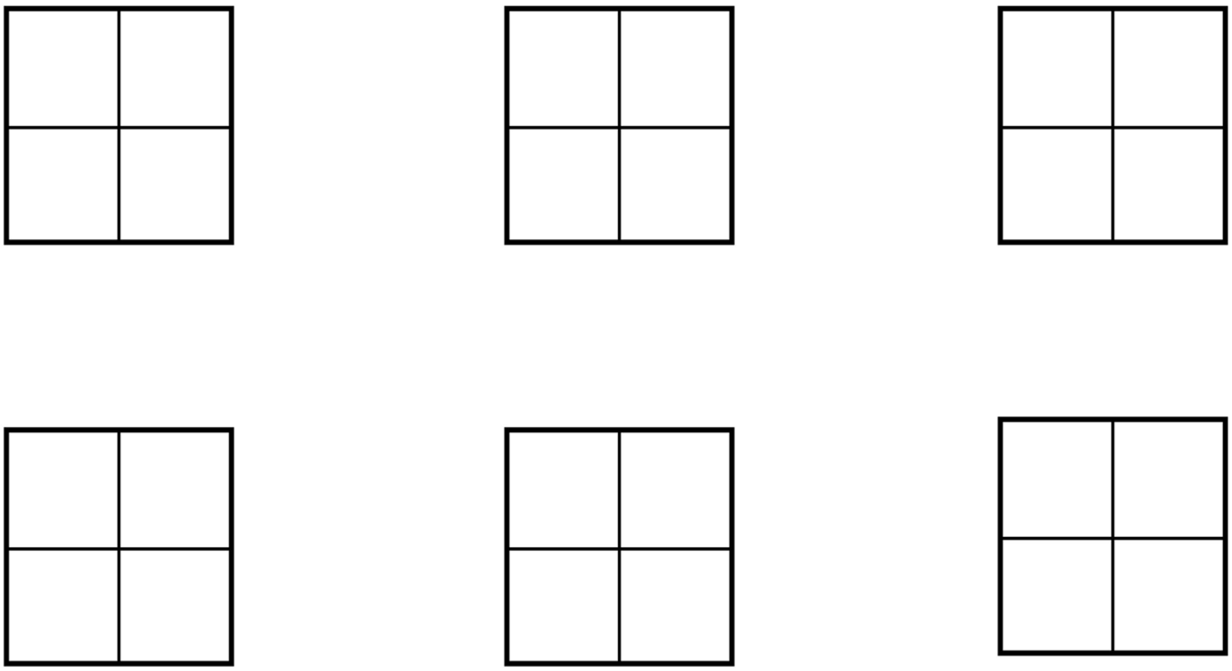
Example 2: Factor $x^2 - 4x - 45$ using the arrays below.

Start with the first and last terms and their signs.

x^2	
	-45

The last term -45 can be…

Example 3: Factor $x^2 + 10x - 24$ using the arrays below.



Section 5.9 Factoring Trinomials with a Lead Coefficient of One

Looking Back 5.9

In previous sections, we factored special trinomials. They had patterns that were related to their names. These special patterns factored trinomials into the sum or difference of binomial squares, or the difference of squares factored into a sum and difference of the square roots of the terms. We also saw that there is no such thing as the sum of squares in factored form.

Moreover, we used an array to factor other trinomials that did not have the previous special patterns, but could be factored into two different binomials. The signs may be the same or different. The constants will not be the same in each factor. In this section, all the trinomials are of the form $ax^2 + bx + c$ in which a, b, and c are real numbers and $a = 1$. That means the lead coefficient is 1 because a is the coefficient of the quadratic term which is the lead term.

Looking Ahead 5.9

The trinomial $x^2 - 5x + 6$ from Problem 6 in the previous Practice Problems section can be factored, but not using any of the previously learned methods. You know that multiplying two binomials often results in a trinomial; therefore, a trinomial that can be factored into integers for the constant term can often be factored into two binomials:

$$x^2 - 5x + 6 = (\_\_\_\_\_\_\_\_\ \_\_\_\_\_\_\_\_\ \_\_\_\_\_\_\_\_)(\_\_\_\_\_\_\_\_\ \_\_\_\_\_\_\_\_\ \_\_\_\_\_\_\_\_)$$

Let us start with the first term, x^2, because it is always the easiest: $x^2 = x \cdot x$. Using reverse thinking: $\sqrt{x^2} = x$.

$$x^2 - 5x + 6 = (x\ \_\_\_\_\_\_\_\_\ \_\_\_\_\_\_\_\_)(x\ \_\_\_\_\_\_\_\_\ \_\_\_\_\_\_\_\_)$$

You are asking yourself: "What term multiplies by itself to result in x^2?" It is x.

Next, let us find the signs. The last terms multiply to result in the last term in each factor and add or subtract to give the middle term. The products of the outer and inner terms result in a sum because they are like terms (both will have an x). The last term is positive. What are the only two signs that multiply to result in a positive number? It can only be a (positive)(positive) or a (negative)(negative). We are getting closer. Which are two that add to result in a negative for the middle term? It must be two negatives because $(-)(-) = +$ and $(-) + (-) = -$.

$$x^2 - 5x + 6 = (x - \_\_\_\_\_\_\_\_)(x - \_\_\_\_\_\_\_\_)$$

Now you are asking yourself: "What two factors multiply to result in a product of $+6$ and a sum of -5?" The factors of 6 are 1 and 6, or 2 and 3. Only -2 and -3 add up to -5 so those are the last terms.

$$x^2 - 5x + 6 = (x - 2)(x - 3) \text{ or } x^2 - 5x + 6 = (x - 3)(x - 2)$$

It does not matter which order you put the binomial factors in because multiplication is commutative.

To check your work, multiply the factors and see if you get the original problem.

$$(x - 2)(x - 3) =$$
$$x(x) + x(3) + (-2)(x) + (-2)(-3)$$
$$x^2 + -3x + -2x + 6$$
$$x^2 + -5x + 6$$
$$x^2 - 5x + 6$$

Example 1:	Factor the trinomial $x^2 + 7x + 10$ and multiply the binomials to check your work.

Example 2:	Factor the trinomial $x^2 - 2x - 3$ and multiply the binomials to check your work.

Example 3:	Factor the trinomial $x^2 + x - 20$ and multiply the binomials to check your work.

Sometimes trinomials cannot be factored into two binomials. Below is an example of this.

Example 4:	Factor the trinomial $x^2 - 2x - 1$. (I know I already told you it is not possible, but try it anyways!)

Section 5.10 Factoring Trinomials with a Lead Coefficient Not Equal to One

Looking Back 5.10

As we have investigated polynomials and factoring, we have mostly explored trinomials (polynomials with three terms) to the second degree (which are called “quadratics”). We call this form $ax^2 + bx + c$ in which $a = 1$. Changing the coefficient of x^2 to another number makes factoring a little more complicated. The guess-and-check method using a rectangular array or parenthesis is a lot of work and can get very confusing.

Looking Ahead 5.10

In the trinomial $2x^2 - 3x - 2$, the coefficient of x^2 is 2. There are two possible arrangements for the first term and two possible arrangements for the last term; this is because the only factors are 2 and 1. Therefore, the four possible arrangements are as follows:

$$(2x + 1)(1x - 2)$$
$$(1x + 1)(2x - 2)$$
$$(2x + 2)(1x - 1)$$
$$(1x + 2)(2x - 1)$$

Example 1: Use the rectangle puzzles (area model) to factor the trinomial $2x^2 - 3x - 2$.

Put the first and last term in the appropriate squares.

Try all the possible arrangements to see if the diagonal from lower left to upper right is added to get the middle term.

a.

$2x^2$	
	-2

b.

$2x^2$	
	-2

c.

$2x^2$	
	-2

d.

$2x^2$	
	-2

Then find the side lengths:

The factors are: ____________________ and ____________________

Did this arrangement also work?

	$2x$	1
x	$2x^2$	$1x$
-2	$-4x$	-2

Notice, the coefficient of the first term a and the coefficient of the last term c are multiplied and then added to result in the middle term when using the Distributive Property.
Therefore, the middle term can be expanded and a common factor can be found after factoring.

Example 2: Factor the trinomial $2x^2 + 9x + 10$ by expanding the middle term.

Example 3: Factor the trinomial $4y^2 - y - 3$ by expanding the middle term.

Section 5.11 Another Method to Factor Trinomials

Looking Back 5.11

As you can see, factoring trinomials whose lead coefficient is a number other than 1 is not a simple process. When the numbers have more than two factors, the guess-and-check method can be really cumbersome. Even expanding the middle term requires a lot of work and can get complicated.

There is another method that works by using reverse thinking. I call it the *slide-and-divide* method.

Looking Ahead 5.11

Example 1:	Factor the trinomial $2x^2 - 3x - 2$ using the slide-and-divide method.

Remember, $2x^2 - 3x - 2$ is in the form $ax^2 + bx + c$. This will make the steps easier to follow.

First, always look to see if there is a Greatest Common Factor and factor it out if there is.

Slide a over to the end and multiply it by c:

$$x^2 - 3x - 2(2) = x^2 - 3x - 4$$

Now, factor like before:

$$(x-4)(x+1)$$

Now, we undo what we did in the first step and divide the constant in each factor by a:

$$(x - \frac{4}{2})(x + \frac{1}{2})$$

Our first factor is $(x - 2)$. Our factor can only have integers, so we must multiply $(x + \frac{1}{2})$ by 2 in order to clear the denominator. Our second factor becomes $(2x + 1)$.

Check your work:

$$(x-2)(2x+1) = x(2x) + x(1) - 2(2x) - 2(1) = 2x^2 + x - 4x - 2 = 2x^2 - 3x - 2$$

No matter which numbers a and c represent, this method is much easier.

This works because a and c get multiplied when you use the Distributive Property, so we divide or factor by un-multiplying or dividing in the end.

Example 2:	Factor the trinomial $3x^2 + 14x + 8$ using the slide-and-divide method.

Is there a Greatest Common Factor?

No.

List the numbers for a, b, and c.

Slide a to the end and multiply it by c:

$$x^2 + 14x + 8(3) = x^2 + 14x + 24$$

Now, factor as usual:

$$(x+12)(x+2)$$

Now, divide the constants in each factor by 3:

$$(x + \frac{12}{3})(x + \frac{2}{3})$$

Simplifying, the first term is $(x + 4)$.
The second term must be multiplied by 3 to clear the denominator of the fraction:

$$3\left(x + \frac{2}{3}\right) = (3x + 2)$$

The factors are $(x + 4)(3x + 1)$,

Multiply and check your work to make sure it is correct:

$$(x + 4)(3x + 2)$$
$$x(3x) + x(2) + 4(3x) + 4(2)$$
$$3x^2 + 2x + 12x + 8$$
$$3x^2 + 14x + 8$$

Example 3:	Factor the trinomial $3x^2 + 5x - 8$ using the slide-and-divide method.

Is there a Greatest Common Factor?

Slide:

Factor:

Divide:

Simplify:

Check your solution:

Section 5.12 Completing the Square

Looking Back 5.12

We have been using polynomial blocks to factor trinomials into two binomials. However, in Section 5.8, we could not always factor trinomials into two binomials. Completing the square is a method that allows us to factor to a trinomial, but the factored form looks different than anything we have seen before. Completing the square results in a factored form that makes it easy to graph quadratic equations. We will learn this method for factoring in this section and use it for graphing in the module about quadratics (quadratic equations have a lead term to the second power).

Completing the square is exactly what it sounds like. The polynomial pieces are used to form a square. There may not be enough pieces or there may be too many pieces. These missing, or leftover pieces, are factored into the equation. A trinomial in standard form is converted to a perfect square that has equal sides. So, to complete the square, a constant is added or subtracted from an equation so the quadratic equation can be factored out into the perfect square.

Looking Ahead 5.12

Example 1:	Factor the trinomial $x^2 - 4x + 7$ by completing the square. Sketch your solution.

To begin finding the polynomial pieces you need to fit as many possible into the t-frame to try to form a square.

x

Example 2:	Factor the trinomial $x^2 + 4x + 1$ by completing the square.

x

Just remember, as in Example 1, when you have pieces left over, you must add them to the perfect square. If, as in Example 2, you have missing pieces, then you should subtract them from the perfect square.

Completing the square allows you to solve for the variable in a quadratic equation. It can be done algebraically following the steps below. To begin, set the quadratic equation equal to zero: $\underline{ax^2 + bx + c = 0.}$

Step 1: Rewrite the quadratic equation in the form $\underline{ax^2 + bx = -c}$

Step 2: Take $\frac{1}{2}b$ and square it

Step 3: Add that number to both sides of the equation to form a perfect square trinomial and simplify

Step 4: Rewrite the left side of the equation as a binomial square

Step 5: Rewrite the equation to equal 0

Example 3: Try Example 1, $x^2 - 4x + 7$, using the algebraic method shown above. Let $x^2 - 4x + 7 = 0$.

Step 1: Rewrite $x^2 - 4x + 7 = 0$

Step 2: $b = -4; \frac{1}{2}b = \frac{1}{2}(4) = -2; (\frac{1}{2}b)^2 = (-2)^2 = 4$

Step 3: $x^2 - 4x + 4 = -7 + 4$

Step 4: $(x - 2)^2 = -3$

Step 5: $(x - 2)^2 + 3 = 0$

Did you get the same solution as in Example 1?

Example 4: Factor the trinomial $x^2 + 6x - 1$ by completing the square. Let $x^2 + 6x - 1 = 0$.

Step 1: $x^2 + 6x = 1$

Step 2: $b = 6; \frac{1}{2}b = \frac{1}{2}(6) = 3;$ $(\frac{1}{2}b)^2 = (3)^2 = 9$

Step 3: $x^2 + 6x + 9 = 1 + 9$

Step 4: $(x + 3)^2 = 10$

Step 5: $(x + 3)^2 - 10 = 0$

Example 5: These steps work when the coefficient of the first term is 1. If the coefficient is not 1, you must divide through the equation with the coefficient to get the quadratic into that form. Try to factor the trinomial below.

$$3x^2 + 18 - 30$$

$3x^2 + 18x - 30 = 0$

$\frac{3}{3}x^2 + \frac{18x}{3} - \frac{30}{3} = \frac{0}{3}$

$x^2 + 6x - 10 = 0$

This is the same as taking out the GCF and factoring, $3(x^2 + 6x - 10 = 0)$. Sometimes the coefficient (a) does not divide with no remainder into each term. These problems will be explored in *Algebra 2*.

Step 1: $x^2 + 6x = 10$

Step 2: $\frac{1}{2}b^2 = (\frac{1}{2}(6))^2 = (3)^2 = 9$

Step 3: $x^2 + 6x + 9 = 10 + 9$

Step 4: $(x + 3)^2 = 19$

Step 5: $3 \cdot (x + 3)^2 = 3 \cdot 19$

Example 6: Sometimes the factors are not integers. Use completing the square and a t-frame to factor the trinomial:

$$x^2 + 5x + 1; \text{ Let } x^2 + 5x + 1 = 0$$

Step 1: $x^2 + 5x = -1$

Step 2: $(\frac{5}{2})^2 = \frac{25}{4}$

Step 3: $x^2 + 5x + \frac{25}{4} = -1 + \frac{25}{4}$

Step 4: $(x + \frac{5}{2})^2 = -\frac{4}{4} + \frac{25}{4}$

Step 5: $(x + \frac{5}{2})^2 = \frac{21}{4}$

$(x + \frac{5}{2})^2 - \frac{21}{4} = 0$

x	

Section 5.13 Factoring Review

Looking Back 5.13

We have covered multiplying and factoring polynomials in great detail. We will now put it all together. Anytime we begin to factor a polynomial, we start by looking for a Greatest Common Factor (GCF) and factoring it out if there is none.

$$2x^2 - 4x + 2$$
$$2(x^2 - 2x + 1)$$

Next, we look at the polynomial left in the parenthesis and look for any patterns that indicate this is a special case. This would be the sum of a binomial square, the difference of a binomial square, and the difference of squares. If we find none, then we see if we can do the trinomial factoring. If there are no special patterns and trinomial factoring does not work, we expand the middle term to factor the trinomial or use the slide-and-divide method.

Looking Ahead 5.13

Completing the square may also be used to factor trinomials. It allows you to factor the equation into graphing form. This will be explored further in the module concerning quadratic equations when we learn to graph them.

Example 1:	Factor completely $5x^2 - 25x + 20$.

Example 2:	Factor completely $3xy^3 - 27xy$.

Example 3:	Factor completely $2a^2 - 4ab + 2b^2$.

Example 4:	Factor completely $3x^2 - 3x + 18$.