Algebra 2 Module 6 Powers and Polynomials

Section 6.1 Defining Polynomials
Looking Back 6.1

A polynomial is an expression that is made up of many terms. A monomial is an expression that is made up of one term that is a variable, constant, or product of a variable or constant.

-3, x or $-3x$

A binomial is an expression that is made up of two terms separated by an addition or subtraction sign. A trinomial consists of three terms. After trinomials, all expressions are polynomials, which do not have a fraction or negative exponent. The exponent of each term must be an integer equal to or greater than zero.

$x^{\frac{1}{2}}$ is $\sqrt[2]{x}$

(not a polynomial)

x^{-1} is $\frac{1}{x}$

(not a polynomial)

Looking Ahead 6.1

Standard Form

$y = ax + b$ (linear)

$y = ax^2 + bx + c$ (quadratic)

$y = ax^3 + bx^2 + cx + d$ (cubic)

The lead term is the highest degree of the exponents, and from there it descends in order. If there is more than one variable they are written in alphabetical order.

$g(x) = 3x^2 + 5y - 3xy$

is written

$g(x) = 3x^2 - 3xy + 5y$

The degree of the lead coefficient is the exponent or sum of the exponents.

$3x^2 - 3x + 5y$ (degree is 2)

$5x^2y^3 + 2xy - 6$ (degree is 5)

$6x^0 = 6$ (degree is 0)

Example 1: a) Write the given equation in standard form.

b) Name the equation by the number of terms.

c) Tell the degree of the polynomial.

d) Name the polynomial by its degree and number of terms.

$$f(x) = 3 + 8x - 3x^3 + 4x^2$$

Example 2: Circle each expression that is a polynomial. If the expression is not a polynomial, explain why.

a) $\frac{1}{2x}$

b) $3x^4 + 5x^{\frac{1}{2}} - 2$

c) $5x^2 - 3x + 4$

d) $\frac{2+x}{x-2}$

Section 6.2 Combining Like Terms

Looking Back 6.2

Polynomials are of the form $a_n x^n + a_{n-1}x^{n-1} \ldots a_1 x + a_0$ where n is a nonnegative integer and $a_n \neq 0$.

Like terms have the same base and exponent and can be combined by adding or subtracting the coefficients of the terms.

Example 1: Combine like terms in the expression.

$$x + x^2 + y + y + 3x + x^2$$

Example 2: Combine like terms in the expression.

$$-5 - 4y^3 - 8y^2 + 4y + y^2 - 7 + 5y^4 + 4y + 8y^5$$

Looking Ahead 6.2

When adding or subtracting polynomials, the like monomial terms are combined, and the answer is written in standard form.

Example 3: Find the sum of the expressions.

$$(7x^2 + 2 - x) + (4x^2 - 2x - 3)$$

Example 4: Find the difference of the expressions.

$$(4x^2 + 9x + 2) - (3x - 3)$$

Section 6.3 Multiplying Polynomials

Looking Back 6.3

Since polynomials are made up of terms of powers to the zero, first, second, third degree, etc., the rules of exponents apply. When adding or subtracting polynomials, like terms are combined. In order to combine the like terms, the bases and exponents must be the same. All work is done by combining the coefficients using addition and subtraction, depending on the sign of the coefficient. Exponents are added when finding the product of like bases and exponents are subtracted when finding the quotient of the like bases. All multiplication or division is done with the coefficients.

Example 1: Multiply the polynomials.

a) $3y^2(-4x - 6xy + y)$

b) $-\frac{1}{4}x(8x^3 + x^2 - 5)$

The multiplication of polynomials may be used to check factoring (division). The distributive property is used to multiply polynomials and then it may be simplified by combining like terms.

Example 2: Multiply the polynomial and simplify.
$(3x + 2)(-4x^2 + 3)$

Example 3: Multiply the polynomial using a rectangular array or long multiplication.

$$(x^3 + 5)(3x^2 - 4x - 6)$$

Example 4: Multiply the polynomials using the distributive property, long multiplication and a rectangular array.

$$(3x^3 + 2x^2 + x + 6)(-5x^2 + x - 1)$$

Section 6.4 Factoring Polynomials

Looking Back 6.4

To factor a polynomial means to divide in order to solve for any unknown variable, usually *x*. The first step to simplifying any polynomial is to find the Greatest Common Factor, or in this case, the Greatest Common Monomial.

Example 1:	Simplify the polynomial by finding the Greatest Common Monomial.

a) $f(x) = 9x^2 + 3x + 18$

b) $g(x) = 2x^3 + 10x^2 - 4x$

c) $h(x) = 8x^3y^3 + 12xy^2 - 4xy$

Looking Ahead 6.4

Often the reason we factor or simplify polynomials is to solve for the unknown variable. To factor and solve polynomials, follow the steps below:

1. Bring all terms to one side of the equation, put it in standard form, and set it equal to 0.
2. Factor by taking out the Greatest Common Monomial (GCM).
3. Factor further using any previously learned methods.
4. Use the zero-product property by setting each factor equal to 0.
5. Solve for the variable.
6. Check the solution.

Example 2:	Solve for the unknown variable in the polynomial $5y^3 + 20y = -20y^2$.

Example 3:	Solve for the unknown variable in the polynomial $5x^2 = 3x$.

Example 4:	Solve for the unknown variable in the polynomial $5x^2 + x - 6 = 0$.

Section 6.5 Special Cases of Factoring

Looking Back 6.5

In the last section, we investigated Greatest Monomial Factors and polynomials that are factorable, but it was mostly a review of factorable quadratic equations. In Algebra I we investigated special cases of binomials for quadratics.

Perfect square trinomials factor into the square of a binomial.

Example 1: Factor the perfect-square trinomial.

a) $2x^2 - 8x + 8$

b) $16a^2 - 24ab + 9b^2$

The difference of squares factors to two binomials with like terms with different signs.

Example 2: Factor the expression using the difference of squares.

a) $a^2 - b^2$

b) $16x^2 - 49y^2$

c) $y^2 - 100$

Looking Ahead 6.5

There are also special factoring patterns for cubic equations:

Sum of Two Cubes:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Difference of Two Cubes:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Example 3: Factor the expression using the sum of two cubes and check your work.
$x^3 + 64$

Example 4:	Factor the expression using the difference of two cubes and check your work.
	$8y^3 - 27$

Cubic equations can also be factored by grouping. Again, it is best to check your work using multiplication.

Example 5:	Factor by grouping and check your work by multiplying.
	$4y^3 - 6y^2 + 10y - 15$

Example 6:	Factor by grouping and check your work by multiplying.
	$2y^3 + 8y^2 - 3y - 12$

Section 6.6 Synthetic Division

Looking Back 6.6

In Algebra 1, you learned how to divide polynomials using long division. Let us review this here.

Example 1: Divide the expressions using long division.

$$(3x^3 + 12x^2 + 2x + 8) \div (x + 4)$$

Example 2: Divide the expressions using long division.

$$(14x^3 + 7x^2 + 10x + 5) \div (2x + 1)$$

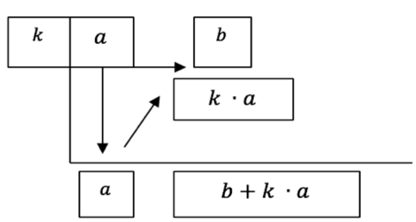
Example 3: Divide the expressions using long division.

$$(27x^3 + 3x^2 + 1) \div (3x + 2)$$

Looking Ahead 6.6

When the divisor is $x \pm k$ where k is a constant, synthetic division can be used as a shortcut for long division. There is another method for dividing lengthy polynomials that is not quite as cumbersome as most. It can be used to find the factors of polynomials for a cubic. It is called synthetic division.

$$ax^3 + bx^2 + cx + d \qquad \text{(Dividing by } x - k\text{)}$$



If the divisor is $x - k$, then k is positive. If the divisor is $x + k$, then k is negative since $x - (-k) = x + k$.

Example 4:	Divide the expressions using synthetic division.
	$(x^2 + 7x + 10) \div (x + 5)$

Example 5:	Divide the expressions using synthetic division.
	$(x^2 + 3x + 4) \div (x + 1)$

The Remainder Theorem states that if a polynomial $f(x)$ is divided by $(x - k)$, then the remainder is $f(k) = r$. In Example 5 the remainder was 2 so $r = 2$ and $k = -1$. That means $f(-1) = 2$. Let us try it.

Example 6:	For $f(x) = x^2 + 3x + 4$ demonstrate that $f(-1) = 2$.

The Factor Theorem states that if a polynomial $f(x)$ has a factor $(x - k)$, then $f(k) = 0$. If the remainder is zero, then $(x - k)$ is a factor. If the remainder is not zero, then $(x - k)$ is not a factor of the polynomial $f(x)$.

Example 7:	Demonstrate that $f(x) = x^3 + 2x^2 - 5x - 6$ has a zero of 2 using synthetic division.

Section 6.7 Solving Polynomials Using the Zero-Product Property

Looking Back 6.7

Given any polynomial function, the zero-product property states that if $ab = 0$, either $a = 0$, $b = 0$, or both equal zero (given that a and b are real numbers). Therefore, if $abc = 0$, either $a = 0$, $b = 0$, or $c = 0$ (given that a, b, and c are real numbers). For any number of factors, to give a product of zero, at least one of the factors must be zero, but one or more of the factors may be equal to zero.

Let us review the factors of the last few problems in the previous "Practice Problems" section.

Example 1: $f(x) = x^3 + x^2 - 6x + x^2 + x - 6$ This may also be simplified to $f(x) = x^3 + 2x^2 - 5x - 6$.

Both of these factors to

$f(x) = (x + 3)(x - 2)(x + 1)$

What are the zeroes of the function?

Looking Ahead 6.7

In Section 6.2, you solved less complex polynomial equations using the zero-product property. You used the method of factoring that you learned in Algebra 1. Now that you know how to use synthetic division to factor more complex polynomials, you can use the zero-product property to find the zeroes of more complex equations.

Example 2: Solve the expression by factoring.

$$x^3 + 27$$

The Fundamental Theorem of Algebra states that any polynomial function f with a degree $n > 0$ and with real coefficients has, at most, n roots. At least one zero is in the set of complex numbers since any real number a can be written as the complex number $a + bi$ where $b = 0$.

Example 3:	Use synthetic division to find the other factors of $f(x) = 2x^3 + 7x^2 - 5x - 4$ if one factor is $x + 4$.

Section 6.8 The Rational Zero Test

Looking Back 6.8

The last few problems of the previous Practice Problems section included a cubic equation that modeled an equation given the volume of a box. That cubic equation was not factorable by special factoring, such as the sum or difference of cubes. Finding the factors of third- and fourth-degree equations can be challenging, but there is a test that makes it a little less difficult.

When multiplying, for example: $(ax - r)(bx - s)(cx - t)$, etc., the first terms in the series multiply to give the lead coefficient, $a \cdot b \cdot c$, and the degree of the polynomial. Real numbers are represented by the multiplication of the second term in each set of parentheses and $r \cdot s \cdot t$ is a constant.

Looking Ahead 6.8

Let $\frac{p}{q}$ be the rational zero of the function f. Remember that a rational number is the quotient of two integers. Let $a_0, a_1, \ldots\ a_n$ be integer coefficients of the polynomial $f(x) = a_n x^n + \ldots a_1 x + a_0$. In this case, a_n represents the lead coefficient and a_0 represents the constant. If a zero is not rational, it is irrational.

Therefore, the rational zero of the function f is $\frac{p}{q} = \frac{a\ factor\ of\ a_0}{a\ factor\ of\ a_n}$.

The remainder theorem says we can evaluate the polynomial in terms of the divisor and then evaluate the polynomial at $x = a$. When $x = a$, the factor $x - a$ is just zero. Evaluating the polynomial gives us $p(a) = (a - a)q(a) + r(a)$ where $q(a)$ is the quotient and $r(a)$ is the remainder.
Then $q(a) = 0 \cdot q(a) + r(a)$, and $p(a) = 0 + r(a)$. Therefore $p(a) = r(a)$. The remainder must be zero, for the zero to be a factor. If the remainder is not 0, then the zero is not a factor.

Example 1: Find all the possible rational zeroes of $x^4 - 2x^3 - 13x^2 + 14x + 24$.

$a_n = 1$ and $a_0 = 24$

Example 2: Use synthetic division with the Remainder Theorem to test the first four positive possible rational zeroes using the function in Example 1 to see if they are actual rational zeroes.

Example 3: Try the opposite of the values tested in Example to find other possible zeroes. Write these as factors and multiply to check your solutions.

Example 4: Divide the original polynomial by the product of the factors found from the first three rational zeroes to see if it gives you the final factor and matches the last rational zero.

Example 5: $f(x) = x^3 + x^2 - 5x - 5$

Find all the factors of a_n and a_0 to get $\frac{p}{q}$, the possible rational zeroes. Use synthetic division to find the rational zeroes and write the factors of the polynomial.

Section 6.9 Sketching Graphs of Polynomial Functions

Looking Back 6.9

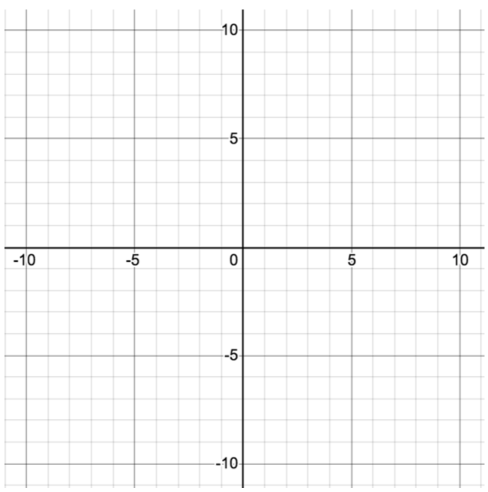
Previously, you have learned that polynomials whose highest degree is even start and end the same way; the graph of $y = x^2$ starts up and ends up; the graph of $y = -x^2$ reflects the parent function over the x-axis when a is negative so it starts down and ends down.

However, $y = x^3$ has an odd power so its graph starts down and ends up, while $y = -x^3$ starts up and ends down. A polynomial whose highest degree is odd starts and ends in opposite ways. We will call this "end behavior." This is the behavior of the graph of the polynomial $f(x)$ as x approaches either positive or negative infinity. The degree of the lead coefficient of the polynomial function determines the end behavior.

Looking Ahead 6.9

The rational zeroes are real numbers that appear as x-intercepts on the graph. A polynomial $g(x)$ that has three factors, $(x-2)(x+4)(x-5)$, has three zeroes: $x = 2$, $x = -4$ and $x = 5$, which appear as x-intercepts. There are three linear factors, so the polynomial is of degree 3. The function is odd, so its graph starts down and ends up. The graph crosses the x-axis at the three zeroes. Sketch the graph below.

Example 1: Graph $g(x) = (x-2)(x+4)(x-5)$.



x-intercepts: ______________________

Degree: ______________________

Odd or Even: ______________________

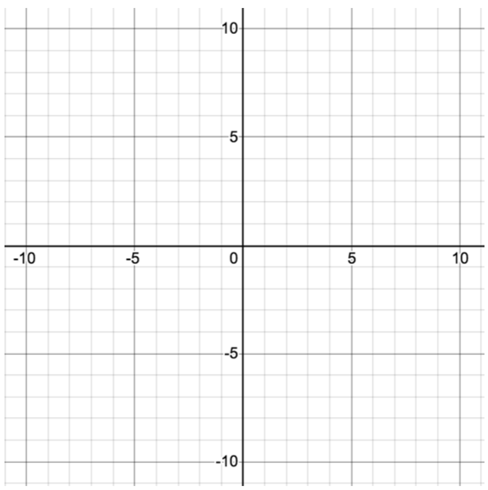
End Behavior: ______________________

y-intercept: ______________________

The mathematical notation for end behavior looks like this: as $x \to +\infty$, $f(x) \to +\infty$; as $x \to -\infty$, $f(x) \to -\infty$. The "→" symbol means "approaches."

Example 2: Describe $f(x) = 3x^3$ verbally and mathematically in terms of end behavior.

Example 3: Sketch the graph of $h(x) = -(x+4)(x-1)^2(x+1.5)$.



x-intercepts: ______________________

Degree: _________________________

Odd or Even: _____________________

End Behavior: ____________________

y-intercept: _______________________

Example 4: A polynomial function is $y = ax(x+1)(x-2)^2$ and goes through the point $(5, -10)$. What is the value of a?

Section 6.10 End Behaviors and Multiplicities

Looking Back 6.10

In the previous section of this module, you were introduced to end behavior of polynomials. In other words, you began to learn what a graph looks like at the front end as well as the back end.

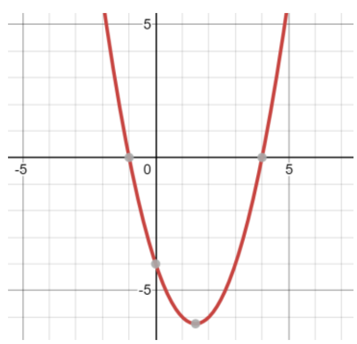
All even degree power functions start up and end up just like quadratics. However, if the lead coefficient is negative, they start down and end down. Either way, the front end is the same as the back end.

All odd degree power functions start down and end up just like cubic equations. However, if the lead coefficient is negative, they start up and end down. Either way, the front end is opposite the back end.

Example 1:	Discuss the end behavior of $4x^9 + 5x^3 + x - 22$.

Looking Ahead 6.10

You learned earlier that the zeroes of the polynomial tell how many times the polynomial graph crosses or touches the x-axis. Each zero has a multiplicity which refers to how many times that factor appears in the polynomial. For example, if $y = (x + 1)(x - 4)$ then the zeroes are at $x = -1$ and $x = +4$. Since $a > 0$, the graph starts up and ends up.

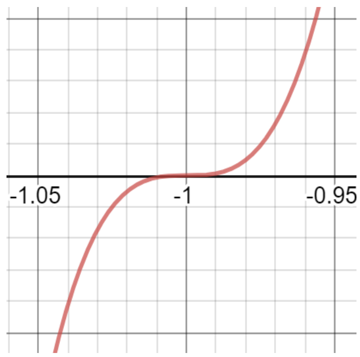


We know $y = (x + 1)^1(x - 4)^1$ and the degree is $1 + 1 = 2$.

However, for $y = (x+1)^3(x-4)^4$ the degree of the polynomial is 7 from the exponents $3 + 4 = 7$. The sum of the multiplicities is the degree of the polynomial.

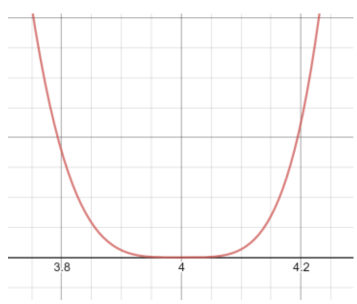
The factor $(x+1)$ is called a triple zero because it has a power and multiplicity of 3. The 3 creates an s-shape that passes through the point $x = -1$ and flattens out a little.

An odd multiplicity crosses or intersects the *x*- axis. Odd degree numbers are both positive and negative and can cause the graph to change from positive to negative when crossing from above the *x*-axis to below it and vice versa.



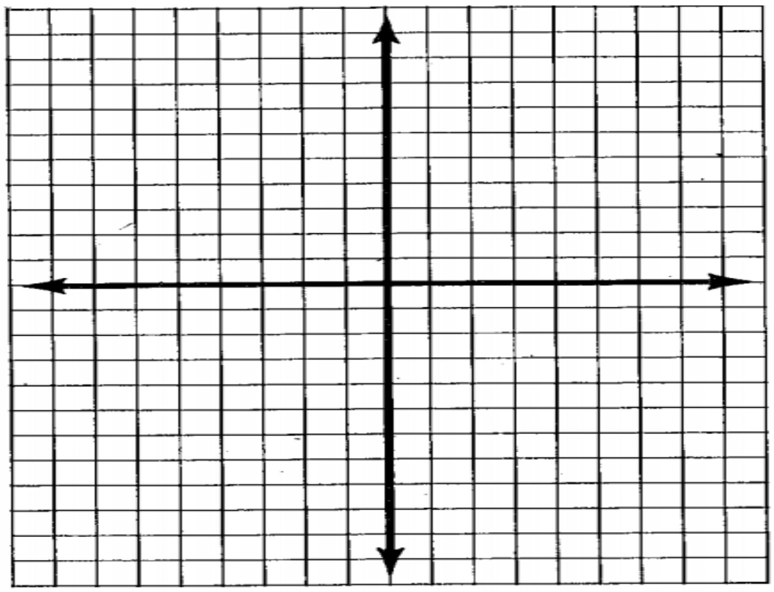
The factor $(x-4)$ has a power and multiplicity of 4. Because 4 is even the graph just touches the *x*-axis at $x = 4$ and returns the same way. We could say the graph bounces off the *x*-axis and bounces back the way it came. There is a flattening near $x = 4$.

An even degree multiplicity touches or bounces off the *x*- axis. Even degree numbers are squares and always positive and cannot cause the graph to change from positive to negative when crossing from above the *x*-axis to below it and vice versa.

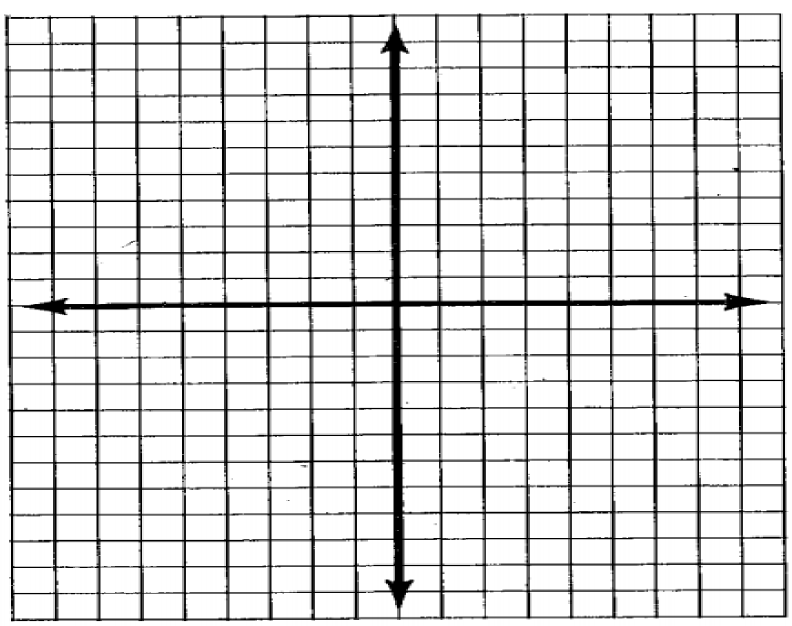


All polynomial functions are odd or even degree and always behave this way.

Example 2: Sketch the graph of $y = (x-6)^2(x+1)^2$.



Example 3: Sketch the graph $y = (x-6)^2(x+1)^3$.

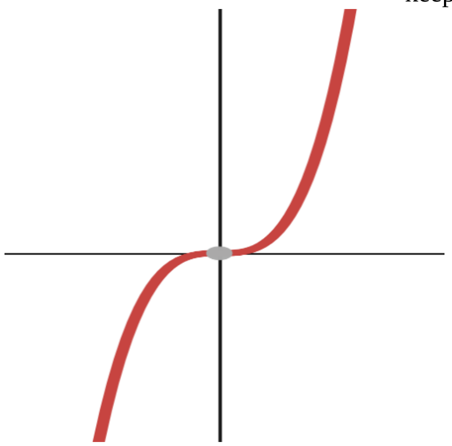


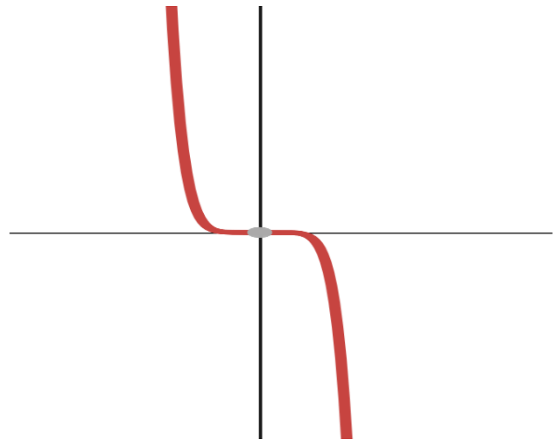
Section 6.11 Curves and Bounces of Graphs

Looking Back 6.11

The curves and bounces we have been discussing have more mathematical names used in Pre-Calculus and Calculus, but for now, the description fits the form. Previously we have said that the graph below on the left starts down and ends up, and the graph on the right starts up and ends down referring to end behavior.

We will define a curve below as a turn where the graph crosses the x-axis and is heading up or down and keeps heading up or down.



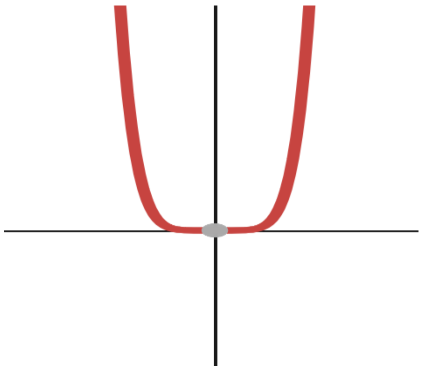


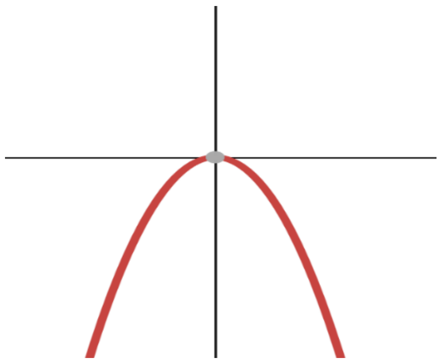
The graph on the left starts down but goes up and continues to go up after it crosses the x-axis. It is a third-degree polynomial. The graph on the right starts up but goes down and continues to go down after it crosses the x-axis. There are small bounces that are not visible where the graph flattens. The point of inflection is at the origin. It is the point where the curvature of the graph changes from concave down to concave up or vice versa.

Concavity refers to curving upward or downward. The curve on the left is concave down and then changes to concave up at the origin. The graph on the right is concave up and then changes to concave down at the origin. Concave up is the slope is increasing and concave down is when the slope is decreasing.

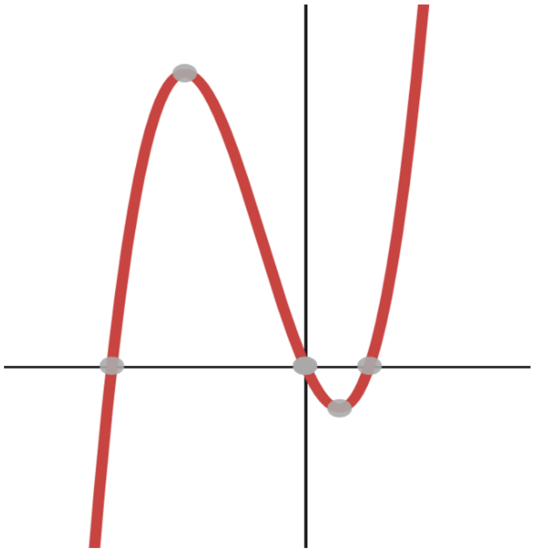
Looking Ahead 6.11

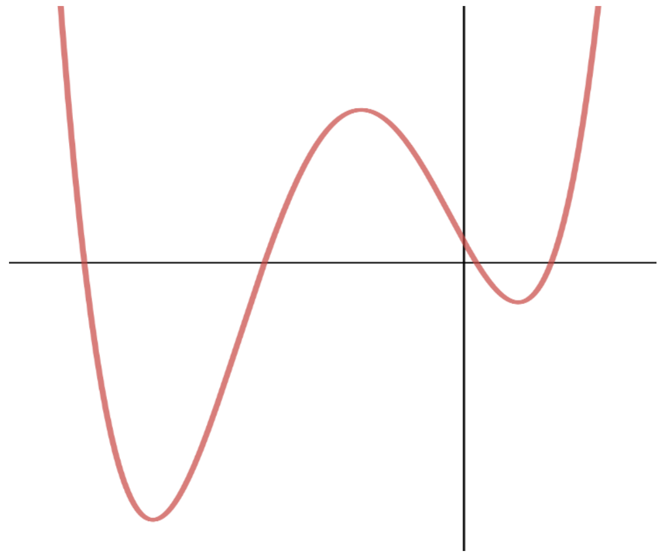
We will refer to a bounce as the point where the graphs barely touches the x-axis and then turns back the way it came. The graph on the left starts up and heads down, then at the origin it bounces and starts heading back up. It is a fourth-degree polynomial and there is a flattening at the origin. This represents a multiplicity of zeroes. Even though it flattens near the origin, it does not touch the x-axis until the origin. The graph on the right starts down and heads up, then at the origin it bounces and starts heading back down. It is a quadratic of degree 2 and has 1 bounce.





Example 1:	Tell how many bounces each graph has. How does the number of bounces relate to the degree?

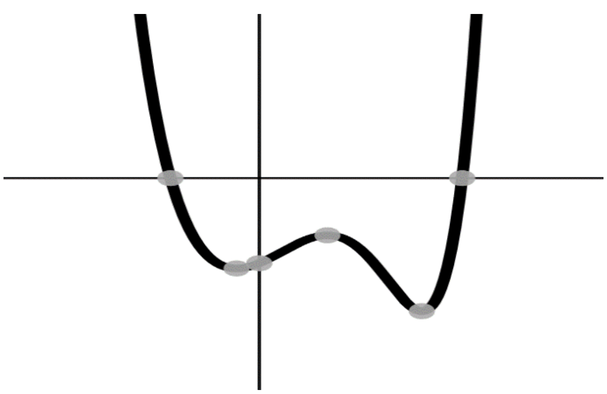


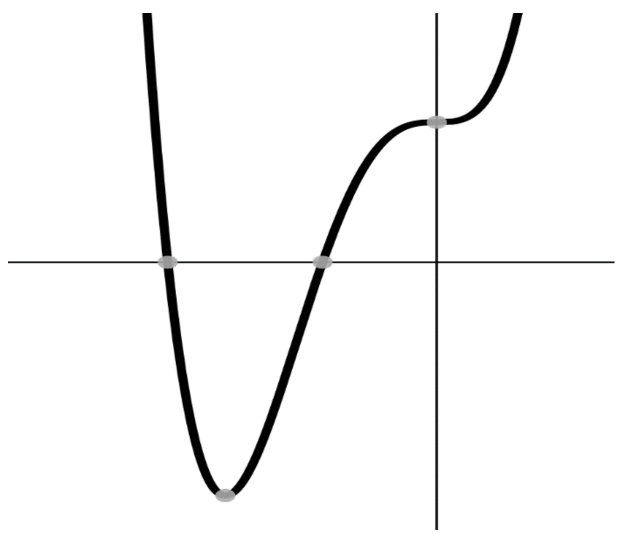


The bounce is less than the degree. At most it is one less than the degree. If n = degree, then $n-1$ is the maximum number of bounces a polynomial graph can have.

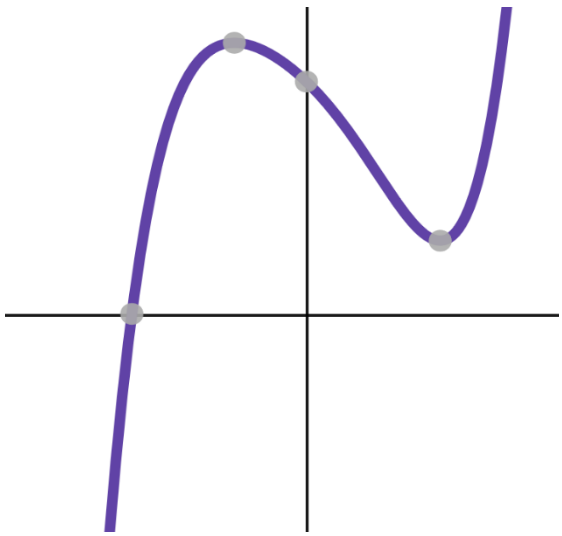
Sometimes these bounces seem to disappear or are not visible. This happens when they flatten out on the graph. A polynomial of degree 5 can have at most 4 bounces but may have less than that visible.

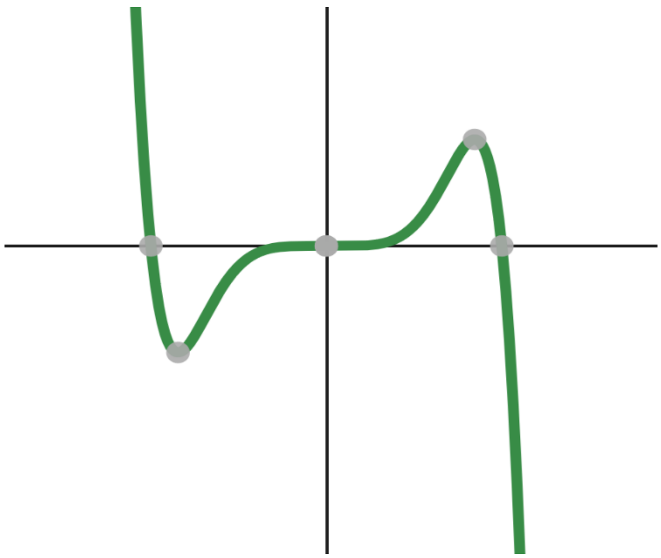
Example 2:	Which graph could be a polynomial of degree 6?

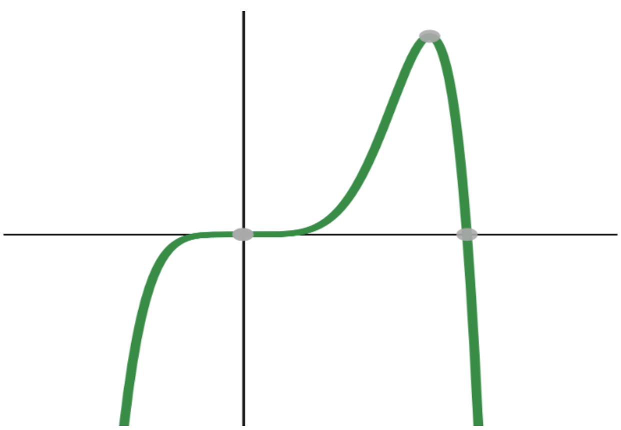


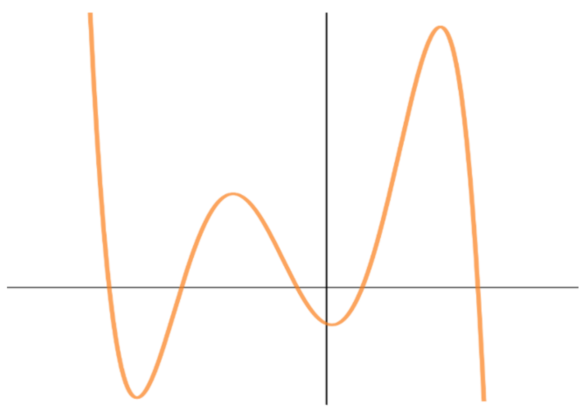


Example 3: Give as much information of each polynomial graph as possible.





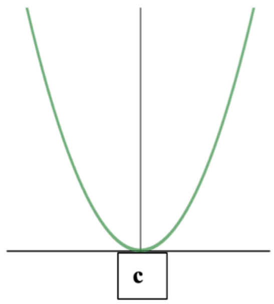


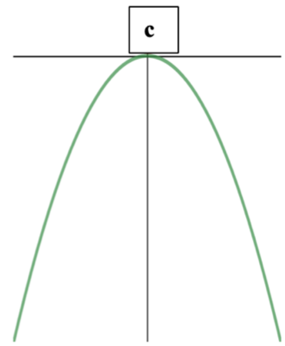


Section 6.12 Concavity, Intervals, and Extrema

Looking Back 6.12

Concavity refers to the direction of a curve. A curve changes direction at a maximum or minimum value. If a curve is concave up as the graph below on the left, it is bent upward like a bowl sitting on a table. If a curve is concave down as the graph below on the right, it is bent downward as if the bowl had been flipped upside-down.





When talking about lenses, the left curve is convex, and the right curve is concave. In math, we use the terms concave up or concave down. There is a second derivative test for concavity and inflection points that is used in Calculus, but for now we will simply use the graph.

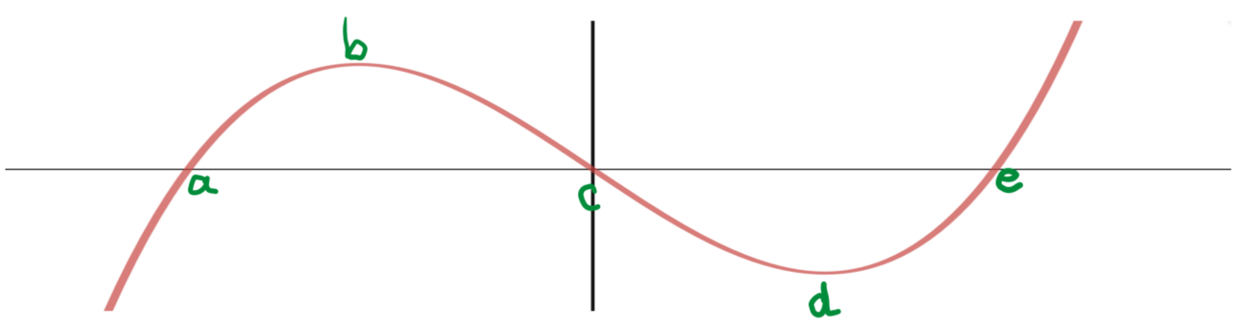
The graph on the left has a minimum value at the origin. This is called a global minimum because it is the minimum value of the entire function, $f(c) \leq f(x)$. The output is decreasing to the left of point *c* and the output is increasing to the right of point *c*. The graph on the right has a maximum value at the origin. This is called a global maximum because it is the maximum value of the entire function, $f(c) \geq f(x)$. The output is increasing to the left of point *c* and the output is decreasing to the right of point *c*.

Looking Ahead 6.12

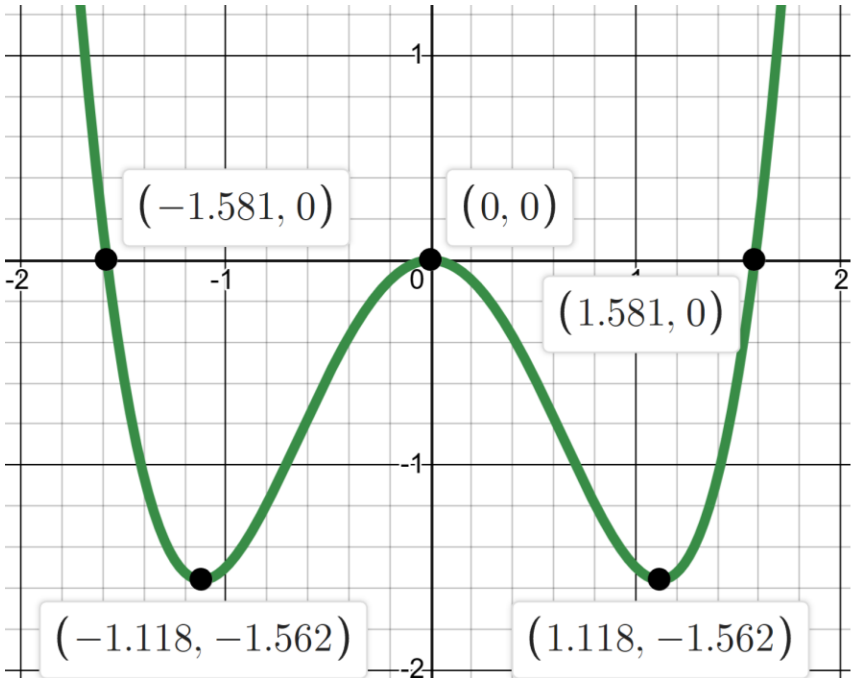
As the input of a polynomial increases, the output of a polynomial increases or decreases. Function values can decrease over an interval and then increase over another interval.

Functions can have a local minimum or maximum over a given interval. The local minimum or maximum can also be referred to as the "relative" minimum or maximum because it is not the minimum/maximum value for the entire function, but rather for the given interval. The plural form for minimum is minima and the plural for maximum is maxima. The local and global minima and maxima are called extrema.

Example 1: Investigate the graph for local and global maxima and minima and increasing and decreasing intervals for the function.

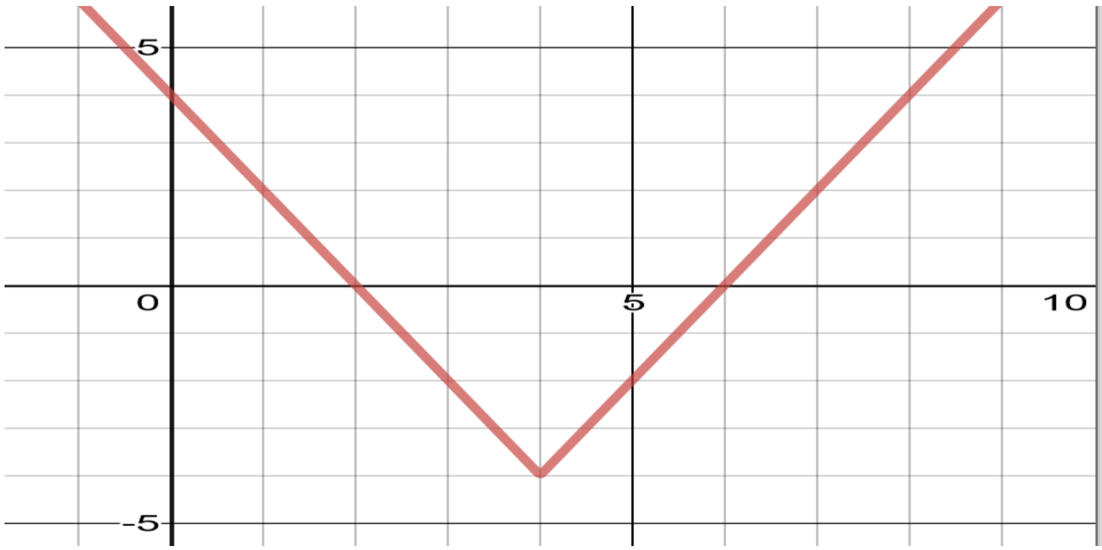


Example 2:	Investigate the graph below to complete the following table.



x-intercept(s)	
y-intercept	
End Behavior	
Increasing Interval	
Decreasing Interval	
Domain	
Range	
Local Minimum	
Local Maximum	
Global Minimum	
Global Maximum	

Example 3:	Complete the table based on the function below.



x-intercept(s)	
y-intercept	
End Behavior	
Increasing Interval	
Decreasing Interval	
Domain	
Range	
Global Minimum	
Global Maximum	

Section 6.13 Writing Polynomial Equations

Looking Back 6.13

The coefficient "a" is the lead coefficient in a standard form polynomial equation. In graphing form, "a" is the scale factor for a vertical stretch or shrink (compression) of the graph.

The coefficient "a" can be found if at least one point (x, y) on the graph is known. This is the same method used to find "a" in quadratic, cubic, and all power functions. It is also the same method used for exponential or logarithmic equations.

Looking Ahead 6.13

Example 1: Given the cubic equation $y = ax^3 + 2x - 4$, find "a" if a point on the line is $(2, 24)$.

Example 2: The graphing form of a cubic equation is $y = a(x-2)^3 + 1$. Find "a" if a point on the line is $(5, 109)$. After you find "a" write the equation in standard form.

Example 3: Given the graph of a polynomial, find "a" and write the equation in standard form.