Algebra 2 Module 8 Exponents and Logarithms

Section 8.1 Introducing Exponential Functions

Looking Back 8.1

At the end of the previous module, you investigated amortization problems. In this section, you will explore the graphs of those problems.

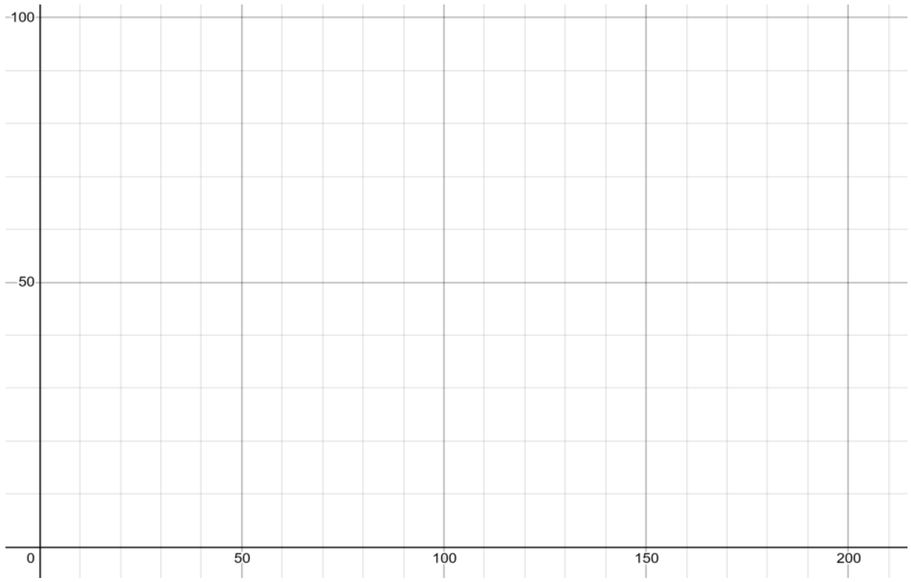
We will begin with an experiment that reviews a function you have investigated and then we will look at another experiment to introduce this present function of study.

Angel and Josh dropped a basketball from 200 cm. and made a chalk mark on the wall for the height it bounced up to on the first bounce (rebound height). They did this trial two more times to find the height of the average rebound. Then they tried it three times from a 175 cm. drop and repeated it at 150, 125, and 100. The results are shown below. They calculated the rebound ratio by comparing the two heights: $\frac{\text{rebound height}}{\text{drop height}} = \text{rebound ratio.}$ Stores often test balls this way to make sure the rebound is consistent with the advertised height.

Example 1: Find the average rebound ratio using the table below.

Start Height (cm.)	**1st**	**2nd**	**3rd**	**Avg. Rebound Height (mean of the measurements of the first three trials)**	**Rebound Ratio**
200	82	99	82	87.6	0.44
175	76	75	75	75.3	0.43
150	65	70	64	66.3	0.44
125	44	50	46	46.6	0.37
100	41	42	45	42.6	0.43

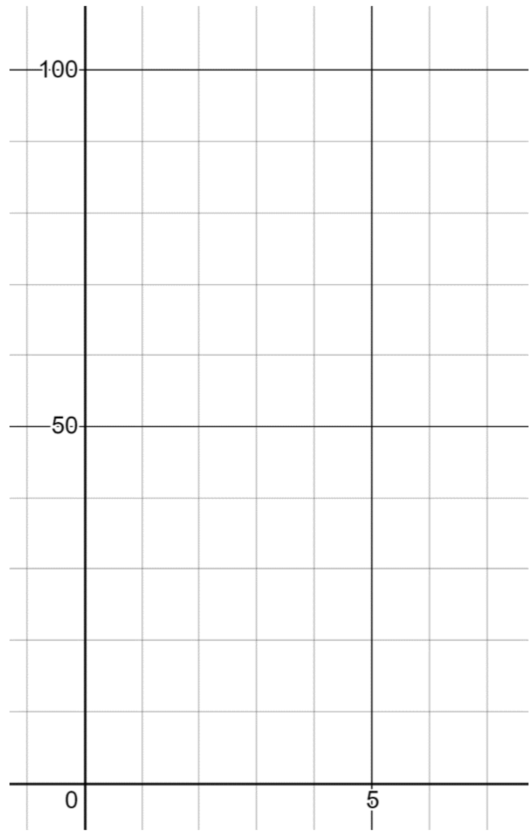
Example 2: Draw the graph of the rebound data from Example 1. Let the x-axis be the start height and the y-axis be the rebound height. What does the constant of variation represent?



Looking Ahead 8.1

Example 3: Once again, Josh and Angel drop a basketball, but this time from a height of 200 cm. They put a tape mark at each bounce height for six consecutive bounces. They marked the tape "1-6" for each bounce. The results are shown below in a table. Graph the results below and you will see the graph is non-linear. It is decreasing exponentially. To get the bounce height, take the original height and multiply by the bounce ratio, which we will say is 0.43. Find the exponential equation for the experiment.

Bounce Number	**Bounce Height**
1	95
2	51
3	23
4	3
5	2
6	1



Section 8.2 Graphs of Exponential Functions

Looking Back 8.2

The equation $y = ab^x$ is the general form of an exponential function where a is the initial value and b is the growth factor. Let us look at various values of b and examine how it affects the exponential curve.

Let $a = 1$. If $b = 0$ in the equation, then 0^0 is indeterminate and 0^n where n is any negative number is undefined. However, 0^n where n is any positive number is 0. So, 0^0 is an open circle on the graph, but $y = 0^x$ is the horizontal line $y = 0$ and the domain is $x > 0$.

Therefore, the graph is a horizontal line and the function is not exponential. There is neither growth nor decay.

Similarly, when $b = 1$, then b^n, where n is any positive or negative number, is just 1. Again, this is just the horizontal line $y = 1$ and is not exponential. So, $y = b^x$ is an exponential function when $b \neq 0$ or $b \neq 1$.

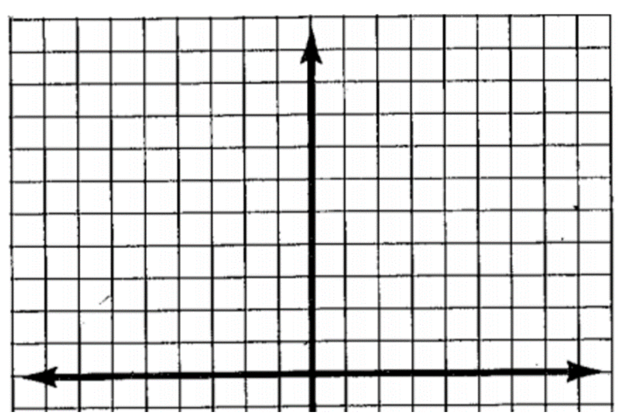
Now let us investigate when $b < 0$ or $0 < b < 1$ or $b > 1$.

Looking Ahead 8.2

Example 1: Investigate the graph of $y = ab^x$ when $a = 1$ and $b = -\frac{1}{2}$.

Example 4: Let $b = \frac{1}{2}$, which satisfies $0 < b < 1$, and complete the table and graph. Then answer the following questions.

x	$f(x)$
-6	64
-5	32
-4	16
-3	8
-2	4
-1	2
0	1
2	0.25
3	0.125



a) What happens to y as x increases?

b) What happens to y as x decreases?

c) What is y when x is 0?

d) Is the exponential function increasing or decreasing as x approaches infinity?

e) Why does this make logical sense? Explain the type of graph you will always get when b is a proper fraction or a decimal number between 0 and 1.

f) Is there a vertical or horizontal asymptote?

Section 8.3 Transformations of Exponential Functions

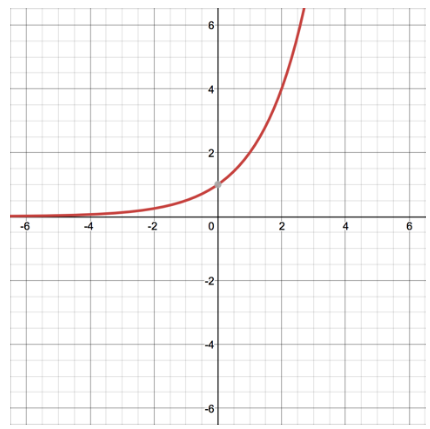
Looking Back 8.3

The standard form of an exponential equation is $y = ab^x$. In the parent function $a = 1$ and b is a value that is greater than 0 but b is not equal to 1. For an increasing exponential, b must be a value that is greater than 1. Two is usually used for b. The table and graph of $f(x) = 2^x$ is given below.

The table and graph for $y = ab^x$ where $a = 1$ and $b = 2$ are shown below.

Substituting in values for x and solving for y will allow a table to be created and then the ordered pairs can be graphed.

x	y or $f(x)$
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8



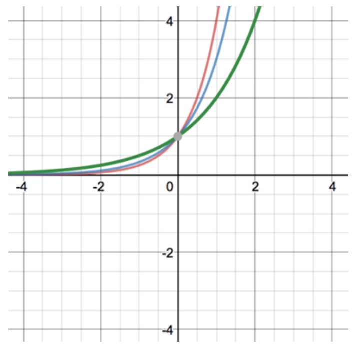
Looking Ahead 8.3

The graphing form for an exponential equation is $y = ab^{x-h} + k$.

In this section, we will investigate the parameters a, h, and k and the effect they have on the parent exponential function $y = 1 \cdot 2^x$ or $y = 2^x$. Let us investigate the parameter a together and then in the practice problems section we will investigate h and k.

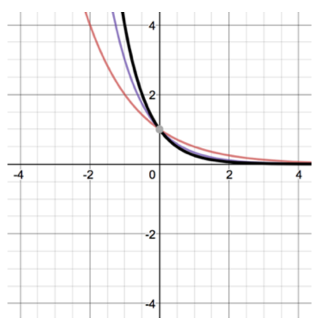
We have already explored b. The base must be greater than zero. If b is between 0 and 1, it is a decreasing exponential. Let us see what happens as b gets increasingly larger.

Example 1:	Compare the graphs of $y = 2^x,\ y = 3^x$ and $y = 4^x$



As b gets bigger, the graph gets steeper or appears to stretch up. We have already seen that when $0 < b < 1$ the graph is decreasing instead of increasing. Let us look at $b = \frac{1}{2}$, $b = \frac{1}{3}$, and $b = \frac{1}{4}$.

Example 2:	Compare the graphs of $y = (\frac{1}{2})^x,\ y = (\frac{1}{3})^x$ and $y = (\frac{1}{4})^x$



As b gets bigger, the graph gets less steep or appears to compress.

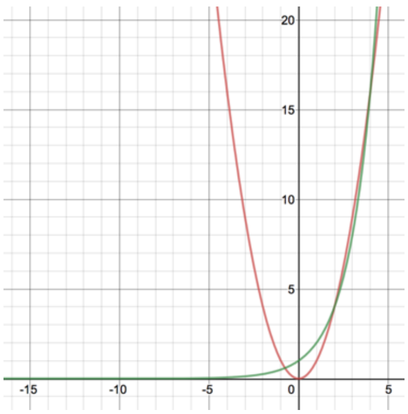
Standard form may be converted to graphing form, $ab^{(x-h)} + k = y$, to graph the equation more readily. You will be investigating the parameters a, h and k and their effects on an exponential equation in today's practice problems.

Section 8.4 Comparing Exponential Functions and Power Functions

Looking Back 8.4

A power function has an exponent that is a number. An exponential function has an exponent that is a variable.

Example 1: Let us compare $y = x^2$ and $y = 2^x$.



The graph of $y = 2^x$ appears steeper than the graph of $y = x^2$. After the intersection point, $(2, 4)$, the graph of $y = x^2$ overtakes the graph of $y = 2^x$. However, after the point $(4, 16)$ the graph of $y = 2^x$ appears to overtake $y = x^2$. In the long run, $y = 2^x$ continues to be steeper.

$\boldsymbol{x}$	$\boldsymbol{2^x}$	$\boldsymbol{x^2}$
0	1	0
1	2	1
2	4	4
3	8	9
4	16	16
5	32	25
6	64	36

In the interval $-\infty < x < -1$, $2^x < x^2$.

When $0 \leq x \leq 1$, $2^x > x^2$.

When $x = 2$, $2^x = x^2$.

When $x = 3$, $2^x < x^2$.

When $x = 4$, $2^x = x^2$

On the interval $x > 4$, $2^x > x^2$ at increasingly greater rates.

Looking Ahead 8.4

Example 2: Let us compare $y = x^3$ and $y = 3^x$.

On what interval is $x^3 < 3^x$?

On what interval is $3^x > x^3$?

x	x^3	3^x
-3	-27	0.03
-2	-8	0.11
-1	-1	0.33
0	0	1
1	1	3
2	8	9
3	27	27
4	64	81
5	125	243

At all other increasing values of x on the graph $3^x > x^3$. In the long run it appears that these exponential functions are greater than these power functions. In the practice problems today, you will look at exponential functions with a large base to determine if they are greater than a power function with a large power for large values of x.

Section 8.5 Problem Solving Using Exponential Equations

Looking Back 8.5

In the past, you have worked on interest rate problems. In the last module, you worked on amortization problems involving interest rates. Now, let us review how to derive the compound interest formula.

Example 1: Camille puts \$10 in her bank at 5% interest per year. How much money will she have after five years?

\$10 at 5% Interest

$a =$ Initial Investment

$b =$ Interest Rate

Year 0: \$10

Year 1: \$10 + \$10 (0.05) = $a + ab = a(a + b)^1$

Year 2: $[\$10 + \$10(0.05)] + [\$10 + \$10(0.05)]0.05$

$$a + a(b) + [a + a \cdot b] \cdot b$$

$$a + a \cdot b + a \cdot b + a \cdot b \cdot b$$

$$a(1 + b + b + b^2)$$

$$a(1 + 2b + b^2)$$

$$a(1 + b)(1 + b)$$

$$a(1 + b)^2$$

Year 3: $a(1 + b)^3$

Year 4: $a(1 + b)^4$

Year n: $a(1 + b)^n$

Looking Ahead 8.5

The formula in the previous module was for amortization, the process of reducing the amount of the loan or the value of an asset based on periodic payments. That formula is $A = P\frac{r(1+r)^n}{(1+r)^{n-1}}$. The formula $A = P(1 + r)^t$ is for calculating compound interest. When the rate is compounded annually, then $t = 1$, where t is time.

The formula $A = a(1 + b)^n$ where A is the total amount of money is often written $A = P(1 + r)^t$ where A is the total amount of money, P is the principal invested, r is the interest rate and t is the amount of time invested. This formula is used to determine the total amount of the investment with an annual percentage rate (APR) in a given period of time.

Example 2: Sevelia deposited $100 in an account that pays 6% annual interest that is compounded quarterly. How much money is in the account after one year?

Compounded quarterly means the interest is compounded each quarter of the year. That means interest is compounded four times in one year, which means the account earns only 1.5% interest per quarter. This comes from dividing the interest rate by the number of times it is compounded, $\frac{r}{n}$, which is $\frac{6\%}{4}$ in this case.

This gives us a compound interest formula $A = P(1+\frac{r}{n})^{nt}$, where P is the principal deposited that pays an interest rate of r, compounded n times per year, and A is the amount in the account after t years.

$$A = 100(1 + \frac{6\%}{4})^{4 \cdot 1}$$

$$A = 100(1 + 1.5\%)^4$$

$$A = 100(1 + 0.015)^4$$

$$A = 100(1.015)^4$$

$$A = \$106.14$$

Example 3: The following exponential equation algorithm can also be used for population growth, bacterial growth, and has many other applications in mathematics and science:

$$A = I(1 + \frac{r}{n})^{nt} \text{ (where } I \text{ is the initial population)}$$

The number of people that smoke in Matlock County has been decreasing at a rate of 7% per year. As of September, in the year 2000, the number of people who smoked in Matlock County was 3,050.

a) Write an equation to represent the number of smokers in Matlock County in the nth year after 2000.

b) Using the model, figure out approximately how many smokers will be in Matlock County in September of 2018.

Section 8.6 Solving the Parts of Exponential Equations

Looking Back 8.6

We have already stated that the standard form of an exponential equation is $y = ab^x$ where a is the initial value, b is the growth factor, x is the input (domain), and y is the output (range). When we previously substituted values for a, b, and x, we were able to solve for y. Now, let us investigate how to solve for a, b, and x.

Looking Ahead 8.6

Example 1: Khali and Amanda dropped a ball from 200 cm. and recorded the height of the first six bounces, shown in the table below. The initial height is $a = 200$ cm. Find b, the base, or multiplier.

Bounce Number (x)	**Bounce Height ($f(x)$)**
0	200
1	95
2	48
3	24
4	11
5	5
6	2

The base, b, is what is being multiplied repeatedly. It is called the growth factor. This is a decreasing exponential so the growth factor is a decimal or fraction. What is being multiplied by 200 to get 95? The equation $200 \cdot m = 95$ means $\frac{95}{200}$ or, using reverse thinking, $m = \frac{95}{200}$. If you multiply 200 by something to get 95, then, going backwards, you can divide 95 by 200 to get the growth factor. This is the bounce ratio as we learned previously. So, our base represents the bounce ratio.

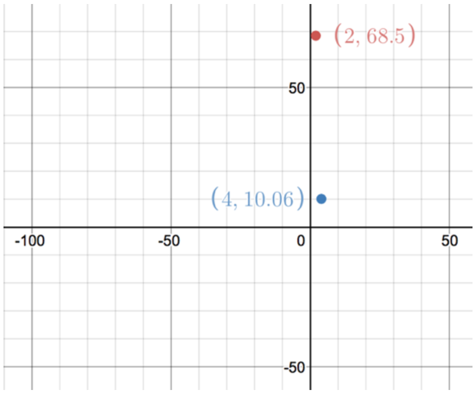
Let us divide each value of $f(x)$ by the previous value to see what we get.

It is an experiment, so the values are not always the same. To get the average bounce ratio add them all and divide the sum by 6.

$$(0.48 + 0.51 + 0.50 + 0.46 + 0.45 + 0.4)/6 \approx 0.47$$

Now we know the equation for the bounce height is $f(x) = 200 \cdot 0.47^x$.

Example 2: Below is a graph of Jerusa and Ruby's experiment. Find b when given only two points on the graph. We can set up two equations and solve for b since we know x, y and we know that $a = 200$ cm.



Example 3: The initial height is unknown for the release of a basketball. Dale and Lisa left their data on the table and water spilled all over it. There are only two points that can be seen clearly on the graph and they are $(2, 35.4)$ and $(4, 12.53)$. Find the initial height.

The two equations are $35.4 = ab^2$ and $12.53 = ab^4$.

Section 8.7 Inverses of Exponential Functions

Looking Back 8.7

When working with power functions, you have learned that $y = x^2$ and $\sqrt{x} = y$ are inverses. In Algebra 1, you found inverses by using tables, graphs and equations.

Solving for x when $8 = 2^x$ can be done mentally:

It is fairly easy because 8 is a perfect cube.

Let us try another:

$$16 = 2^x$$

Multiplying 2 by itself 4 times gives x. However, $16 = 3^x$ is a bit more difficult because x is a decimal number. In Algebra 1 you used the guess and check method to approximate the answer.

By using reverse thinking when taking the inverses, you can solve for x in an exponential equation. The inverse of a square is a square root. The inverse of a cube is a cube root. The inverse of an exponential function is called a logarithm.

A new notation will be used to write the inverse, which is called a logarithm. Logarithms were originally used by John Napier and other mathematicians to do arithmetic with very large numbers.

A method for lattice multiplication that is named after John Napier is called Napier's Bones. You learned about that in Pre-Algebra. This led to his use of and study of logarithms.

As told by Joseph Frederick Scott in his biography of John Napier, the Scottish mathematician would go on to make possibly the greatest contributions to the study of logarithms. Napier not only described the nature of logarithms, but also explained the "method of their construction." Along with being an extraordinary mathematician, Napier was also known for his reputation as a passionate and outspoken Christian.

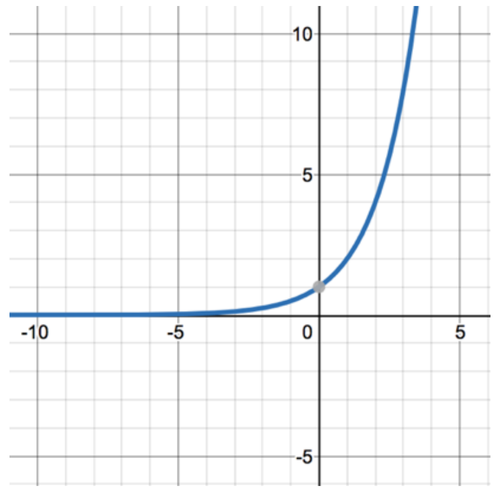
Looking Ahead 8.7

Example 1:	Let us investigate the table, graph and equation for the parent exponential function.

$$y = 1 \cdot 2^x$$

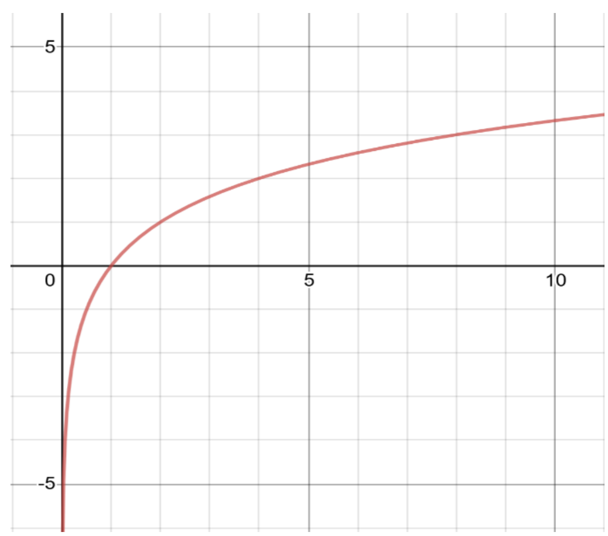
$$y = 2^x$$

x	y
-6	0.0156
-5	0.03125
-4	0.0625
-3	0.125
-2	0.25
-1	0.5
0	1
1	2
2	4
3	8
4	16
5	32
6	64



Example 2: To find the inverse of the function in Example 1, switch the x and y coordinates in the table and then graph the points. It should be a reflection over the line $y = x$ on the graph.

x	y
0.0156	-6
0.03125	-5
0.0625	-4
0.125	-3
0.25	-2
0.5	-1
1	0
2	4
3	8
4	16
5	32
6	64



The inverse of an exponential function is a logarithmic function and the same rules apply to both.

Just as exponential functions do not have a base of 0, 1, or negative numbers, logarithms do not have a base of 0, 1, or negative numbers.

$b > 0$ and $b \neq 1$

If $b^x = y$ then $\log_b y = x$

These are equivalent ways of writing the same thing.

To find the inverse, switch the x and y.

If $\log_b y = x$ then the inverse is $\log_b x = y$.

Example 3:	Find the logarithmic equation that is the inverse of $y = 2^x$.

Another mathematician who worked in the development of exponential notation is Jost Burgi. As told by Kathlee M. Clark and Clemency Montelle in Logarithms: The Early History, Jost developed tables of anti-logarithms (inverses) as well as tables for operations of multiplication, division, square roots and cube roots. He consolidated the tables because he found the separate tables to be "irksome, laborious, and cumbersome."

Today all those tables are consolidated in the functionality of graphing calculators.

Section 8.8 Converting Exponential and Logarithmic Equations

Looking Back 8.8

Previously, you have learned how to write exponential equations as logarithms. In the next section, you will be finding the unknown exponent without using your calculator. The section following that you will use your graphing calculator to solve exponential equations. For now, we will focus on expressing exponential equations in logarithmic form and expressing logarithmic equations in exponential form.

Looking Ahead 8.8

Example 1: Express the logarithmic equation in exponential form.

$$\log_2 32 = 5$$

Example 2: Express the exponential equation in logarithmic form.

$$4^3 = 64$$

Example 3: What two consecutive integers does $\log_2 10$ lie between?

Section 8.9 Solving for the Variable in Logarithmic and Exponential Equations

Looking Back 8.9

Now that you know how to convert exponential form to logarithmic form and vice versa, you will be able to simplify and solve exponential equations.

The equation $\log_5 125 = 3$ can be written $5^3 = 125$. Since this is a true statement, it is correct. Let us try to solve an equation like this when x is the base of the exponential equation.

Looking Ahead 8.9

Example 1:	For the following equation, solve for x. Once you have found the base, check your solution.

$$\log_x 125 = 3$$

Example 2:	For the following equation, solve for x. Check your solution.

$$\log_5 x = 3$$

*This last example can be done mentally because 125 is a perfect cube. This also can be solved using the logarithmic key on your calculator, which is just above the exponent. Demonstrate the numbers are all together on one side of the equation and the variable is on the other side.

Example 3:	For the following equation, solve for x. Once you have found the exponent, check your solution.

$$5^x = 125$$

Example 4: Simplify the logarithm and solve for *x*.

$$\log_6 6\sqrt{6} = x$$

Example 5: Simplify the logarithm and solve for *x*.

$$\log_3 \frac{1}{3} = x$$

Example 6: Simplify the logarithm and solve for *x*.

$$\log_4 \sqrt{2} = x$$

Section 8.10 Change of Base and the Logarithm of a Power

Looking Back 8.10

The logarithms in the previous section could be converted to exponents and solved mentally. However, some logarithms require the use of a calculator to solve. Common logarithms are base 10, which is the base of our number system.

Example 1:	Evaluate the following base 10 logarithms:

$$\log 10 = x$$

$$\log 100 = x$$

$$\log 1{,}000 = x$$

$$\log 10{,}000 = x$$

$$\log 100{,}000 = x$$

$$\log 10^6 = x$$

If no base is shown, it is a common logarithm, which is of base 10. These can be written as equations and solved for the unknown x.

Any positive number can be written as a power of 10: $5 = 10^{0.6989\ldots}$ which can be written as $\log_{10} 5 \approx 0.6989$.

If your calculator has a change of base key, then $5.3^x = 4$ can be converted to $\log_{5.3} 4$ and the expression can be evaluated. If there is not a change of base key, then the base of 5.3 must be converted to base10. To evaluate logarithms in other bases, you can use the change of base formula:

$$\log_a b = \frac{\log_{10} b}{\log_{10} a}$$

The numerator and denominator of the quotient are both written as a common logarithm to be evaluated in a base other than 10.

Example 2:	Evaluate $\log_2 6$ by using the change of base formula.

Looking Ahead 8.10

Logarithms give you an algebraic way to solve complex exponential equations that are not perfect squares, perfect cubes, etc. Below is an example:

$$5.3^x = 4$$

If your calculator has a change of base key, the expression "$\log_{5.3} 4$" may be evaluated directly.

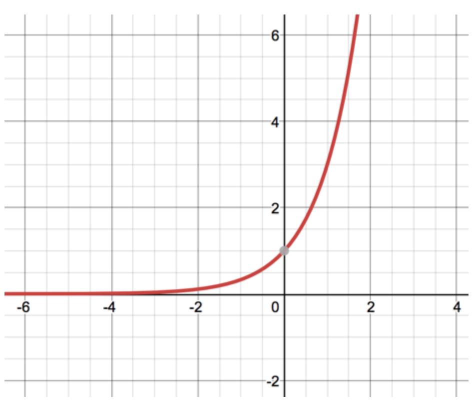
Just as there is a "power to power" property for exponents, there is a log of a power property for logarithms.

Example 3: Find $\log 2$ and $\log 16$. Numerically, show that $\log 16$ is four times $\log 2$.

Therefore, $\log_a b^x = x \log_a b$. These properties help us evaluate logarithmic expressions and solve for the unknown variable in logarithmic equations. Properties of logarithms that involve arithmetic operations will be discussed in the next section.

Example 4: Evaluate $\log 2^7$. (This could be written as $\log_{10} 2^7 = x$.)

Example 5: Evaluate $\log_3 -1$.



Section 8.11 Operations and Properties of Logarithms

Looking Back 8.11

You have already learned "The Log of a Power Property" in the previous section, which states that, "when the base is b, $\log b^x = x \log b$."

You also know that the inverse of an exponential function with base b is a logarithmic function with base b. Let us review the properties for exponents:

Product Property:

$$b^n b^m = b^{n+m}$$

Quotient Property:

$$\frac{b^n}{b^m} = b^{n-m}$$

Power to a Power Property:

$(b^n)^m = b^{nm}$ (where $b \neq 0$)

Looking Ahead 8.11

The properties of logarithms are also in the same three categories and follow similar rules.

Properties of Logarithms

Product Property:

$$\log_b(nm) = \log_b n + \log_b m$$

Quotient Property:

$$\log_b(\frac{n}{m}) = \log_b n - \log_b m$$

Power to a Power Property:

$$\log_b n^m = m\log_b n$$

(This can also be called "The Log of a Power Property" since it is difficult to detect the power of a power when written as a logarithm.)

Example 1:	Write the following equations using Properties of Logarithms and solve them using a calculator.

a) $\log_6(3 \cdot 4)$

b) $\log_6 \frac{3}{4}$

Example 2:	Use $\log_{10} 3$ and $\log_{10} 4$ to demonstrate the Properties of Logarithms.

a) $\log_{10} 12$

b) $\log_{10} 81$

Example 3:	To add logarithms with like bases, multiply the logarithms: $\log_b n + \log_b m = \log_b(mn)$.

a) $\log_2 3 + \log_2 7$

b) $\log_2 x + \log_2 7x$

Example 4:	To subtract logarithms with like bases, divide the logarithms: $\log_b n - \log_b m = \log_b\left(\frac{n}{m}\right)$.

a) $\log_2 6 - \log_2 5$

b) $\log_5 x^4 - \log_5 y^2$

Sometimes multiple properties can be used to expand a logarithmic expression.

Example 5:	Expand the logarithmic expression below using the Properties of Logarithms.

$$\log_{10} 4x^2$$

When logarithms are already written in expanded form, they can be compressed and written as one expression in standard or general form.

Example 6:	Write the expanded logarithm below as a compressed logarithm.

$$\log_{10} x - 2\log_{10} y$$

Section 8.12 Applications of Logarithms

Looking Back 8.12

The concentration of acid in a solution can be found by knowing the pH values. The pH scale goes from 0 to 14: 7 is neutral, less than 7 is acidic, and greater than 7 is basic.

Mrs. Winzler performed the following experiment with her chemistry students to examine the relationship between pH values and acidity in sodas.

1. Pour 100 mL. of soda into a 200 mL. beaker.

2. Stir the soda until most of the bubbles from the carbonation disappear.

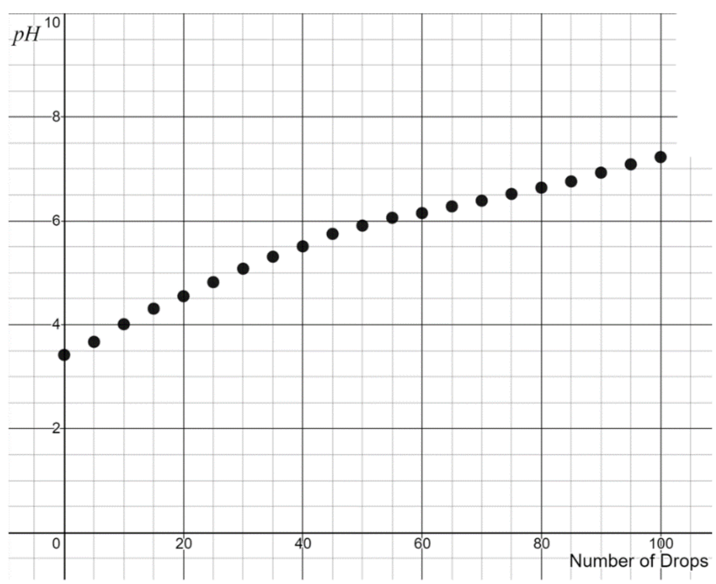
3. Using a pH probe, measure the pH in the soda. The probe should be calibrated to 4.0. Move the pH probe slowly around in the soda to get a stable reading. Record it in the chart below under pH next to 0 for the Number of Drops of solution at the start of the experiment.

4. Using a dropper, add 5 drops of 0.5 M (moles) Na_2CO_3 solution to the soda and measure the pH level. A solution can be made by mixing 2 teaspoon of baking soda with 75 mL. of water and stirring until the baking soda dissolves.

5. Add 5 drops of the solution at a time and measure the new pH level each time until you get to 100 drops. Record your results in the table below and draw the graph.

Below are the results from Vincent and Drake's experiment.

Number of Drops	pH
0	3.42
5	3.67
10	4.01
15	4.31
20	4.55
25	4.82
30	5.08
35	5.31
40	5.51
45	5.75
50	5.91
55	6.06
60	6.15
65	6.28
70	6.39
75	6.52
80	6.64
85	6.76
90	6.93
95	7.09
100	7.23



Example 1: In the table and graph above, the x-axis represents the number of drops and the y-axis represents the pH level. Use this information to answer the following questions.

a) What happened to the pH as drops of solution were added to the soda?

b) What happens to the acidity of the soda as sodium bicarbonate (baking soda) solution is added?

c) Does the pH exhibit a constant change as 5 additional drops keep getting added to the soda?

d) Is the graph linear?

e) What type of function models the graph of this experiment?

The number of hydrogen ions in a solution determine the acidity of the solution. The name for hydrogen ions is $\mathrm{H_3O^+}$, which can be simplified to the $[\mathrm{H^+}]$, the concentration of hydrogen ions. The pH value of a solution is equal to the negative log of $[\mathrm{H^+}]$.

Hydrogen ions are measured in moles per liter. A mole is a unit of measurement that contains the same number of particles for any substance: $6.02 \cdot 10^{23}$. This number is called Avogadro's Constant, named in honor of Amedeo Avogadro, an Italian chemist and physicist who lived in the 1800s. Water has $6 \cdot 10^{16}$ hydrogen ions in one liter.

Example 2: Find the number of moles in 1 L of water.

The pH value is equal to the negative log of $[H^+]$.

Example 3:	Solve for $[H^+]$ using the equation $pH = -\log[H^+]$.

Example 4:	Using the data from Vincent and Drake’s experiment, answer the following questions.

a) What is the change in pH from the beginning of the experiment to the end?

b) What factor of change is this? Divide the beginning $[H^+]$ by the ending $[H^+]$.

c) What does this indicate about the change in the acid concentration from the beginning of the experiment to the end of the experiment?

d) As the acidity of the soda decreased, what happened to the pH? What does this mean?

e) Why does the graph appear to be reaching an asymptote at a pH of 7?

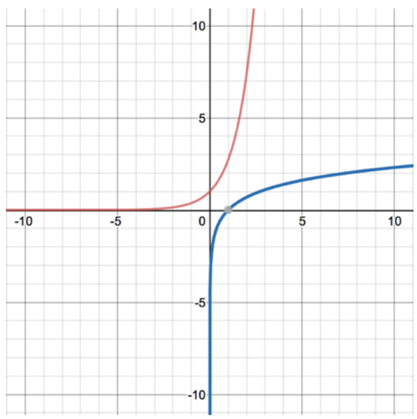
Section 8.13 The Natural Logarithm and its Inverse

Looking Back 8.13

The natural logarithm is $\ln(x)$. The Latin name for the natural logarithm is "logarithmus naturali," giving the abbreviation "ln." The natural logarithm is the time needed to grow x amount with 100% continuous growth.

The inverse of $\ln x$ is e^x, which is the amount of growth after time x, with 100% continuous growth. The base e is an irrational and transcendent number called Euler's Constant, which is approximately 2.71828. The natural logarithm $\ln(x)$ can also be written $\log_e x$, and $f(x) = e^x$ is called the natural exponential function.

If you put $1 in the bank at 100% interest and it is compounded continuously, you will have approximately $2.71 at the end of one year. Therefore, $e^1 \approx 2.71$ or $e \approx 2.71$ and $\ln 2.71 \approx 1$. Also, $e^5 \approx 148$ and $\ln 148 \approx 5$. Moreover, e^x is the amount of growth after a certain amount of time. The natural log (ln) is the amount of time needed to reach a certain amount of growth. The graph of e^x and $\ln x$ is shown below.



The graph of $f(x) = \ln x$ has an asymptote of approximately 2.72.

Looking Ahead 8.13

The number e represents continuous growth. We can combine rate and time using e^x. Five years at 100% growth is the same as one year at 500% growth when compounded continuously. Any rate and time can be converted to a rate of 100% so that only the time component must be considered.

$$e^x = e^{rate \cdot time} = e^{1.0 \cdot time} = e^{time}$$

Therefore, e^x is a scale factor that shows us how much growth we get after x units of time. When we have e^x, x is the amount of time, which gives us $f(x)$, the amount of growth. When we have $\ln x$, x is the amount of growth, which gives us $f(x)$, the amount of time.

Example 1: What is $\ln(1)$?

It is important to understand that $\ln(x) = \log_e x$. The natural logarithm of a number is its logarithm to the base e. It is sometimes written $\log x$ if the base e is implied.

Example 2: If $e^4 \approx 54.6$, what is $\ln(54.6)$?

Example 3: How long will it take to get $\frac{1}{4}$ the current amount of growth? What is $\ln(0.25)$ is equal to?

Example 4: How long does it take to grow quadruple the amount of growth? Show how to get the same result using doubling.

Let us investigate and see if the other rules of logarithms also apply to natural logarithms. Now that we have explored multiplication, let us explore division.

Example 5: $\ln(\frac{7}{5})$ means the length of time needed to grow 7 times the amount of something and then $\frac{1}{5}$ the amount of that. Show a rule of natural logarithms that applies here.

How many years would it take to get 20 times the growth at a 7% return on an investment that was compounded continuously?

We know the time for 20 times growth is represented by $\ln 20$ and that is approximately 3 so it would take 3 units of time to get 20 times the growth:

$$\ln 20 = 3$$

$$e^3 = 20$$

$$e^{100\% \cdot 3} = 20$$

$$e^x = e^{rate \cdot time}$$

How many units of time would it take to get a 7% return on this investment?

$$rate \cdot time = 3$$

$$time = \frac{3}{0.07}$$

$$time \approx 42$$

We can use the same process we used above to mentally calculate how long it takes an investment to double. This is called the Rule of 72.

$$\ln 2 = 0.693$$
$$rate \cdot time = 0.693$$

$$time = \frac{0.693}{rate}$$

$$0.693 \cdot 100 = 69.3$$

The time to double is $\frac{72}{rate}$

The number 72 is used since it is divisible by 2, 3, 4, 6, 8, and 12. It is close to 69 which is not very divisible by other integers.

Example 6:	How long does it take an investment to double at 6% growth?