Discrete Eshelby Inclusions in Amorphous Solids

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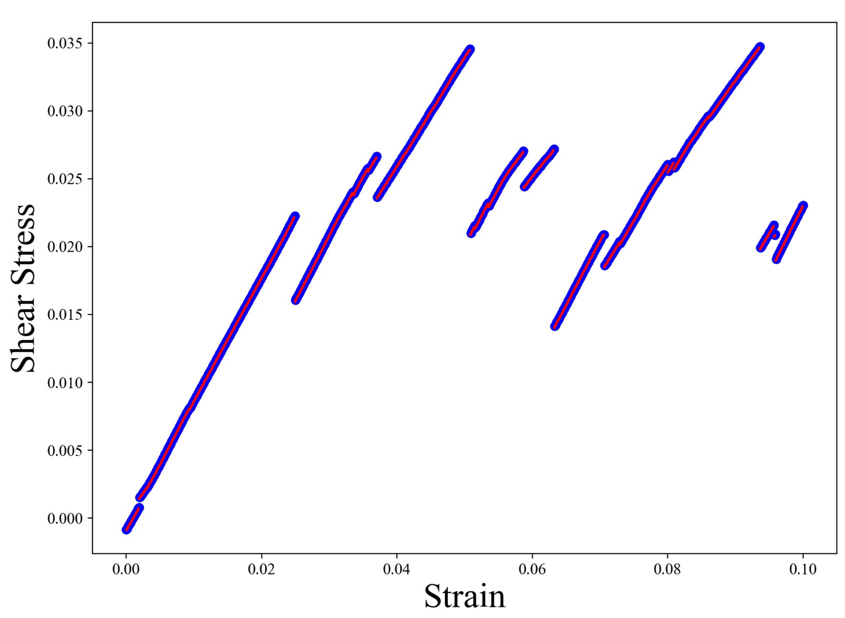
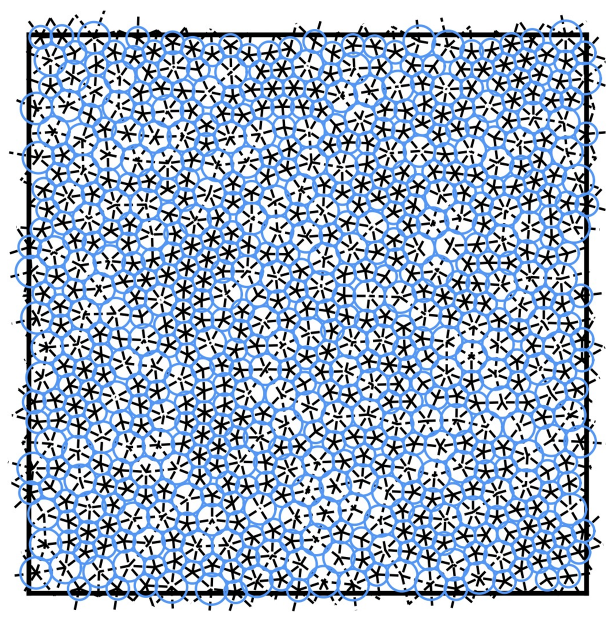
Introduction
-Amorphous Solids under Athermal-Quasistatic Shear (AQS)



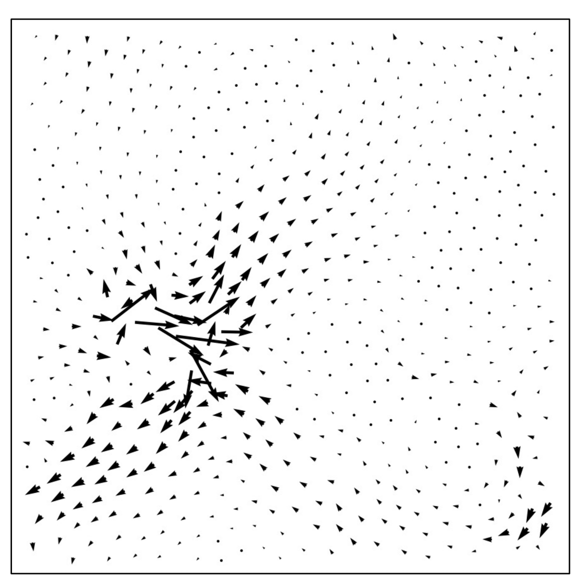


Questions

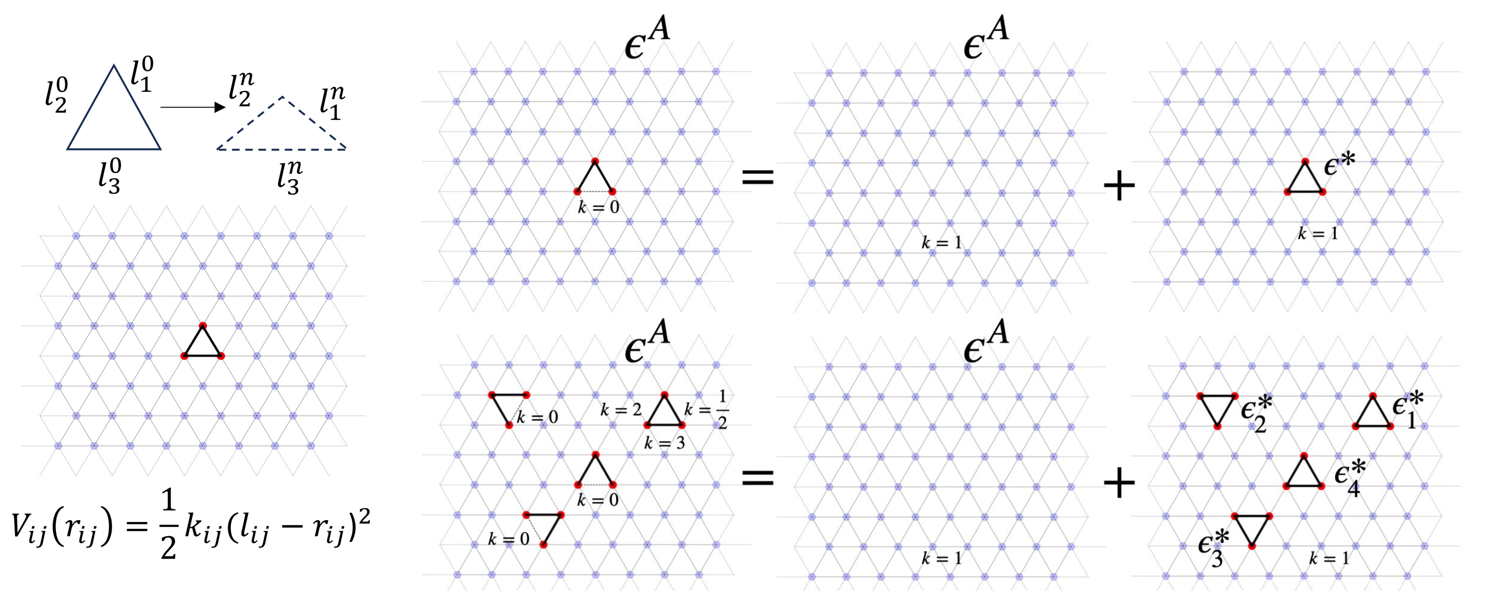
- The mechanical response of amorphous solids under AQS is characterized by smooth elastic segments separated by abrupt stress drops. The **non-affine displacement fields** are generally complex and towards the end often exhibit **quadrupolar-like** features.
- Can we explain the structures observed in these fields in terms of some underlying **defects?**

Solution Approach

- The quadrupolar-like nature of these fields suggests that we search for defects in term of **Eshelby-like inclusions.**



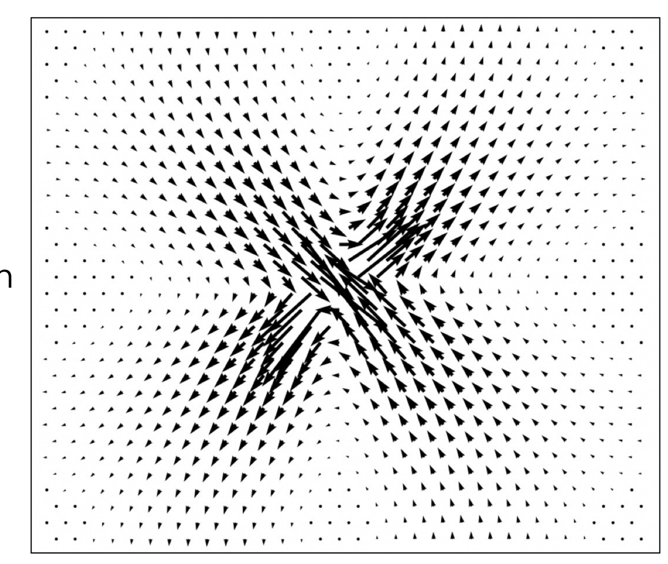
Methods
-Discrete Eshelby Formulation



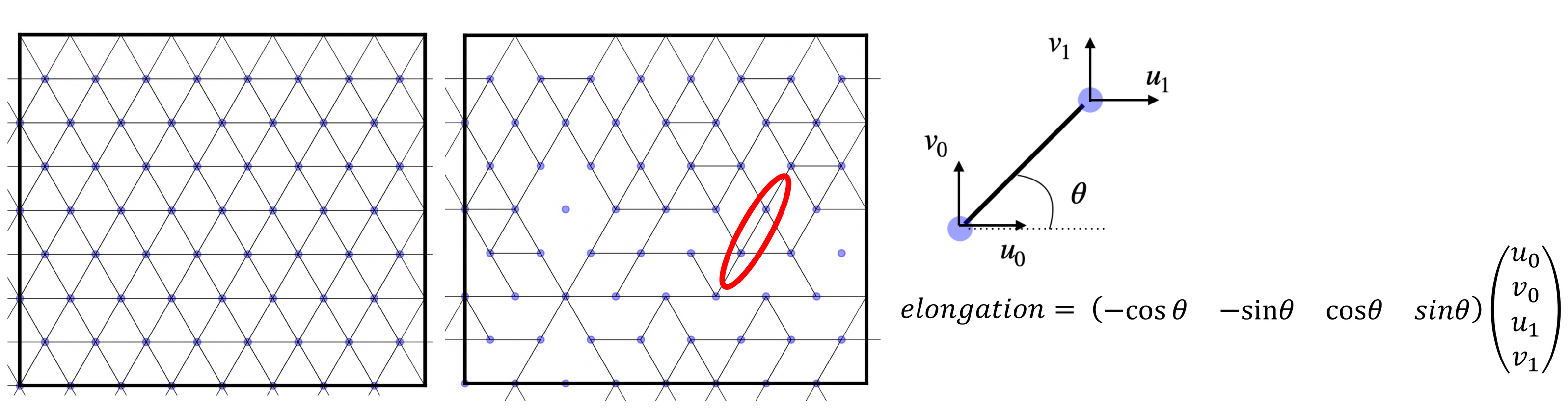
$$V_{ij}(r_{ij}) = \frac{1}{2}k_{ij}(l_{ij} - r_{ij})^2$$

Triangular Inclusions

- We start from a perfect hexagonal lattice and consider triangle units of vertices as discrete Eshelby inclusions as triangles contain the same number of **degrees of freedom** as ellipses.
- We observe that a **pure shear eigenstrain** on a single triangle unit corresponds to an ideal Eshelby quadrupole.
- We apply Eshelby's equivalence principle to decompose an **elastic mismatch under a global strain** to the sum of the global strain solution plus an eigenstrain applied to the triangular inclusion containing the elastic mismatch.



Methods
-Eshelby's Equivalence Relations for Discrete Systems



$$elongation = (-\cos\theta \quad -\sin\theta \quad \cos\theta \quad \sin\theta)\begin{pmatrix} u_0 \\ v_0 \\ u_1 \\ v_1 \end{pmatrix}$$

$$\hat{C}_0 = \begin{pmatrix} k_0 & 0 & \cdots & 0 \\ 0 & k_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k_0 \end{pmatrix} \qquad \hat{C}_0 = \begin{pmatrix} k_0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k_0 \end{pmatrix}$$

Element Stiffness Matrices

$$\hat{A} = \begin{pmatrix} -\cos\theta & -\sin\theta & \cos\theta & \sin\theta & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\cos\theta & -\sin\theta & 0 & \cdots & \cdots & 0 & \cos\theta & \sin\theta & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & -\cos\theta & -\sin\theta & 0 & 0 & \cos\theta & \sin\theta & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & -\cos\theta & -\sin\theta & \cos\theta & \sin\theta \end{pmatrix}$$

Bond Elongation Matrix

$$\vec{b} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ l_x\gamma\cos\theta \\ \vdots \\ 0 \\ \vdots \\ l_x\gamma\cos\theta \\ \vdots \\ 0 \end{pmatrix}$$

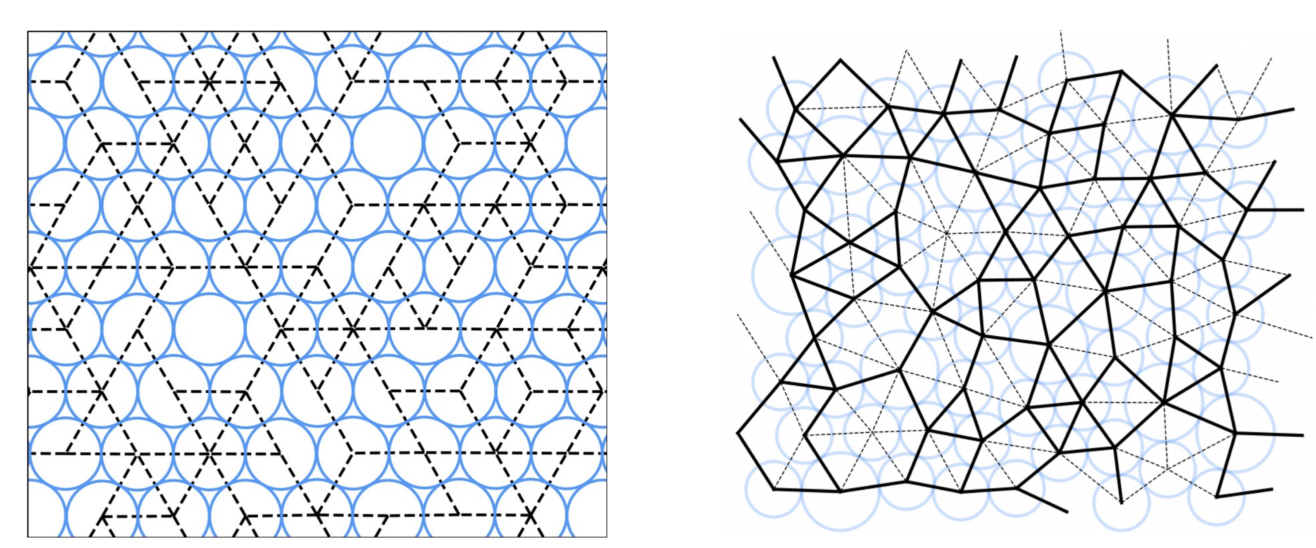
Bond Elongations due to x-extension of the boundary

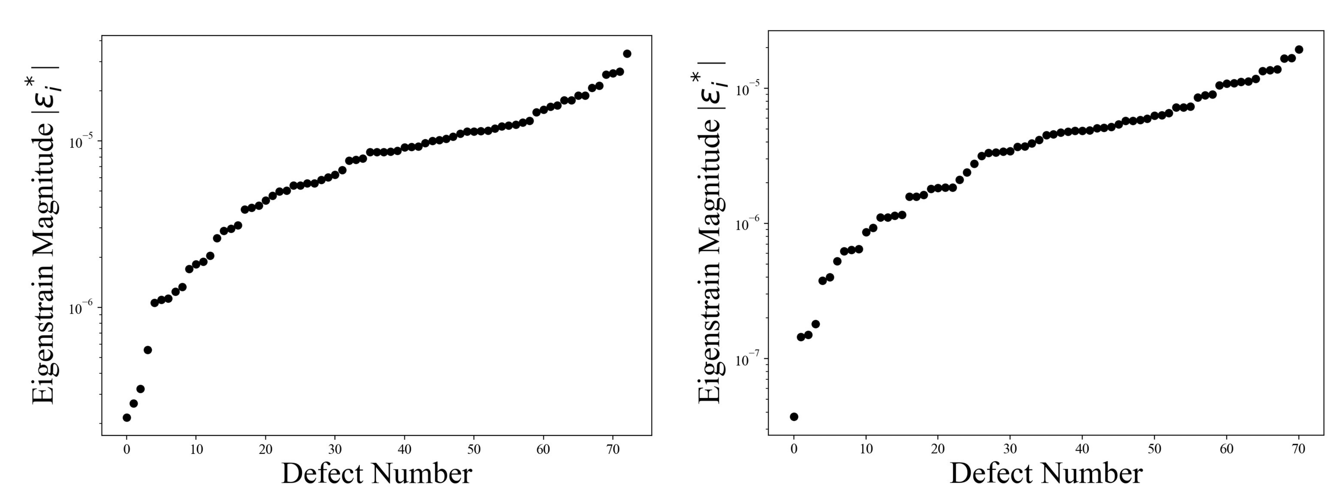
$$\vec{\epsilon}^* = \hat{C}_0^{-1}\left[(\hat{C}_0 - \hat{C})(\hat{A}\vec{u}^A + \vec{b}) + (\hat{C}_0 - \hat{C})\hat{A}(\hat{A}^T\hat{C}\hat{A})^{-1}\hat{A}^T\left(\hat{C}_0(\hat{A}\vec{u}^A + \vec{b}) - \hat{C}(\hat{A}\vec{u}^A + \vec{b})\right)\right]$$

Eigenstrains

$$\vec{u}^A = (\hat{A}^T\hat{C}_0\hat{A})^{-1}\hat{A}^T\hat{C}_0\vec{b}$$

Results
-Applications to initially stress-free isostatic systems

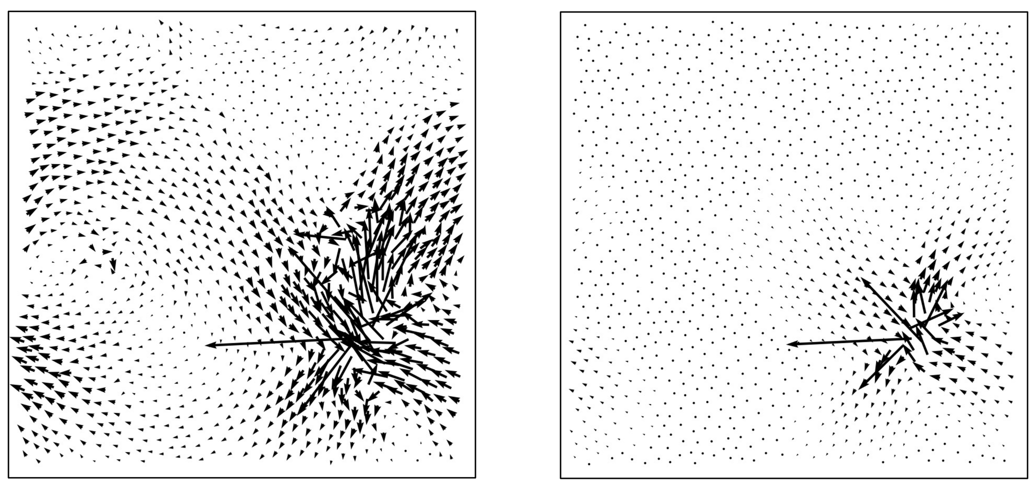




Results
-Applications to pre-stressed systems

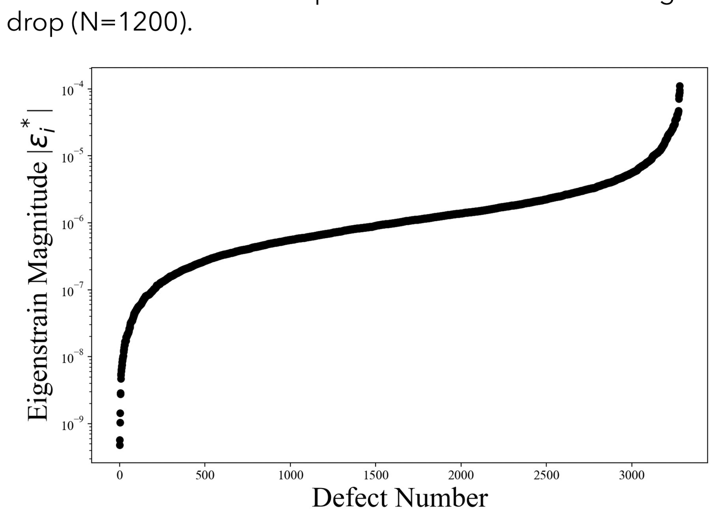
$$\vec{\epsilon}^* = \hat{C}(\hat{A}\vec{u}_{stressed} + \vec{b} + \overrightarrow{\Delta l}) - \hat{C}(\hat{A}\vec{u}_{unstressed} + \vec{b})$$

Tension difference in the bonds due to added pre-stress.



Non-affine displacement field between the two consecutive strain steps before a stress drop (N=1200).

Displacement field produced by the local eigestrains of 21 bonds.



Conclusions

- We developed the discrete analog of the Eshelby inclusion problem by considering triangle units of particles/vertices as inclusions.
- We reformulate Eshelby's equivalence principle for discrete systems to determine what local eigenstrains in a uniform system are needed to reproduce the same displacement field as that of a system with elastic mismatches under a global strain.
- We consider packings as a series of elastic mismatches and apply our reformulated equivalence principle to generate the corresponding eigenstrains.
- These local eigenstrains of triangle units compose a basis for describing the non-affine displacements fields of packings.
- As the jamming regime is approached, more defects are required to reconstruct the general features of the displacement field.

Acknowledgements

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