A Critical State $\mu(I)$-Rheology Model for Cohesive Granular Flows

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Chair of Alpine Mass Movements

1 Introduction

Understanding the mechanical behavior of granular media is crucial in various industrial processes and in geophysical events like landslides, debris flows and avalanches. The number of grains in such flows is usually too high to be modeled in a discrete manner. Hence, continuum models are often used to efficiently simulate such natural phenomena.

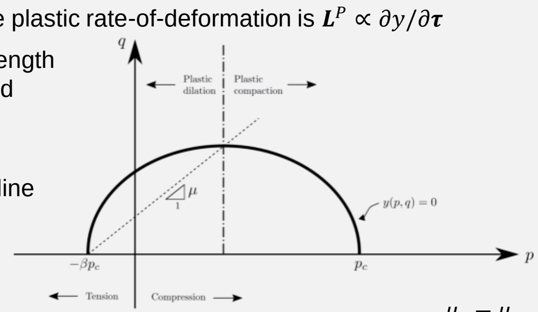
It is challenging to achieve models that can describe the diverse behavior of granular systems – the diversity stems from the varying apparent friction and solid fraction, as well as the effect of possible inter-granular cohesive forces, or even elastic forces in the slow flow regime. The $\mu(I)$-rheology has become well-established for modeling *granular liquids* in a fluid mechanics framework where the apparent friction μ depends on the inertial number I. In the geo-mechanics community, modeling the deformation of *granular solids* typically relies on concepts from Critical State Soil Mechanics (CSSM).

In this work, we develop a continuum model based on CSSM in an elastoplastic framework which recovers the $\mu(I)$-rheology under flow. This model permits a treatment of plastic compressibility in systems with or without cohesion. Implemented in a 2D/3D Material Point Method (MPM), it allows for the trivial treatment of the free surface.

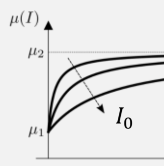
2 The CSSM $\mu(I)$-Rheology Model

In a finite-strain continuum mechanics framework, we propose an elasto-viscoplastic constitutive model for cohesive and compressible granular media. Main ingredients are

1. Elasticity: Hencky's hyperelastic model giving a linear relationship between the Kirchhoff stress $\boldsymbol{\tau}$ and the Hencky strain $\boldsymbol{\varepsilon}$
2. Yield criterion: A cohesive Modified Cam-Clay (MCC) yield surface in the space of the isotropic pressure $p = \frac{1}{3}\mathrm{tr}(\boldsymbol{\tau})$ and equivalent shear stress $q = \frac{1}{\sqrt{2}}||\boldsymbol{\tau}'||$,
$$y(p,q) = q^2 - \mu(p + \beta p_c)(p_c - p) \leq 0$$
where $p_c \geq 0$ is the compressive strength, $\beta \geq 0$ is a dimensionless measure of cohesion and $\mu \geq 0$ is the slope of the critical state line
3. Plastic flow rule: Associative, i.e., the plastic rate-of-deformation is $\boldsymbol{L}^P \propto \partial y / \partial \boldsymbol{\tau}$
4. Hardening law: The compressive strength evolves according to the accumulated plastic volumetric strain ε_V^P through
$$p_c(\varepsilon_V^P) = p_c^0 e^{-\xi \varepsilon_V^P}$$
5. Rate-dependence: The critical state line $\mu = \mu(I)$ is made dependent on a cohesive inertial number defined as
$$I = \frac{\dot{\gamma}_S\, d}{\sqrt{\bar{p}/\rho_*}}$$
where
$\dot{\gamma}_S$ is the shear strain rate
d is the diameter of the grains
ρ_* is the intrinsic density of the grains
$\bar{p} = p + \beta p_c \geq 0$ is an "effective" pressure



$$\mu(I) = \mu_1 + \frac{\mu_2 - \mu_1}{\frac{I_0}{I} + 1}$$



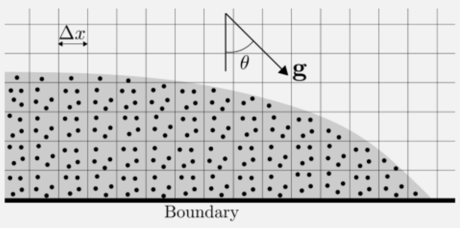
This model can be viewed as a Perzyna/overstress model with a non-constant viscosity

3 Numerical Scheme

The Material Point Method (MPM) is used to numerically approximate solutions to the (weak form of the) momentum conservation equation,
subject to the elasto-viscoplastic constitutive model outlined above
$$\rho \frac{D\boldsymbol{v}}{Dt} = \nabla \cdot \boldsymbol{\sigma} + \rho \boldsymbol{g}$$

In short, MPM consists of
- Tracking moving particles (or "MPs")
- A background Eulerian grid
- Interpolation btw particles and grid nodes

Mass conservation automatically fulfilled



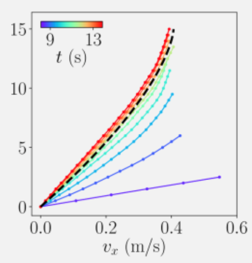
We implemented a 2D/3D parallelized C++ MPM code featuring quadratic B-spline interpolation, Affine FLuid-Implicit Particle (AFLIP) transfer scheme, using an adaptive explicit time-integration.

4 Validation: Steady State Flows

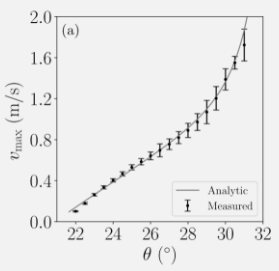
Simulations on inclined planes shows the ability of the model to capture steady-state flow on inclinations θ between $\tan^{-1}(\mu_1) < \theta < \tan^{-1}(\mu_2)$ with the expected Bagnold flow profiles, i.e.,

$$v_x(z) = v_{\max}(\theta,h) \cdot \left(1 - \left(1 - \frac{z}{h}\right)^{\frac{3}{2}}\right)$$
where h is the height of the flow and where the surface velocity $v_{\max}$ also depends on the rheological paramaters

Convergence of velocity profile over time on $\theta = 24°$

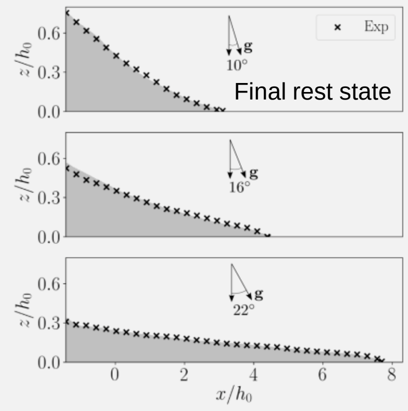


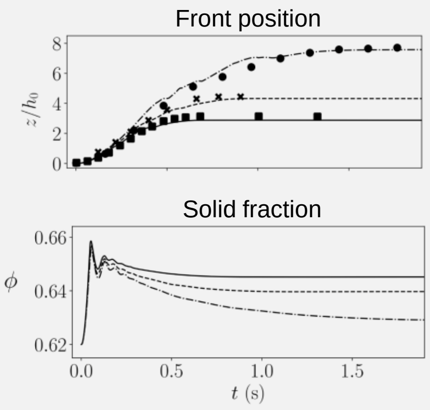
Surface velocity on various inclinations θ



5 Validation: Granular Collapse

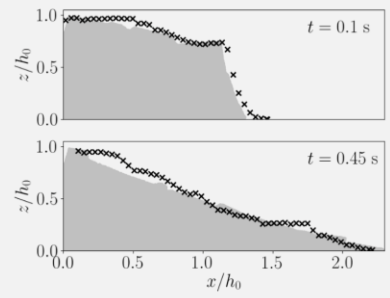
Cohesionless granular collapse: comparing to spherical glass beads experiments by Mangeney et al. (2010), *JGR*, on various inclinations $(10°, 16°, 22°)$

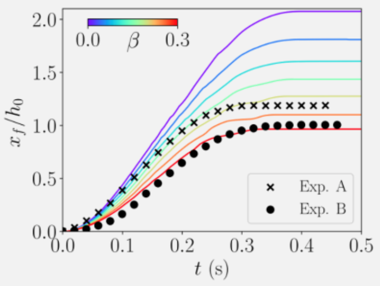


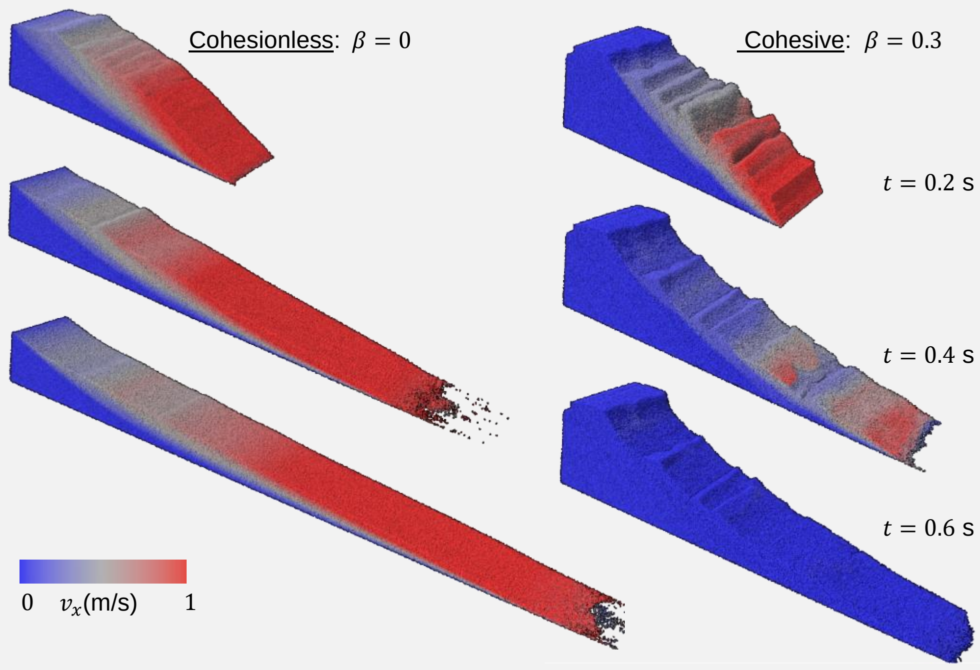


Cohesive granular collapse: comparing to two experiments by Gans et al. (2023), *JFM,* where they coated spherical glass beads with polyborosiloxane, providing a stable cohesive force

We can tune the cohesion β to approximately reproduce the experiments

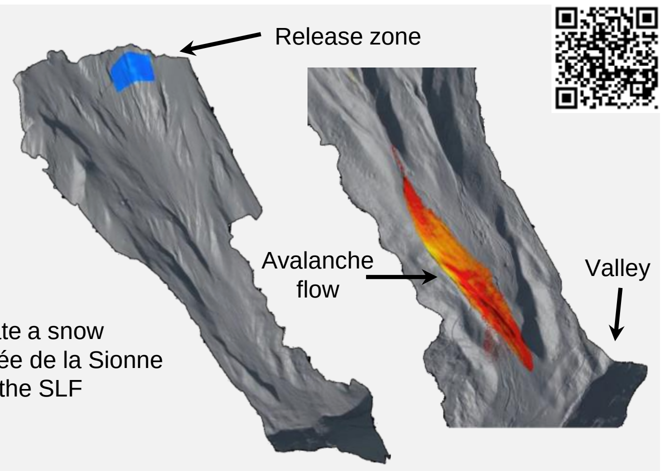






6 Snow Avalanche Simulation

- Effect of cohesion in granular flows has become an increasingly relevant topic in the snow avalanche community as climate change induces an increase in wet snow avalanches (highly cohesive and compressible) compared to dry ones
- We applied our model to simulate a snow avalanche on the terrain of Vallée de la Sionne in Valais, Switzerland, which is the SLF avalanche test site



6 Conclusions

- The flow and collapse simulations presented here provide a small glimpse at what is feasible with the presented theoretical and computational framework
- The proposed model considers varying friction, solid fraction, and takes into account cohesive and elastic forces, implemented in a computational framework able to capture the solid-to-fluid transition
- We expect the proposed approach to be used to evaluate snow avalanche impact pressures, runout distances and more generally risk management and mitigation
- Outlook: nonlocal effects implementation (Gaume et al. 2020, PRL); snow collapse experiments; $\phi(I)$ implementation through a modification of the hardening law.

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