Dense, Inclined Flow of Water and Spheres of Two Sizes over an Erodible Bed [1]

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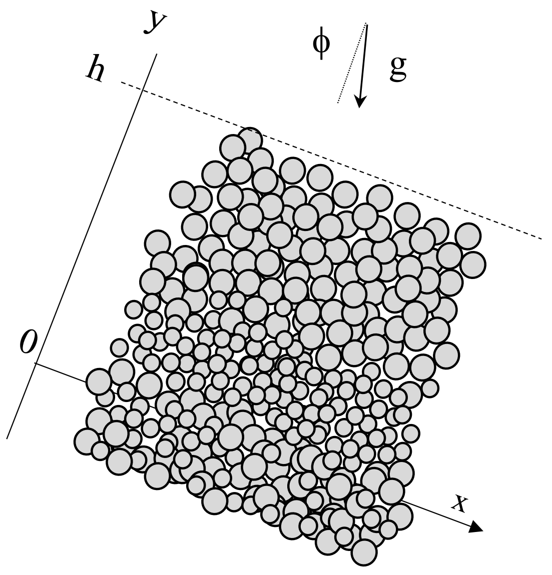
Outline a theory for dense, inclined shearing flows of a binary mixture of inelastic, frictional spheres and water: include velocity correlations and viscous collisional dissipation.

Phrase and solve a one-dimensional boundary-value problem for steady, fully-developed, inclined flow of a mixture of spheres of two sizes over an erodible bed between vertical sidewalls



Spheres of radii r_A and r_B, $d \equiv r_A + r_B$ mass density ρ_p mixture concentration c.

Fluid of mass density ρ_f, kinematic viscosity ν_f.

Density ratio $\sigma \equiv \rho_p/\rho_f$, buoyant gravity $\hat{g} \equiv (1 - 1/\sigma)g$.

Dense flow: $c > 0.49$.

Saturated: water depth equal to particle depth.

Mixture Momentum Balance

Lengths dimensionless by d, velocity by $(\hat{g}d)^{1/2}$, stresses by $\rho_p \hat{g} d$, energy flux by $\rho_p (\hat{g}d)^{3/2}$.

Pure fluid pressure: $\tilde{P}$

$$0 = -(1-c)\tilde{P}' - \frac{1-c}{\sigma-1}\cos\phi$$

Average collisional particle pressure: $p \equiv c(\tilde{p} - \tilde{P})$

$$0 = -p' - c\tilde{P}' - c\frac{\sigma}{\sigma-1}\cos\phi, \quad 0 = -p' - c\cos\phi$$

Average fluid shear stress:

$$0 = S' + \frac{1-c}{\sigma-1}\sin\phi - \frac{c}{\sigma}D(\bar{U} - u)$$

Average collisional particle shear stress:

$$0 = s' + c\frac{\sigma}{\sigma-1}\sin\phi + \frac{c}{\sigma}D(\bar{U} - u),$$

Drag: $D \equiv \left[(3/10)\left|\bar{U} - u\right| + 18/R\right]/(1-c)^{3.1}$, Reynolds number: $R = d(d\hat{g})^{1/2}/\nu_f$

Inelastic, frictional spheres with mean diameter d; particle restitution: e; particle friction: μ.

Incorporate friction in the particle restitution: $e_\mu = e - (3/2)\mu e^{-3\mu}$ [2, 3]

Incorporate viscosity in the particle restitution: [4, 5]

Reynolds number: $R = d(\hat{g}d)^{1/2}/\nu_f$; Stokes number: $St = \sigma T^{1/2} R/9$

Viscous restitution: $\varepsilon = \max\left[0, e_\mu - 6.96(1+e_\mu)/St\right]$

Idealize the flow as one-dimensional with variation normal to the flow

Sidewall friction: μ_w; Cell width: W; Resistive force: $2\mu_w p/W$ [6]

Assume that the transport coefficients become singular at a concentration c_∞ at which particles first touch: [2] $c_\infty = 0.58 + (0.636 - 0.58)e^{-4.5\mu}$

Particles

Retain only the terms in the kinetic theory that result from collisions:

$$p = 2c(1+\varepsilon)GT, \qquad G = 5.69c\frac{0.64 - 0.49}{c_\infty - c},$$

$$s = \frac{8\hat{J}}{5\pi^{1/2}}c(1+\varepsilon)GT^{1/2}u', \quad \hat{J} = \frac{1}{2} + \frac{\pi}{4}\frac{(3\varepsilon-1)(1+\varepsilon)}{24-(1-\varepsilon)(11-\varepsilon)}, \quad s = \frac{4\hat{J}}{5\pi^{1/2}}\frac{p}{T^{1/2}}u' \equiv \mu_p u'$$

Fluid

Assume that the fluid viscosity results from fluctuations that are correlated with the particle fluctuations, and incorporate added mass: [7]

$$S = \frac{1}{\sigma}\frac{1+2c}{2(1-c)}\mu_p U'$$

In the preceding, U is uniform across the cell; because the fluid does not slip at the walls,

$$\bar{U} \doteq \frac{1}{2}U \text{ and } S = 2\frac{1}{\sigma}\frac{1+2c}{2(1-c)}\mu_p \bar{U}'.$$

Particle Energy Balance

$$-q' + su' - \gamma = 0$$

$$q = -\frac{4\hat{M}}{\pi^{1/2}}(1+\varepsilon)GT^{1/2}T', \qquad \hat{M} = \frac{1}{2} + \frac{9\pi}{8}\frac{(2\varepsilon-1)(1+\varepsilon)}{16-7(1-\varepsilon)}, \qquad q = -\frac{2\hat{M}}{\pi^{1/2}}\frac{p}{T^{1/2}}T'$$

Introduce a correlation length in place of particle diameter in the rate of dissipation. [8, 9]

$$\gamma = \frac{12}{\pi^{1/2}}\left(1-\varepsilon^2\right)\frac{GT^{3/2}}{L} = \frac{6}{\pi^{1/2}}(1-\varepsilon)\frac{pT^{1/2}}{L},$$

$$L = f\frac{u'}{T^{1/2}} = f\frac{5\pi^{1/2}}{4\hat{J}}\frac{s}{p}, \quad f = \left[\frac{2\hat{J}}{15(1-\varepsilon)}\right]^{1/2}\left[1 + \frac{26}{15}(1-\varepsilon)\left(\frac{c-0.49}{0.64-c}\right)\right]^{3/2}.$$ [10]

$w = T^{1/2}$: Fluctuation velocity. $$w'' + \frac{p'}{p}w' + \frac{3}{2\hat{M}}\left[\frac{5\pi}{24\hat{J}}\left(\frac{s}{p}\right)^2 - \frac{1-\varepsilon}{L}\right]w = 0$$

In the bed, $q' + \gamma = 0$ or $w'' - \lambda^2 w = 0$, with $\lambda^2 = \frac{3}{2\hat{M}}\frac{1-\varepsilon}{L_0}$

Boundary Conditions: Free surface

Assume that the pressure at the top of the collisional flow balances the weight of a single layer of particles above it.

$$\text{At } y = h: \; p = c\cos\phi, \; s = 0, \; q \propto w' = 0, \quad S = 0.$$

Boundary Conditions: Erodible bed

Assume that the surface of the bed is at the concentration at which the particles first touch.

$$\text{At } y = 0: \; c = c_\infty, \; u = 0, \; w' = \left(\frac{3}{2\hat{M}}\frac{1-\varepsilon}{L}\right)^{1/2}w, \quad U = 0.$$

Segregation

In general, with

$\delta r \equiv (r_A - r_B)/r_B$, $\delta m \equiv (m_A - m_B)/(m_A + m_B)$, and $2X \equiv (n_A - n_B)/(n_A + n_B)$,

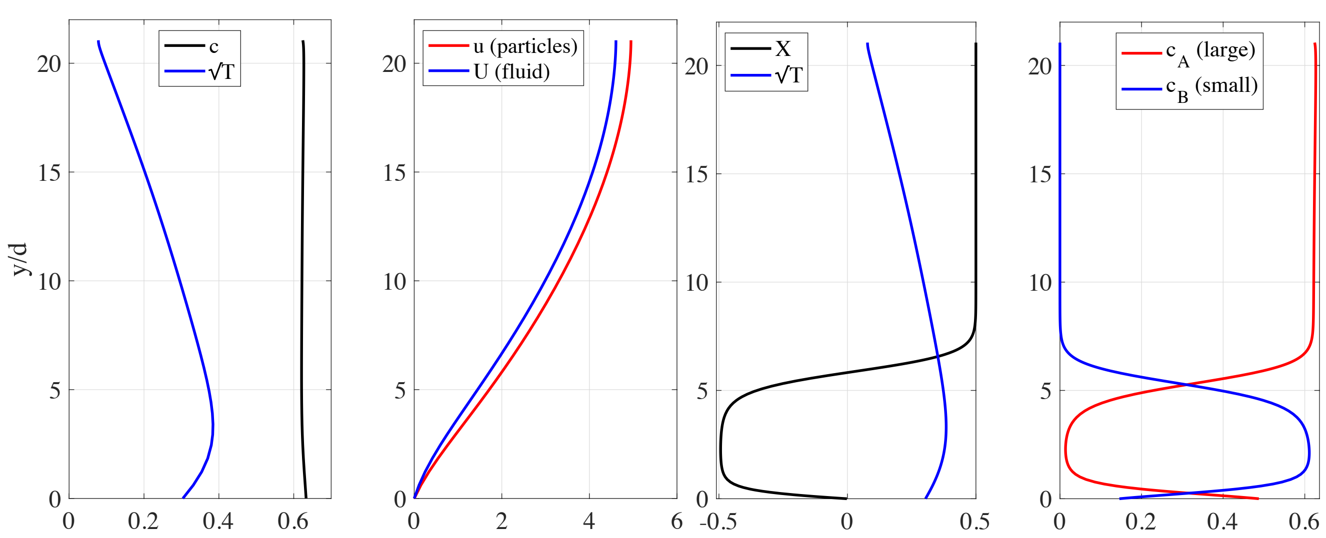
$$v_A - v_B = -\frac{\pi}{6}\frac{r_A + r_B}{G}\left(\frac{2T}{m_A + m_B}\right)^{1/2}$$
$$\times\left[\left(\Gamma_1\delta m + R_1\delta r\right)\frac{\nabla T}{T} - \left(\Gamma_2\delta m + R_2\delta r\right)\frac{(m_A + m_B)\mathbf{g}\cos\phi}{2T} + \frac{\nabla X}{0.25 - X^2}\right]$$

Steady, uniform, dense, with $G \equiv 5.69c(c_\infty - 0.49)/(c_\infty - c)$,

$$\frac{dX}{dy} = -\left(\frac{1-4X^2}{2}\right)\left[(6.21\delta m - 4.35\delta r)G\frac{1}{w}\frac{dw}{dy} - (2\delta m - 3\delta r)\frac{g\cos\phi}{2w^2}\right].$$

Results

Equal numbers of large and small spheres, with $r_A/r_B = 3/2$.



For spheres made of the same material, $2\delta m - 3\delta r = 0$, and gravity influences segregation only through its influence on the distribution of the granular temperature. Segregation is driven by the temperature, and the large spheres concentrate near the base.

[1] Jenkins & Larcher, Water 15, 2629 (2023); [2] Chialvo, et al., Phys. Rev. E 85 (2012);
[3] Larcher & Jenkins, Phys. Fluids 25 (2013); [4] Barnocky & Davis, Phys. Fluids 31 (1988);
[5] Berzi & Fraccarollo, Phys. Fluids 25 (2013); [6] Taberlet, et al., Phys. Rev. Lett. 91 (2003);
[7] Berzi & Fraccarollo, Phys. Rev. Letts. 115, 2015; [8] Jenkins, Phys. Fluids 18 (2006);
[9] Jenkins & Berzi, Gran. Matt. 12 (2010); [10] Berzi & Vescovi, Phys. Fluids 27 (2015).