The Danger of Using Math "Tricks"

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Outline

Table of Contents

Basics 1: General Number Properties

1 is not a prime number!

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If it was, then the Fundamental Theorem of Algebra, which states that every integer can be uniquely written as a product of primes, would be violated.

Basics 1: General Number Properties

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These define evens and odds *constructively*, instead of defining evens by division and odds by what they *aren't* ("an odd number isn't divisible by 2").

Basics 2: Definitions

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They are not (necessarily) *lines that the function can't pass through*. Many functions pass through horizontal asymptotes.

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Factoring is rewriting an expression as a product. For example,

$$12 = 3(4)$$
$$2x - 4 = 2(x - 2)$$
$$x^2 - x - 6 = (x - 3)(x + 2)$$
$$3x^2 - 3x - 9 = 3(x^2 - x - 3)$$

are all examples of factoring.

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are all examples of factoring.

When we teach factoring like this, it helps students remember that not *all* factoring is simply rewriting a quadratic trinomial as the product of two linear binomials.

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This will become important when students transform graphs (dilate by $|a|$), compute vector magnitudes, and express absolute value functions as piecewise functions.

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- Parentheses that just represent grouping (not multiplication)
- Expressions with variables that aren't x
- Functions being represented by their name ($f(x), s(t)$, etc.) and not $y =$

Table of Contents

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Example	What Students Do
$6\left(\frac{12}{3}\right)$	Calculator $\rightarrow \frac{72}{3} = 24$
$3x^2\left(\frac{14}{15}\right)\left(\frac{10}{6x}\right)$	Calculator $\rightarrow \frac{420x^2}{90x}$

The Alternative

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$$3x^2\left(\frac{14}{15}\right)\left(\frac{10}{6x}\right) = x^2\left(\frac{14}{5}\right)\left(\frac{5}{3x}\right) = \frac{14}{3}x$$

Arithmetic 2: “Keep-Change-Flip”

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$\frac{\frac{3}{4}}{\frac{5}{8}}$	What do I do again?

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Example	What Students Do
$\frac{\frac{3}{4}}{\frac{5}{8}}$	What do I do again?
$\frac{\frac{2}{3}}{7}$	How do I do this? OR $\frac{14}{3}$

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Division by x is multiplication by the reciprocal of x.

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Therefore,

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$$\frac{\frac{2}{3}}{7} = \frac{2}{3} \cdot \frac{1}{7} = \frac{2}{21}$$

Arithmetic 3: The "Butterfly" Method

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Example	What Students Do
$\frac{9}{16} - \frac{5}{24}$	$\frac{9(24)-16(5)}{16(24)} \to \text{Calculator} \to \frac{136}{384}$

The Alternative

Find the least common multiple of the denominators (ideally through factoring).

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Therefore,

$$\frac{9}{16} - \frac{5}{24} = \frac{9}{2(8)} - \frac{5}{3(8)} = \frac{9}{2(8)}\left(\frac{3}{3}\right) - \frac{5}{3(8)}\left(\frac{2}{2}\right)$$
$$= \frac{27-10}{48}$$
$$= \frac{17}{48}$$

Table of Contents

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$\frac{3}{3x}$	The 3's cancel $\rightarrow \frac{3}{3x} = x$
$\frac{3x+6}{3}$	The 3's cancel $\rightarrow x + 6$
$\frac{1}{x+2} + \frac{3}{2x+1}$	$\frac{(2x+1)+3(x+2)}{(x+2)(2x+1)} = 1 + 3 = 4$ (or just 3)

The Alternative

"Cancelling" is hard to eradicate, but at least emphasize the following

- We are typically dividing to make 1.
- Division must distribute over addition, but does not have to over multiplication, i.e.

$$\frac{3x^2}{3y} = \frac{x^2}{y} \text{ is legal}$$

$$\frac{3x^2 + y}{3} \text{ requires } \frac{3x^2}{3} + \frac{y}{3} = x^2 + \frac{y}{3}$$

Algebra 2: "Anything you do to the top, do to the bottom"

The rule/trick in question: doing "something" to the top and bottom always maintains equality

Example	What Students Do
$\frac{3}{x+1} + \frac{5}{(x+1)^2}$	

Algebra 2: "Anything you do to the top, do to the bottom"

The rule/trick in question: doing "something" to the top and bottom always maintains equality

Example	What Students Do
$\frac{3}{x+1} + \frac{5}{(x+1)^2}$	$\frac{3^{2}}{(x+1)^{2}} + \frac{5}{(x+1)^2} = \frac{14}{(x+1)^2}$

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$\frac{3}{x+1} + \frac{5}{(x+1)^2}$	$\frac{3^2}{(x+1)^2} + \frac{5}{(x+1)^2} = \frac{14}{(x+1)^2}$
$\frac{9}{x} + \frac{1}{\sqrt{x}}$	

Algebra 2: "Anything you do to the top, do to the bottom"

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Example	What Students Do
$\frac{3}{x+1} + \frac{5}{(x+1)^2}$	$\frac{3^{2}}{(x+1)^{2}} + \frac{5}{(x+1)^2} = \frac{14}{(x+1)^2}$
$\frac{9}{x} + \frac{1}{\sqrt{x}}$	$\frac{\sqrt{9}}{\sqrt{x}} + \frac{1}{\sqrt{x}} = \frac{4}{\sqrt{x}}$

The Alternative

Be precise with students: the only operations that, when simultaneously performed in the numerator and denominator of a fraction, preserve value are multiplication and division.

Algebra 3: Factoring - multiply to c, add to b

The rule/trick in question: factoring $x^2 + bx + c$ into $(x+p)(x+q)$ where p, q multiply to c and add to b

Example	What Students Do
$3x^2 + 19x - 14$	

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Example	What Students Do
$3x^2 + 19x - 14$	"Nothing multiplies to make -14 and adds to 19."

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$-3t^3 + 24t$	

Algebra 3: Factoring - multiply to c, add to b

The rule/trick in question: factoring $x^2 + bx + c$ into $(x + p)(x + q)$ where p, q multiply to c and add to b

Example	What Students Do
$3x^2 + 19x - 14$	"Nothing multiplies to make -14 and adds to 19."
$-3t^3 + 24t$	$(-3t + 8)(t^2 + 3t)$ or similar attempts

The Alternative

Have students become fluent with all kinds of polynomial multiplication, including with varied letters and exponents.

Always intermix factoring problems: GCF (positive, negative, include variables), monic quadratic, non-monic quadratic, in non-standard form

The Alternative

My own book's first 16 factoring problems:

Factor each as much as possible, if possible.

1. $y^2 - 3y - 54$
2. $5y - xy$
3. $x^2 - 49$
4. $x^3 - 1$
5. $6x^2 - 13x + 5$
6. $x^3 + x^2 - 6x$
7. $11x^2 - 11x + 22$
8. $32x - 8$
9. $x^2 + 7x - 10$
10. $\frac{1}{4}x^2 + \frac{3}{8}x + \frac{1}{8}$
11. $6 - 24x^2$
12. $8x^6 + 13x^3 - 6$
13. $9x^4 - 16$
14. $3x^2 + 5x - 4$
15. $x^4 - 1$
16. $27 - y^3$

Algebra 4: Distributing

The rule/trick in question: “don’t forget to distribute,” “you always have to distribute”

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$(x+3)^2$	

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$\sqrt{x^2+9}$	$x+3$

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$(x+3)^2$	x^2+6x+9
$\sqrt{x^2+9}$	$x+3$
$\left(2x^2\right)^2$	

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Example	What Students Do
$(x+3)^2$	x^2+6x+9
$\sqrt{x^2+9}$	$x+3$
$\left(2x^2\right)^2$	$2x^4$

The Alternative

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Multiplication <u>does</u> distribute over addition and subtraction, but does <u>not</u> over multiplication.

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Emphasize distribution of operations over other operations. For instance:

Multiplication does distribute over addition and subtraction, but does not over multiplication.

Exponentiation and radicals do distribute over multiplication and division, but do not over addition or subtraction.

Table of Contents

Thank You!

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