Building Vocabulary in AP Precalculus & Advanced Algebra

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Georgia Association of Advanced Placement Math Teachers

Outline

Rather than saying "x and y" or even "x and $f(x)$"... say **inputs** and **outputs.**

Motivation for Changing Vocabulary

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- Inputs and outputs give us a common language with which to discuss every single function in the course (and any course!). This especially helps with choosing function models; inverse, trigonometric, and polar functions; and as log and exponent rules. It's the gift that keeps on giving.

Other Vocabulary Changes & Enhancements

Besides using "inputs and outputs", there is some other language you may want to consider changing or enhancing.

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"$y =$"	$f(x) =$, $g(x) =$, etc.	Functions!!

Common New Expressions

- Over consecutive/successive equal-length intervals of inputs/outputs...
- Increasing/decreasing at an/a increasing/decreasing rate
- Intersection of terminal ray of angle in standard position and circle centered at origin

Lessons and Applications of Vocabulary

Let's look at a few ways we can utilize this vocabulary to our advantage in the classroom and help our students build their own mathematical communication skills.

Rates of Change Vocabulary Lesson

Function // $f(x)$		**Rate of Change**		**Rate of Rate of Change**		**Concavity**
Positive		Any		Any		Any
Negative		Any		Any		Any
Increasing	⟷	Positive		Any		Any
Decreasing		Negative		Any		Any
Any		Increasing	⟷	Positive	⟷	Up
Any		Decreasing		Negative		Down

The Biggest Change
Motivation
Other Vocabulary Changes & Enhancements
Lessons & Applications
Rates of Change
Inverse Functions
Log Rules
Closing Thoughts
Raffle

EXAMPLE 1: The function f is increasing with a graph that is concave down for all x. Outputs of f for select values of x are in the table.

x	1	2	3	4	5	6	Behavior	Behavior
$f(x)$	1	15	26	32	34	35	of f:	N/A
1st diff. (changes)	N/A						of ROC:	of ROC:
2nd diff. (changes in changes)	N/A	N/A					N/A	of ROROC:

(a) Fill in "increasing" in the first "Behavior" column in the $f(x)$ row.

(b) Compute the 1st differences. Then, in the first "Behavior" column, write either "Positive" or "Negative" in the "1st diff." row.

(c) Look at how the first differences are changing. Write in either "increasing" or "decreasing" in the 2nd "Behavior" column in the "1st diff." row.

(d) Compute the 2nd differences. Then, in the second "Behavior" column, write either "Positive" or "Negative" in the "2nd diff." row.

Rates of Change Vocabulary Lesson

EXAMPLE 2: The function g is decreasing and its graph is concave up for all x. Outputs of g for select values of x are in the table.

x	1	2	3	4	5	6	Behavior	Behavior
$f(x)$							of f:	N/A
1st diff. (changes)	N/A						of ROC:	of ROC:
2nd diff. (changes in changes)	N/A	N/A					N/A	of ROROC:

Complete the table - you get to choose the outputs values so long as they agree with the description of g!

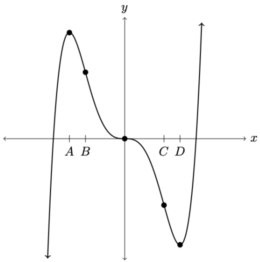
Rates of Change Vocabulary Lesson

NOTES

- We never talk about the rate of change increasing or decreasing *at an increasing or decreasing rate*.
- It matters whether you say "the function" or "the rate of change." These are entirely different things!

PRACTICE CONTINUED

1. Consider the graph of f below. The graph of f has extrema at $x = A$ and $x = D$ and inflection points at $x = B$ and $x = C$.



(a) Use the table to fill in the following.

i. For $x > C$, the graph of f is concave up, so the rate of change of f is ______________.

ii. On the interval $(B, 0)$, the graph is above the x-axis, so ______________ is positive. However, as x increases on this interval, the output values decrease, so ______________________ is negative. The graph of f is concave up on this interval, though, so ______________________ is positive.

iii. The function changes from positive to negative at the x-value/s ______________.

iv. The rate of change of f changes from positive to negative at the x-value/s ______________.

v. The rate of rate of change of f changes from positive to negative at the x-value/s ______________.

vi. f goes from increasing to decreasing at the x-value/s ______________.

vii. The rate of change of f goes from decreasing to increasing at the x-value/s ______________.

(b) Describe how the rate of change is changing on the interval $0 \leq x \leq C$.

Rates of Change Vocabulary Lesson

2. Correct each of the following incorrect statements about a generic function g by changing the underlined portion.

    (a) When g is increasing, the <u>function</u> is positive.

    (b) When the graph of g is concave down, the rate of change of g is <u>decreasing at a decreasing rate</u>.

    (c) If g is increasing, then the <u>rate of rate of change</u> is positive.

    (d) When the <u>rate of change</u> changes from positive to negative, the graph of g has an inflection point.

    (e) A local maximum occurs when the <u>rate of change</u> changes from increasing to decreasing.

    (f) A local maximum occurs when the rate of change changes from <u>increasing to decreasing</u>.

    (g) A local minimum occurs when the <u>function</u> changes from negative to positive.

    (h) A local minimum occurs when the function changes from <u>negative to positive</u>.

    (i) If the rate of change of a function is <u>negative and increasing</u>, then the graph of g is concave down.

    (j) If the rate of change of a function is negative and increasing, then the graph of g is <u>concave down</u>.

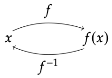
Inverse Functions Activity

In Topic 2.7, you investigated function composition. One function that came up was the *identity function* $i(x) = x$. This leads to a definition.

Definition. The <u>INVERSE</u> of a function $f(x)$ is the function $f^{-1}(x)$ that

- Returns an output $f(x)$ back to x, and
- satisfies $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

This can be visualized as follows.



When a function has an inverse (on a certain domain), it is called *invertible*.

Try using the definition of an inverse function to answer the following.

1. Given an invertible function $f(x)$ with $f(2) = 3$, what is $f^{-1}(3)$?
2. The graph of the invertible function g contains the point $(4, -7)$. What point must the graph of $g^{-1}(x)$ contain?
3. Because an inverse function $f^{-1}(x)$ is defined strictly by its "undoing" of outputs of $f(x)$, we have a number of relationships that arise. Try completing the following.

Relationship 1	The inputs of $f(x)$ are the _________ of $f^{-1}(x)$, and the outputs of $f(x)$ are the _________ of $f^{-1}(x)$.
Relationship 2	The domain of $f(x)$ is the _________ of $f^{-1}(x)$, and the range of $f(x)$ is the _________ of $f^{-1}(x)$.*
Relationship 3	If a point (x, y) is on the graph of $f(x)$, then the point _______ is on the graph of $f^{-1}(x)$.

Inverse Functions Activity

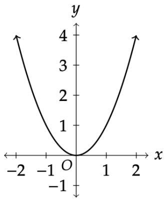
In Relationship 2, there was an asterisk. Let's investigate.

4. Consider $f(x) = x^2$.
   (a) What is $f(4)$? How about $f(-4)$?
   (b) Explain why you can't determine $f^{-1}(16)$.

5. Try to generalize the result of the previous question below.

   If a function f has ________________ with the ______________,
   then the inverse function isn't ______________.

Problem 5 can be visualized graphically. Consider $f(x) = x^2$ below.



6. Visually, how can you see that multiple inputs have the same output?

7. This "problem" can be fixed with a *domain restriction*. What is the largest domain x that you could restrict $f(x)$ to such that *no* two inputs would have the exact same output?

So, in general, functions may only be invertible on a specific domain: this is often called the *invertible domain*. In this case, we have to modify Relationship 2 to state that

The INVERTIBLE DOMAIN of $f(x)$ is the range of $f^{-1}(x)$,
and the range of $f(x)$ *on its invertible domain* is the DOMAIN of $f^{-1}(x)$.

Inverse Functions Activity

Now, you've looked at finding input-output pairs of an inverse function and even looked at when an inverse function exists (or doesn't). How do you actually *find* an inverse function, though? It comes down to Relationship 1. If $y = f(x)$, then the inputs of f are x and the outputs are y; for an inverse function, these inputs and outputs simply swap places, meaning x becomes y and y becomes x (analytically, this appears as $f^{-1}(y) = x$ – see how x and y have "swapped places"?). Let's try an example.

Use the box at the right to answer the following.

8. Let $f(x) = \dfrac{3x-1}{5}$. First, replace $f(x)$ with y.
9. Now, swap x and y – i.e., reverse the roles of the inputs and outputs.
10. Now, solve for y. This resulting y will actually be $f^{-1}(x)$.

Finding the inverse of $f(x) = \dfrac{3x-1}{5}$

This process will work in general. Now, what about the *graph* of $f^{-1}(x)$?

12. We have stated that, for an inverse function, the inputs and outputs are reversed. If you were to graph $f(x)$ in the coordinate plane with x and y axes, how do you think you could graph $f^{-1}(x)$?

13. You have a separate piece of patty paper with the graph of $f(x) = \dfrac{3x-1}{5}$. To swap the x- and y-axes, you can actually *fold* the x-axis onto the y-axis.

14. Take the patty paper and fold the x-axis onto the y-axis. Press down to make a nice crease. Then, thickly trace over the graph of $f(x)$.

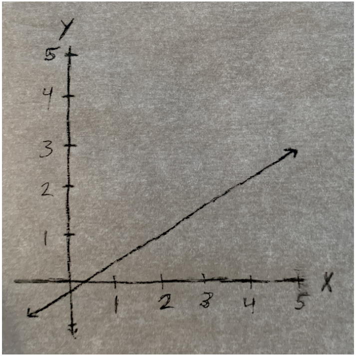
15. Unfold the paper. You should now see a new line graphed (where you drew) – this line is the graph of $f^{-1}(x)$!

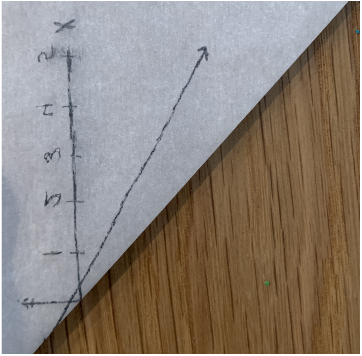
16. You should also see a line where your crease was made. What is the equation of this line?

17. Complete the following:

For an invertible function f, the graph of $f^{-1}(x)$ is simply
the graph of f ____.

Inverse Functions Activity





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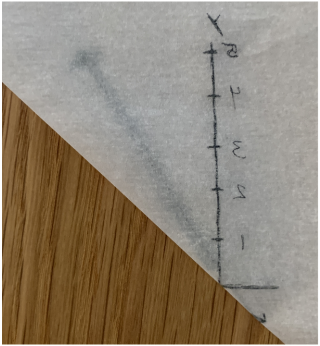

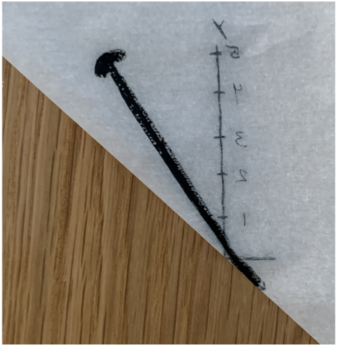

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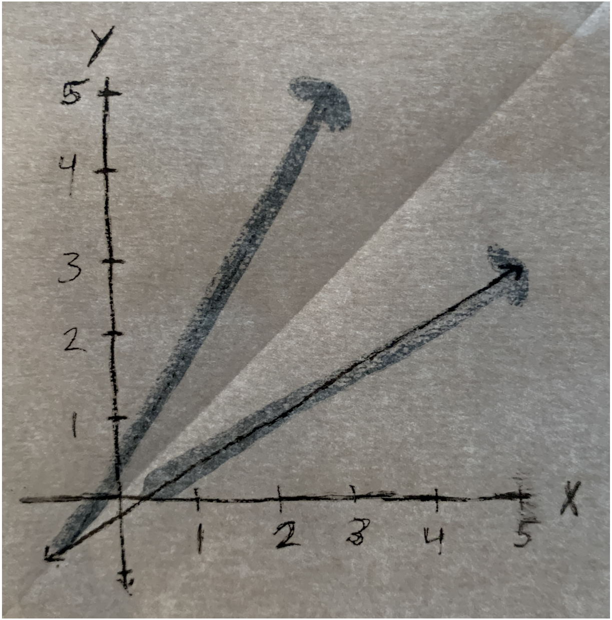

Log Rules

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$$\log_b(xy) = \log_b x + \log_b y$$

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Let $f(x) = b^x$. Then $(f(x))^n = (b^x)^n = b^{xn} = f(xn)$. Therefore, exponentiation of an output equals a dilation of an input.

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$$\log_b(x^n) = n \cdot \log_b x$$

Closing Thoughts

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- Precision in vocabulary is important: the more you model it for your students, the more they'll pick up on it.
- "Don't knock it 'til you try it."
- AP Precalculus spirals beautifully and effectively - if you start with precise vocabulary early, it will pay dividends in the long run.

Raffle

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