

## Absolute Value Functions

Recall that the ABSOLUTE VALUE of a number  $x$ , written  $|x|$ , is the *magnitude* of  $x$ , or its distance from 0. Therefore, for all  $x$ ,

$$|x| = |-x|$$

This is a critical fact: the absolute value function is symmetric about  $x = 0$ , and can be written as the piecewise function

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

More generally, every absolute value function has one or more “breaking points,” or input values in which the function changes signs. In many absolute value problems, the strategy is

Step 1. Find the inputs for the arguments of the absolute value functions equal 0. These are the *breaking points*  $x_1 < x_2 < \dots < x_n$ .

Step 2. Do casework: solve the inequality/equation on the sub-intervals

$$(-\infty, x_1), (x_1, x_2), \dots, (x_n, \infty)$$

Let's try some problems.

1. Find the sum of the solutions to  $|x - 2026| = 4599$ .
2. How many integers  $n$  satisfy  $|3n - 7| < 10$ ?
3. How many real solutions does the equation  $|2x - 5| = x + 3$  have?
4. Find the number of real values of  $x$  that satisfy  $||x - 1| - 2| = 3$ .
5. Let  $S = |x - 2| + |x + 1|$ . What is the minimum possible value of  $S$ ?
6. (AMC 10B 2020 #5) What is the sum of all real numbers  $x$  for which  $|x^2 - 12x + 34| = 2$ ?
7. (AMC 10A 2018 #12) How many ordered pairs of real numbers  $(x, y)$  satisfy the following system of equations?

$$\begin{aligned} x + 3y &= 3 \\ ||x| - |y|| &= 1 \end{aligned}$$

8. (AMC 12A 2022 #5) The taxicab distance between points  $(x_1, y_1)$  and  $(x_2, y_2)$  in the coordinate plane is given by

$$|x_1 - x_2| + |y_1 - y_2|.$$

For how many points  $P$  with integer coordinates is the taxicab distance between  $P$  and the origin less than or equal to 20?

9. (AMC 10B 2024 #14) A dartboard is the region  $B$  in the coordinate plane consisting of points  $(x, y)$  such that  $|x| + |y| \leq 8$ . A target  $T$  is the region where  $(x^2 + y^2 - 25)^2 \leq 49$ . A dart is thrown and lands at a random point in  $B$ . The probability that the dart lands in  $T$  can be expressed as  $\frac{m}{n} \cdot \pi$ , where  $m$  and  $n$  are relatively prime positive integers. What is  $m + n$ ?
10. (AMC 10A 2021 Spring #19) The area of the region bounded by the graph of

$$x^2 + y^2 = 3|x - y| + 3|x + y|$$

is  $m + n\pi$ , where  $m$  and  $n$  are integers. What is  $m + n$ ?

11. (AMC 10A 2021 Fall #14) How many ordered pairs  $(x, y)$  of real numbers satisfy the following system of equations?

$$\begin{aligned}x^2 + 3y &= 9 \\ (|x| + |y| - 4)^2 &= 1\end{aligned}$$

12. (AMC 12B 2016 #18) What is the area of the region enclosed by the graph of the equation  $x^2 + y^2 = |x| + |y|$ ?