AM, GM, Sequences, & Series Introduction

An <u>ARITHMETIC SEQUENCE</u> is one in which each consecutive terms share a common difference, i.e.

$$a, a+d, a+2d, a+3d, a+4d, a+5d, a+6d, \ldots \qquad (\bigstar)$$

The terms in an arithmetic sequence satisfy an interesting property.

1. Pick any term in the arithmetic sequence ($\bigstar$). How is it related to the term immediately before it and the term immediately after it?

It turns out, this has a name: for any two numbers x and y, the quantity $\frac{x+y}{2}$ is the _____.

$$\text{AM} = \frac{x+y}{2}$$

More generally, for n numbers $a_1, \ldots, a_n$, the arithmetic mean is

$$\text{AM} = \frac{a_1 + a_2 + \cdots + a_n}{n}$$

2. (Competition Example) Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. If you randomly select 3 unique numbers from S, what is the probability the three numbers form an arithmetic progression?

A <u>GEOMETRIC SEQUENCE</u> is one in which each term is a constant multiple of the previous term, i.e.

$$a, ar, ar^2, ar^3, ar^4, ar^5, \ldots ar^n, \ldots \qquad (\diamondsuit)$$

The terms in a geometric sequence satisfy an interesting property.

3. Pick any term in the geometric sequence ($\diamondsuit$). How is it related to the *product* of the term immediately before it and the term immediately after it?

It turns out, this has a name: for any two numbers x and y, the quantity $\sqrt{xy}$ is the _____.

$$\text{GM} = \sqrt{xy}$$

More generally, for n numbers $a_1, \ldots, a_n$, the geometric mean is

$$\text{GM} = \sqrt[n]{a_1 a_2 \cdots a_n}$$

4. (Competition Example) A geometric progression has 5 terms, two of which are known.

$$g_n = \{3, g_2, g_3, g_4, 27\}$$

Find g_4.

There is an interesting relationship between the arithmetic and geometric means of any given numbers. It is:

$$\text{AM} \geq \text{GM}$$

or

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

and the two expressions will only be equal if _____. This theorem is of major use whenever we are trying to *minimize* or *maximize* a quantity.

5. (Competition Example) Let a, b, c be positive real numbers with $abc = 1$. Find the minimum possible of

$$S = (a+1)(b+1)(c+1)$$

When we take a sequence and add all the terms, it is called a <u>SERIES</u>; arithmetic sequences become arithmetic series, and geometric sequences become geometric series. What primarily concerns us right now is finding values for these series quickly. Let's do some deriving.

<u>Arithmetic Series Formulas</u>

1. (General Arithmetic Series) $\displaystyle\sum_{k=0}^{n}(a + dk) =$

2. (Sum to n) $\displaystyle\sum_{k=1}^{n} k =$

3. (Finite Geometric Series) $\displaystyle\sum_{k=0}^{n} ar^k =$

4. (Infinite Geometric Series with $|r| < 1$: $\displaystyle\sum_{k=0}^{\infty} ar^k =$

Before some easier competition style problems, let's look at how the techniques from our derivations can solve a hard problem.

Example: Compute $\sum_{n=1}^{\infty} \frac{n}{5^n}$.

Finally, some more practice.

6. Compute $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \ldots$.
7. Rebekah wanted to stump her teacher, so she whipped out a calculator and added up the integers from 12 to 88. Before she could finish saying "I just added from up 12 to 88, Mr. Smart Teacher Man. I bet you can't tell me...", the teacher said the correct answer. What was it?
8. (Competition Example) The numbers $x, 18, y$, in order, form an arithmetic progression. The numbers $18, y, 90$, in order, also form an arithmetic progression. What is x?
9. (Competition Example) The terms of a geometric sequence a_n are all positive. Given that

$$a_2 + a_5 = 90 \qquad \text{and} \qquad a_3 a_7 = 6400,$$

find a_4.

Homework: Go to Alcumus (artofproblemsolving.com → Resources (dropdown) → Alcumus), select "Algebra," and play any of

- Arithmetic Sequences
- Arithmetic Series
- Geometric Sequences
- Geometric Series

For more of a challenge, try the "Intermediate Algebra" focus and any of

- Arithmetic Sequences and Series
- Geometric Sequences and Series
- Telescoping Series (not covered today)
- General Sequences and Series
- Advanced Sequences and Series

Solutions

An <u>ARITHMETIC SEQUENCE</u> is one in which each consecutive terms share a common difference, i.e.

$$a, a+d, a+2d, a+3d, a+4d, a+5d, a+6d, \ldots \tag{$\bigstar$}$$

The terms in an arithmetic sequence satisfy an interesting property.

1. Pick any term in the arithmetic sequence ($\bigstar$). How is it related to the term immediately before it and the term immediately after it? **It is the average.**

It turns out, this has a name: for any two numbers x and y, the quantity $\dfrac{x+y}{2}$ is the <u>ARITHMETIC MEAN</u>.

$$\text{AM} = \frac{x+y}{2}$$

More generally, for n numbers $a_1, \ldots, a_n$, the arithmetic mean is

$$\text{AM} = \frac{a_1 + a_2 + \cdots + a_n}{n}$$

2. Let the numbers selected be $x < y < z$. Then $y = \dfrac{x+z}{2}$, so $x+z$ must be even and hence x and z must have the same parity. Therefore, out of the $\dbinom{10}{3} = \dfrac{10(9)(8)}{3(2)(1)} = 120$ ways to choose 3 numbers, we can select any of the values of $x+z$, which will have a unique y and hence triplet (x, y, z), and record how many possibilities there are for x and z:

$$\begin{aligned} x+z = 4 &\implies 1 \text{ options } ((1,3)) \\ x+z = 6 &\implies 2 \text{ options } ((1,5),(2,4)) \\ x+z = 8 &\implies 3 \text{ options } ((1,7),(2,6),(3,5)) \\ x+z = 10 &\implies 4 \text{ options } ((1,9),(2,8),(3,7),(4,6)) \end{aligned}$$

By symmetry, there are also 10 options for the sums $x+z$ of 12, 14, 16, and 18. Therefore, the probability of getting an arithmetic progression is $\dfrac{20}{120} = \dfrac{1}{6}$.

A <u>GEOMETRIC SEQUENCE</u> is one in which each term is a constant multiple of the previous term, i.e.

$$a, ar, ar^2, ar^3, ar^4, ar^5, \ldots ar^n, \ldots \tag{$\diamond$}$$

The terms in a geometric sequence satisfy an interesting property.

3. Pick any term in the geometric sequence ($\diamond$). How is it related to the *product* of the term immediately before it and the term immediately after it? **It is the square root of the product.**

It turns out, this has a name: for any two numbers x and y, the quantity $\sqrt{xy}$ is the <u>GEOMETRIC MEAN</u>.

$$\text{GM} = \sqrt{xy}$$

More generally, for n numbers $a_1, \ldots, a_n$, the geometric mean is

$$\text{GM} = \sqrt[n]{a_1 a_2 \cdots a_n}$$

4. We can quickly find $g_3 = \sqrt{3(27)} = 9$ and $g_4 = \sqrt{9(27)} = 9\sqrt{3}$.

There is an interesting relationship between the arithmetic and geometric means of any given numbers. It is:

$$\text{AM} \geq \text{GM}$$

or

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

and the two expressions will only be equal if _____. This theorem is of major use whenever we are trying to *minimize* or *maximize* a quantity.

5. We use the AM-GM inequality on each of the terms $a+1, b+1, c+1$.

$$\frac{a+1}{2} \leq \sqrt{a(1)} \implies a+1 \geq 2\sqrt{a}$$
$$\frac{b+1}{2} \leq \sqrt{b(1)} \implies b+1 \geq 2\sqrt{b}$$
$$\frac{c+1}{2} \leq \sqrt{c(1)} \implies c+1 \geq 2\sqrt{c}$$

Therefore, $S \geq (2\sqrt{a})(2\sqrt{b})(2\sqrt{c}) = 8\sqrt{abc} = 8$. The minimum value is **8**.

When we take a sequence and add all the terms, it is called a <u>SERIES</u>; arithmetic sequences become arithmetic series, and geometric sequences become geometric series. What primarily concerns us right now is finding values for these series quickly. Let's do some deriving.

<u>Arithmetic Series Formulas</u>

1. (General Arithmetic Series) $\displaystyle\sum_{k=0}^{n}(a+dk) = \underline{(a+(a+dn)) \cdot \frac{n}{2}} = \underline{(\text{first} + \text{last}) \cdot \frac{\#\text{ terms}}{2}}$
2. (Sum to n) $\displaystyle\sum_{k=1}^{n} k = \underline{\frac{n(n+1)}{2}}$
3. (Finite Geometric Series) $\displaystyle\sum_{k=0}^{n} ar^k = \underline{a \cdot \frac{1-r^{n+1}}{1-r}}$
4. (Infinite Geometric Series with $|r| < 1$: $\displaystyle\sum_{k=0}^{\infty} ar^k = \underline{a \cdot \frac{1}{1-r}}$

Before some easier competition style problems, let's look at how the techniques from our derivations can solve a hard problem.

<u>Example</u>: Let's call our sum S. Then

$$S = \frac{1}{5} + \frac{2}{25} + \frac{3}{125} + \frac{4}{625} + \cdots$$

On first inspection, this series looks almost geometric, except for the pesky numerators, which won't stay constant. If we want to utilize the semi-geometric nature of the series, perhaps it would be useful to multiply it by $\frac{1}{5}$.

$$\frac{1}{5}S = \frac{1}{5}\left(\frac{1}{5} + \frac{2}{25} + \frac{3}{125} + \frac{4}{625} + \cdots\right) = \frac{1}{25} + \frac{2}{125} + \frac{3}{625} + \cdots$$

Aha! It looks just the original sum S, only the numerators have all been decreased by 1. This means, if we subtract the sums, we should get some useful simplification.

$$\begin{aligned} S - \frac{1}{5}S &= \left(\frac{1}{5} + \frac{2}{25} + \frac{3}{125} + \frac{4}{625} + \cdots\right) - \left(\frac{1}{25} + \frac{2}{125} + \frac{3}{625} + \cdots\right) \\ &= \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \frac{1}{625} + \cdots \end{aligned}$$

We have our geometric series! We can now easily sum this and find S.

$$\begin{aligned} \frac{4}{5}S &= \frac{1}{5} \cdot \frac{1}{1 - \frac{1}{5}} \\ S &= \frac{5}{4} \cdot \frac{1}{5} \cdot \frac{5}{4} \\ S &= \frac{5}{16} \end{aligned}$$

Our series has a sum of $\mathbf{\frac{5}{16}}$.

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6. This is a geometric series with initial term 1 and ratio $\frac{2}{3}$, so it equals $\dfrac{1}{1 - \left(-\frac{2}{3}\right)} = \mathbf{\frac{3}{5}}$.

7. $(12 + 88) \cdot \dfrac{(88 - 12 + 1)}{2} = \mathbf{3850}$

8. We have $18 = \dfrac{x + y}{2}$ and $y = \dfrac{18 + 90}{2}$. Therefore, $y = 54$ and $\boldsymbol{x = -18}$.

9. The second equation, by the geometric mean, yields $a_5 = \sqrt{6400} = 80$. Plugging this into the first equation yields $a_2 = 10$. Then, the common ratio is clearly 2, so $a_4 = 10(2)(2) = \mathbf{40}$.