Factorials, Permutations, & Combinations

Definition: The expression $n!$ is called n ______________ and is defined by

$$n! = n(n-1)(n-2)\cdots 3(2)(1)$$

NOTE: By convention, $0! = 1$.

Practice

1. Compute $n!$ for $n \in \{1, 2, 3, 4, 5, 6, 7\}$.

2. If $9! = 362,880$, what is $10!$?

3. Compute $\frac{2025!}{2023!}$.

4. Compute $\frac{101! - 100!}{99!}$.

Factorials are incredibly useful for counting permutations and combinations.

Definition: A permutation of n objects is a reordering of those objects where the order of the objects ________ matter. A combination of n objects is simply a set of n objects where the order ___________ matter.

Examples

1. Suppose you have 4 books - call them A, B, C, and D - to arrange on a shelf. How many ways can the books be arranged?

2. Suppose you have 7 books - call them A, B, C, D, E, F, and G - and you are going to select 4 to place on a shelf. How many possible orderings of books on the shelf are there?

One way to answer the previous question is using a formula for permutations. If you have n objects and you want to select r of them, where the order of the r objects matters, then there are

$$_nP_r = \frac{n!}{(n-r)!}$$

possibilities.

For instance, to answer the last question, we computed

$$_7P_4 = \frac{7!}{(7-4)!} = \frac{7!}{3!} = \frac{7(6)(5)(4)(3)(2)(1)}{3(2)(1)} = 7(6)(5)(4)$$

NOTE: It's not worth memorizing this formula - you likely won't use it in practice, and it's more important that you know where it comes from and what it means!

If we simply want to *select* r objects from n objects **without considering the order**, then the formula above would overcount every permutation of the r objects. For instance, we computed that, if there are 7 books and you want to place 4 of them on a shelf, there are $\frac{7!}{3!} = 7(6)(5)(4)$ ways to do this. However, if we just want to select 4 books, then the formula above overcounts every permutation of the 4 books. We showed that there are $4!$ ways to arrange these books, so we need to divide our result above by $4!$ to get

$$\frac{\frac{7!}{3!}}{4!} = \frac{7!}{3!4!} = \frac{7(6)(5)(4)}{4(3)(2)(1)} = 70$$

This leads to a formula for the number of **combinations** of selecting r objects from n objects:

$$_nC_r = \binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

The notation $_nC_r$ is rarely used. Instead, you will frequently see $\binom{n}{r}$, which is called **the binomial coefficient** and is said as "***n* choose *r***".

The numerator seems abstract, but the key to note is that there are **exactly *r* terms in it**. For example, if we wanted to compute $\binom{11}{5}$, we would write 5 consecutive decreasing integers starting at 11 in the numerator and put $5!$ in the denominator:

$$\binom{11}{5} = \frac{11(10)(9)(8)(7)}{5(4)(3)(2)(1)}$$

Simplifying quotients gives this value as 462.

Practice

1. 10 runners are entering a race. Only 3 of these runners can win gold, silver, and bronze medals. How many different ways can the 3 medals be awarded?
2. Let any string of 5 unique characters be called a *5-string*. For instance, *ABRQT* and *RTQBA* are unique 5-strings, while *AACLM* is not a 5-string. How many 5-strings are there using characters from the English alphabet?
3. How many 5-strings are there that start with a vowel and end with a Z?
4. 7 people are going to sit down in a row of 7 chairs. How many ways can the 7 people be seated?
5. 7 people are going to sit at a round table. If rotations of the table are considered identical, how many ways can the 7 people be seated?
6. 7 people are going to sit at a round table, but two of the people are married to each other and would like to sit next to each other. How many ways can the 7 people be seated that satisfy the married couple's request?
7. 7 people are going to sit at a round table, but two of the people are mortal enemies and cannot sit next to each other. How many ways can the 7 people be seated that will keep the enemies from sitting next to one another?
8. Compute $\binom{7}{2}$, $\binom{8}{3}$, and $\binom{10}{4}$.
9. Every row in Pascal's Triangle, shown below, is actually a binomial coefficient.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

   (a) Compute $\binom{4}{0}$, $\binom{4}{1}$, $\binom{4}{2}$, $\binom{4}{3}$, and $\binom{4}{4}$. Which row of Pascal's Triangle do these values correspond to?

   (b) If the row 1-4-6-4-1 is the "fourth" row, what is the row with just a 1 in it?

10. Compute $\binom{9}{3}$ and $\binom{9}{6}$. Can you explain in terms of combinations of objects why these are equal?

11. Explain why $\binom{n}{r} = \binom{n}{n-r}$ for all $n, r \in \mathbb{N}$ with $r \leq n$.

12. Suppose there are 10 students in a certain school club.

    (a) If 3 students are to be randomly chosen to be the president, vice president, and secretary, respectively, how many ways can the leadership positions be chosen?

    (b) If 3 students are to be randomly chosen to simply be "officers," in how many ways can the leadership positions be chosen?

    (c) Explain the difference between (a) and (b).

13. Let S be the set $S := \{1, 2, 3, \ldots, 19\}$. Randomly select 2 unique numbers from S.

    (a) What is the probability their product will be odd?

    (b) What is the probability their sum will be even?

14. One known identity with binomial coefficients is $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$.

    (a) Test the identity out with a couple of small values of n and r.

    (b) Can you explain the identity in terms of Pascal's Triangle?

    (c) Explain the identity in terms of counting. (HInt: Think of selecting $r+1$ officers from a club of $n+1$ students and whether or not one particular student is an officer or not.)

15. A drawer has 8 identical blue socks and 6 identical red socks in it. If 4 socks are randomly selected from the drawer, what is the probability that one pair of blue socks and one pair of red socks are selected?

16. Refer back to the previous scenario. If 4 socks are randomly selected from the drawer, what is the probability that 2 pairs of matching socks are selected?

17. If you rearrange the letters in the word SANDWICH, how many different "words" can be formed? (Any sequence of letters counts as a "word.")

18. If you rearrange the letters in the word SEQUOIAS, how many different "words" can be formed? (Hint: Don't forget that the order of the two S's doesn't matter.)

19. If you rearrange the letters in the word IMPOSSIBILITY, how many different "words" can be formed?

20. Compute $\binom{12}{3}$ and $\binom{15}{6}$.

21. Compute $\binom{2024}{1991} - \binom{2024}{33}$.

22. Eight runners run a race. One runner, Steve, either finishes second or third. How many different finishes to the race can there be?

23. Mr. Hornbeck recently went to a wedding where he was seated at a table of 8 people. He was seated next to his wife and one of her friends. The other 5 people did not know each other. In how many ways could the table have been arranged if Mr. Hornbeck, his wife, and her friend had to be seated together?

24. A math team consists of 6 seniors, 8 juniors, 5 sophomores, and 9 freshmen. If 5 members are to be randomly selected to serve as officers, then the probability that the officers are two seniors and three juniors can be written as $\frac{m}{n}$, where m and n are relatively prime. Compute $m + n$.

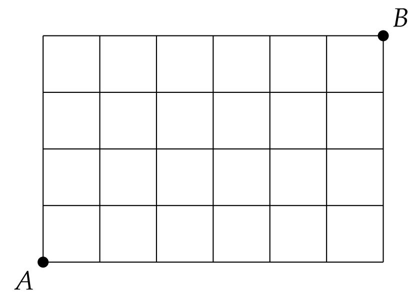
25. Consider the rectangular grid below.

    (a) Draw 2 different paths from A to B using only moves directly up or directly to the right.

    (b) Count the number of total moves in each of your two paths.

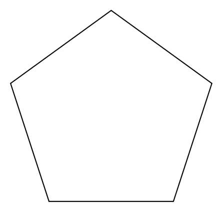
    (c) Count the number of "up" moves in each of your two paths.

    (d) How many total paths are there from A to B using only moves directly up or directly to the right?



26. Fifteen people are at a party. Every person shakes hands with every other person, but no one can shake hands with themselves. How many handshakes occur? (Hint: How many people does it take to form a handshake, and does order matter?)

27. Consider the pentagon below.



    (a) Draw all of the diagonals of the pentagon. How many are there?

    (b) One way to count the diagonals is as follows: A diagonal consists of 2 vertices, where order doesn't matter, and there are 5 vertices. Therefore, there are $\binom{5}{2} = 10$ diagonals. What's wrong with this argument?

    (c) Can you find a way to use the number of vertices to arrive at your answer from (a)?

28. How many diagonals does a hexagon have? How about an octagon? How about an n-gon (polygon with n sides)?

Solutions

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NOTE: By convention, 0! = 1.

Practice

1. Compute $n!$ for $n \in \{1,2,3,4,5,6,7\}$.
   Answer: $\mathbf{1! = 1, 2! = 2, 3! = 6, 4! = 24, 5! = 120, 6! = 720, 7! = 5040}$

2. If $9! = 362,880$, what is $10!$?
   Answer: $\mathbf{10! = 10 \cdot 9! = 10(362880) = 3628800}$

3. Compute $\frac{2025!}{2023!}$.
   Answer: $\mathbf{\frac{2025 \cdot 2024 \cdot 2023!}{2023!} = 2025(2024) = 4098600}$

4. Compute $\frac{101! - 100!}{99!}$.
   Answer: $\mathbf{\frac{101!}{99!} - \frac{100!}{99!} = 101(100) - 100 = 100(101-1) = 100(100) = 10000}$

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Definition: A permutation of n objects is a reordering of those objects where the order of the objects does matter. A combination of n objects is simply a set of n objects where the order does not matter.

Examples

1. Suppose you have 4 books - call them A, B, C, and D - to arrange on a shelf. How many ways can the books be arranged?
   Answer: 4! = 24

2. Suppose you have 7 books - call them A, B, C, D, E, F, and G - and you are going to select 4 to place on a shelf. How many possible orderings of books on the shelf are there?
   Answer: 7! = 5040

One way to answer the previous question is using a formula for permutations. If you have n objects and you want to select r of them, where the order of the r objects matters, then there are

$${}_nP_r = \frac{n!}{(n-r)!}$$

possibilities.

For instance, to answer the last question, we computed

$${}_7P_4 = \frac{7!}{(7-4)!} = \frac{7!}{3!} = \frac{7(6)(5)(4)(3)(2)(1)}{3(2)(1)} = 7(6)(5)(4)$$

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Simplifying quotients gives this value as 462.

Practice

1. 10 runners are entering a race. Only 3 of these runners can win gold, silver, and bronze medals. How many different ways can the 3 medals be awarded?
   Answer: $10 \cdot 9 \cdot 8 = 720$

2. Let any string of 5 unique characters be called a *5-string*. For instance, *ABRQT* and *RTQBA* are unique 5-strings, while *AACLM* is not a 5-string. How many 5-strings are there using characters from the English alphabet?
   Answer: $26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 = 7893600$

3. How many 5-strings are there that start with a vowel and end with a Z?
   Answer: There are 5 choices for the first letter, 24 choices for the second letter (as it can't be the first letter or a Z), 23 choices for the third letter, 22 choices for the fourth letter, and 1 choice for the last letter, so there are $5 \cdot 24 \cdot 23 \cdot 22 \cdot 1 = 60720$ such strings.

4. 7 people are going to sit down in a row of 7 chairs. How many ways can the 7 people be seated?
   Answer: $7! = 5040$

5. 7 people are going to sit at a round table. If rotations of the table are considered identical, how many ways can the 7 people be seated?
   Answer: It's still 7!permutations, only each arrangement is repeated 7 times (for 7 equivalent rotations), so there are $\frac{7!}{7} = 6! = 720$ unique arrangements.

6. 7 people are going to sit at a round table, but two of the people are married to each other and would like to sit next to each other. How many ways can the 7 people be seated that satisfy the married couple's request?
   Answer: Treat the two seats the married couple are at as one seat. Then, there are $\frac{6!}{6} = 5! = 120$ unique arrangements of 6 seats. For each of these arrangements, there are 2 ways the married couple can be seated (who is on the left and who is on the right), so there are $2 \cdot 120 = 240$ seatings.

7. 7 people are going to sit at a round table, but two of the people are mortal enemies and cannot sit next to each other. How many ways can the 7 people be seated that will keep the enemies from sitting next to one another?
   Answer: From #9, there are 720 arrangements of 7 people at a table, and from #10, there are 240 ways that 2 specific people can be seated next to each other. Therefore, there are $720 - 240 = 480$ ways in which 2 specific people are *not* seated next to each other.

8. Compute $\binom{7}{2}$, $\binom{8}{3}$, and $\binom{10}{4}$.
Answer: $\binom{7}{2} = \frac{7(6)}{2} = 21$, $\binom{8}{3} = \frac{8(7)(6)}{3(2)(1)} = 56$, $\binom{10}{4} = \frac{10(9)(8)(7)}{4(3)(2)(1)} = 210$

9. Every row in Pascal's Triangle, shown below, is actually a binomial coefficient.

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&3&&3&&1& \\ 1&&4&&6&&4&&1 \end{array}$$

   (a) Compute $\binom{4}{0}$, $\binom{4}{1}$, $\binom{4}{2}$, $\binom{4}{3}$, and $\binom{4}{4}$. Which row of Pascal's Triangle do these values correspond to?
   Answer: $\binom{4}{0} = 1$, $\binom{4}{1} = 4$, $\binom{4}{2} = 6$, $\binom{4}{3} = 4$, $\binom{4}{4} = 1$

   (b) If the row 1-4-6-4-1 is the "fourth" row, what is the row with just a 1 in it?
   Answer: The 0th row

10. Compute $\binom{9}{3}$ and $\binom{9}{6}$. Can you explain in terms of combinations of objects why these are equal?
Answer: $\binom{9}{3} = \frac{9(8)(7)}{3(2)(1)} = 84$, $\binom{9}{6} = \frac{9(8)(7)(6)(5)(4)}{6(5)(4)(3)(2)(1)} = \frac{9(8)(7)}{3(2)(1)} = 84$. **Choosing 3 objects out of 9 objects is the same as *not choosing* $9 - 3 = 6$ objects.**

11. Explain why $\binom{n}{r} = \binom{n}{n-r}$ for all $n, r \in \mathbb{N}$ with $r \leq n$.
Answer: Choosing r objects from n objects is the same as *not choosing* $n - r$ objects from n objects.

12. Suppose there are 10 students in a certain school club.

   (a) If 3 students are to be randomly chosen to be the president, vice president, and secretary, respectively, how many ways can the leadership positions be chosen?
   Answer: $10(9)(8) = 720$

   (b) If 3 students are to be randomly chosen to simply be "officers," in how many ways can the leadership positions be chosen?
   Answer: $\binom{10}{3} = \frac{10(9)(8)}{3(2)(1)} = 120$

   (c) Explain the difference between (a) and (b).
   Answer: Order matters in (a), but not in (b).

13. Let S be the set $S := \{1, 2, 3, \ldots, 19\}$. Randomly select 2 unique numbers from S.

   (a) What is the probability their product will be odd?
   Answer: To get an odd product, 2 odd numbers must be selected. There are 10 odd numbers to choose from, so there are $\binom{10}{2} = \frac{10(9)}{2(1)} = 45$ ways to select the 2 odd numbers. There are $\binom{19}{2} = \frac{19(18)}{2(1)} = 171$ ways to select 2 numbers overall, so the probability is $\frac{45}{171} = \frac{5}{19}$.

   (b) What is the probability their sum will be even?
   Answer: To get an even sum, two odd numbers or two even numbers must be selected. There are 45 ways to get 2 odd numbers and $\binom{9}{2} = \frac{9(8)}{2(1)} = 36$ ways to get 2 even numbers, so the probability is $\frac{45+36}{171} = \frac{9}{19}$.

14. One known identity with binomial coefficients is $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$.

   (a) Test the identity out with a couple of small values of n and r.
   Example answer: For $n = 6$ and $r = 2$, we get $\binom{6}{2} + \binom{6}{3} = 15 + 20 = 35 = \binom{7}{3}$.

   (b) Can you explain the identity in terms of Pascal's Triangle?
   Answer: Each entry in the triangle is the sum of the entries directly above it.

(c) Explain the identity in terms of counting. (HInt: Think of selecting $r+1$ officers from a club of $n+1$ students and whether or not one particular student is an officer or not.)
Answer: Consider a particular student X. If X is an officer, then there are n members left to choose from and r officers left to choose for a total of $\binom{n}{r}$ possibilities. If X is not an officer, then there are n members left to choose from and $r+1$officers left to choose for a total of $\binom{n}{r+1}$ possibilities. Since the $\binom{n+1}{r+1}$ ways to choose $r+1$ officers can only occur if X either is or is not an officer, $\binom{n+1}{r+1}$ is equivalent to $\binom{n}{r}+\binom{n}{r+1}$.

15. A drawer has 8 identical blue socks and 6 identical red socks in it. If 4 socks are randomly selected from the drawer, what is the probability that one pair of blue socks and one pair of red socks are selected?
Answer: There are $\binom{8}{2}=28$ ways to get 2 blue socks, $\binom{6}{2}=15$ ways to get 2 red socks, and $\binom{14}{4}=\frac{14(13)(12)(11)}{4(3)(2)(1)}=7(13)(11)$ total ways to get 4 socks. Therefore, the probability is $\frac{28(15)}{7(13)(11)}=\frac{4(15)}{13(11)}=\frac{60}{143}$.

16. Refer back to the previous scenario. If 4 socks are randomly selected from the drawer, what is the probability that 2 pairs of matching socks are selected?
Answer: We could get 2 pairs of blue socks, 2 pairs of red socks, or one pair of each. We already computed there are 15(28) ways to get one pair of each. There are $\binom{8}{4}=70$ ways to get 4 blue socks (2 blue pairs) and $\binom{6}{4}=15$ ways to get 4 red socks (2 red pairs), so the probability is $\frac{15(28)+70+15}{7(13)(11)}=\frac{505}{1001}$.

17. $8! = \mathbf{40,320}$

18. The answer would be the same as before, but each permutation involving two S's is double counted, as the order of the S's doesn't matter. Therefore, we must divide the answer in #1 by 2, giving us an answer of $8!/2 = \mathbf{20,160}$.

19. There are 13! ways to rearrange the letters, but we will overcount the permutations of the 4 I's and the 2 S's. Therefore, there are $\frac{13!}{4!2!}=\mathbf{129,729,600}$ possible "words."

20. $\binom{12}{3}=\frac{12(11)(10)}{3(2)(1)}=\mathbf{220}$, $\binom{15}{6}=\frac{15(14)(13)(12)(11)(10)}{6(5)(4)(3)(2)(1)}=\mathbf{5005}$

21. Since $1991+33=2024$, $\binom{2024}{1991}=\binom{2024}{33}$. Therefore, the difference is **0**.

22. If Steve finishes second, then there are $7!=5040$ ways the other runners can finish. If Steve finishes third, there are also 5040 ways. Therefore, there are $2(5040)=\mathbf{10080}$ possible finishes to the race.

23. Pretend the seats Mr. Hornbeck, his wife, and her friend are in are one seat. Then there are 6 total seats, and there are $\frac{6!}{6}=5!=120$ arrangements. For each of these arrangements, there are $3!=6$ ways that Mr. Hornbeck, his wife, and her friend can be arranged amongst themselves. Therefore, there are **720** possible seatings.

24. There are $\binom{6}{2}=15$ ways to pick the seniors and $\binom{5}{3}=10$ ways to pick the juniors. There are $\binom{6+8+5+9}{5}=\binom{28}{5}$ ways to choose the five officers. Therefore, the probability is

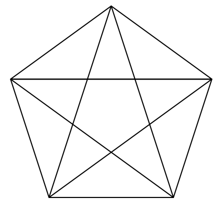
$$\frac{15(10)}{\binom{28}{5}}=\frac{15(10)}{\frac{28(27)(26)(25)(24)}{5(4)(3)(2)(1)}}=\frac{15(10)}{28(9)(26)(5)(3)}=\frac{5}{14(9)(26)}=\frac{5}{3276}$$

Thus $m=5$ and $n=3276$, so $m+n=\mathbf{3281}$.

25. (b) 10 total moves
(c) 4 up moves
(d) Each path is simply a different choice of 4 up moves among 10 total moves, so there are $\binom{10}{4}=\mathbf{210}$ total paths.

26. Each handshake that occurs is simply a selection of 2 people, so $\binom{15}{2}=\mathbf{105}$ handshakes occur.

27.



(a) There are **5** diagonals.

(b) This includes pairs of adjacent vertices.

(c) Each vertex can be connected to any other vertex except for itself and the 2 adjacent vertices. Therefore, each vertex can form a diagonal with 2 other vertices. This leads to $\frac{5}{2}$ diagonals, only we have double counted (since the order of the vertices doesn't matter). There are thus $\frac{5(2)}{2} = 5$ diagonals.

28. A hexagon has $\frac{6(6-3)}{2} = \mathbf{9}$ diagonals; an octagon has $\frac{8(8-3)}{2} = \mathbf{20}$ diagonals; an n-gon has $\boldsymbol{\frac{n(n-3)}{2}}$ diagonals.