Logarithmic Manipulation

Recall the following properties of logarithms.

1. $\log_b x = y \Rightarrow$
2. $\log_b 1 =$
3. $\log_b x^k =$ _____
4. $\log_b x + \log_b y =$ _____
5. $\log_b x - \log_b y =$ _____
6. (Change of Base) $\log_b x =$ _____
7. (Special Case of #3) $\log_b \frac{1}{x} =$ _____
8. (Special Case of #6) $\log_b a =$ _____

These problems were pulled from a Western PA ARML practice.

1. Let $\log_{10} 70 = m$ and $\log_{10} 20 = p$. Given that $\log_{10} 14 = Am + Bp + c$, where A, B, and C are integers, compute $A + B + C$.
2. Compute $(\log_{125} 16)(\log_4 27)(\log_3 625)$.
3. What is the value of
$$\log_3 7 \cdot \log_5 9 \cdot \log_7 11 \cdot \log_9 13 \cdots \log_{21} 25 \log_{23} 27?$$
4. (2000 AIME II #1) The number $\frac{2}{\log_4 2000^6} + \frac{3}{\log_5 2000^6}$ can be written as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.
5. If $\log_b(a) \log_c(a) \log_c(b) = 25$ and $\frac{a^2}{c^2} = c^k$, what is the sum of all possible values of k?
6. (2010 AMC 12A #11) The solution of the equation $7^{x+7} = 8^x$ can be expressed in the form $x = \log_b 7^7$. What is b?
7. (2005 AMC 10B #17) that $4^a = 5, 5^b = 6, 6^c = 7$, and $7^d = 8$. What is $abcd$?
8. (2008 AMC 12A #16) The numbers $\log\left(a^3b^7\right), \log\left(a^5b^{12}\right)$, and $\log\left(a^8b^{15}\right)$ are the first three terms in an arithmetic sequence, and the 12th term of the sequence is $\log b^n$. What is n?
9. (2019 AMC 12A #15) Positive real numbers a and b have the property that
$$\sqrt{\log a} + \sqrt{\log b} + \log \sqrt{a} + \log \sqrt{b} = 100$$
and all four terms on the left are positive integers, where log denotes the base 10 logarithm. If $ab = 10^k$, what is k?
10. (2003 AMC 12B #17) If $\log(xy^3) = 1$ and $\log(x^2y) = 1$, what is $\log(xy)$?

Hints

1. Use properties of logs.
2. Use change of base.
3. Use change of base.
4. Use properties #3, #8, #3 again, then #4; or rewrite from the beginning (introduce more logs).
5. Use change of base.
6. Split 7^{x+7} first.
7. Take $7^d = 8$ and substitute in $7 = 6^c$. Repeat.
8. Use the arithmetic mean.
9. Let $x = \sqrt{\log a}$ and $y = \sqrt{\log b}$.
10. Use property #4 to split the logs *or* do a variable substitution like in #9.

Answers: 1. 0 2. 8 3. 6 4. 7 5. -4 6. 8/7 7. 3/2 8. 112 9. 164 10. 3/5