Some Algebraic Identities

This lesson is dedicated to a few identities and tricks related to polynomials.

Square of a Binomial & Difference of Squares

Two of the simplest, but most useful identities involves squares.

$$(x \pm y)^2 = x^2 + y^2 \pm 2xy(x+y)(x-y) = x^2 - y^2$$

Note that the first identity contains, for numbers x and y, (a) the sum/difference, (b) the product, and (c) the sum of squares. The second identity contains (a) the sum, (b) the difference, and (c) the difference of squares. This means that, if two of these quantities are known in conjunction, then the third can be computed!

1. The area of a right triangle is 50. If the longer leg is 4 more than the shorter leg, then the length of the hypotenuse can be written as $p\sqrt{q}$, where q is not the square of any prime. Find $p+q$.

2. If $3x - 4y = 9$ and $9x^2 - 16y^2 = 3$, compute $8^{3x} \cdot 64^{2y}$.

These identities can also be used for *computation*.

3. Find the largest prime factor of 3551.

4. Compute 1025^2.

Cubic Identities

Similar identities exist for cubics; note the alternative rewriting of $(a+b)^3$!

$$a^3 \pm b^3 =$$

$$(a+b)^3 =$$

Powers of $x + \frac{1}{x}$

Expansions of powers of $\left(x + \frac{1}{x}\right)$ are quite common in math competitions.

5. Examining the structure of powers of $\left(x + \frac{1}{x}\right)$

   (a) Expand $\left(x + \frac{1}{x}\right)^2$.

   (b) Expand $\left(x + \frac{1}{x}\right)^3$. Use the alternative rewriting.

   (c) Expand $\left(x + \frac{1}{x}\right)^4$. Then, factor out a coefficient from the two terms with the same coefficient.

Now for some problems.

6. If $x + \frac{1}{x} = 7$, what is $x^2 + \frac{1}{x^2}$?

7. Suppose $a + \frac{1}{a} = 4$. Compute $a^3 + \frac{1}{a^3}$.

8. If $x + \frac{1}{x} = 5$, what is $x^4 + \frac{1}{x^4}$?

9. If $x^3 + \frac{1}{x^3} = 8$, what is $x^5 + \frac{1}{x^5}$?

Sums of Coefficients of Polynomials

10. Examining structure
    (a) Expand $f(x) = (x+1)^2$ and find the sum of the coefficients of f.
    (b) Evaluate $f(1)$.
    (c) Expand $g(x) = (x+2)^3$ and find the sum of the coefficients of g.
    (d) Evaluate $g(1)$.
    (e) What do you notice from (a)-(d)?

Now for some problems.

11. Find the sum of the coefficients of $(2x-3)^8$.
12. Find the sum of the coefficients of $(x-3)^5$.
13. Find the sum of the coefficients of $(x+2y)^4$.

Using Vièta's relations with cubics

14. Suppose p and q are the roots of $f(x) = x^2 + 8x - 3$. Compute $p^3 + q^3$.

15. Suppose m and n are the solutions of $x^2 + x + 2024 = 0$. Compute $m^3 + n^3$.

Manipulating Systems of Equations

Many systems of equations ask you to compute something involving *multiple* terms. In these cases, it is often wise not to compute the individual terms, but rather to manipulate the system. For instance, suppose you were asked to compute $x + y$ with the system

$$9x - 4y = 5$$
$$5x - 8y = 22$$

What could we do instead of solving for x or y?

Now for some problems.

16. If $a + b = 5, b + c = 8,$ and $a + c = 13$, what is $a + b + c$?

17. If $x + 7y = 2$ and $3x + 34y = 8$, what is $2x + y$?

18. Find $x - y + z$ if

$$\begin{aligned} 9x - 5z &= 4 \\ 4x - y &= 11 \\ y - 2z &= -7 \end{aligned}$$

Solutions

This lesson is dedicated to a few identities and tricks related to polynomials.

Square of a Binomial & Difference of Squares

Two of the simplest, but most useful identities involves squares.

$$(x \pm y)^2 = x^2 + y^2 \pm 2xy(x+y)(x-y) = x^2 - y^2$$

Note that the first identity contains, for numbers x and y, (a) the sum/difference, (b) the product, and (c) the sum of squares. The second identity contains (a) the sum, (b) the difference, and (c) the difference of squares. This means that, if two of these quantities are known in conjunction, then the third can be computed!

1. Let the legs be a and b, with $a > b$. Then we have $a - b = 4$ and $\frac{1}{2}ab = 50$, or $ab = 100$. Now, we use the identity for a square of a binomial.

$$\begin{aligned}(a-b)^2 &= a^2 + b^2 - 2ab \\ (4)^2 &= a^2 + b^2 - 2(100) \\ 204 &= a^2 + b^2\end{aligned}$$

The hypotenuse is precisely $\sqrt{a^2+b^2} = \sqrt{204} = 2\sqrt{51}$. Therefore, $p = 2$, $q = 51$, and $p + q = 53$.

2. From the difference of squares identity, we have

$$\begin{aligned}(3x-4y)(3x+4y) &= 9x^2 - 16y^2 \\ 9(3x+4y) &= 3 \\ 3x + 4y &= \frac{1}{3}\end{aligned}$$

It would be great if we could find a way to use the value of $3x + 4y$. Investigating $8^{3x} \cdot 64^{2y}$, we realize that we can rewrite 64 as 8^2 to get

$$8^{3x} \cdot \left(8^2\right)^{2y} = 8^{3x} \cdot 8^{4y} = 8^{3x+4y} = 8^{1/3} = 2$$

These identities can also be used for *computation*.

3. Noticing that 3551 is close to 3600 (a perfect square), we get

$$3551 = 3600 - 49 = 60^2 - 7^2 = 67(53)$$

Therefore, 67 is the largest prime factor.

4. We use binomial expansion.

$$1025^2 = (1000+25)^2 = 1000^2 + 2(25)(1000) + 25^2 = 1000000 + 50000 + 625 = 1050625$$

Cubic Identities

Similar identities exist for cubics; note the alternative rewriting of $(a+b)^3$!

$$\begin{aligned}a^3 \pm b^3 &= (a \pm b)(a^2 \mp ab + b^2) \\ (a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + b^3 + 3ab(a+b)\end{aligned}$$

Powers of $x + \frac{1}{x}$

Expansions of powers of $\left(x + \frac{1}{x}\right)$ are quite common in math competitions.

5. (a) $x^2 + \frac{1}{x^2} + 2x\left(\frac{1}{x}\right) = x^2 + \frac{1}{x^2} + 2$

(b) $x^3 + 3x^2\left(\frac{1}{x}\right) + 3x\left(\frac{1}{x}\right)^2 + \left(\frac{1}{x}\right)^3 = x^3 + \frac{1}{x^3} + 3x + \frac{3}{x} = x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right)$

(c) $x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4} = x^4 + \frac{1}{x^4} + 6 + 4\left(x^2 + \frac{1}{x^2}\right)$

6. From 1(a), $x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2 = 7^2 - 2 = 47$.

7. From 1(b), $x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right) = 4^3 - 3(4) = 52$.

8. From 1(c), $x^4 + \frac{1}{x^4} = \left(x + \frac{1}{x}\right)^4 - 6 - 4\left(x^2 + \frac{1}{x^2}\right)$. We can compute $x^2 + \frac{1}{x^2} = 5^2 - 2 = 23$, so $x^4 + \frac{1}{x^4} = 5^4 - 6 - 4(23) = 527$.

9. Expanding $\left(x + \frac{1}{x}\right)^5$ gives $x^5 + \frac{1}{x^5} + 5\left(x^3 + \frac{1}{x^3}\right) + 10\left(x + \frac{1}{x}\right)$. Following a similar procedure to #3 gives $\boldsymbol{x^3 + \frac{1}{x^3} = 198}$. Therefore,

$$x^5 + \frac{1}{x^5} = \left(x + \frac{1}{x}\right)^5 - 5\left(x^3 + \frac{1}{x^3}\right) - 10\left(x + \frac{1}{x}\right) = 6^5 - 5(198) - 10(6) = 6726$$

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Sums of Coefficients of Polynomials

10. (a) $f(x) = x^2 + 2x + 1$, with sum of coefficients equal to 4.
    (b) $f(1) = (1 + 1)^2 = 4$
    (c) $g(x) = x^3 + 3x^2(2) + 3x(2)^2 + 2^3 = x^3 + 6x^2 + 12x + 8$, with sum of coefficients equal to 27.
    (d) $g(1) = (1 + 2)^3 = 27$
    (e) The sum of coefficients of a polynomial $p(x)$ equals $p(1)$.
11. $(2(1) - 3)^8 = (-1)^8 = 1$
12. $(1 - 3)^5 = (-2)^5 = -32$
13. $(1 + 2(1))^4 = 3^4 = 81$

Using Vièta's relations with cubics

14. We know from Vièta's relations that $p + q = -8$ and $pq = -3$. Since $(p + q)^3 = p^3 + q^3 + 3pq(p + q)$, we can solve for $p^3 + q^3$: $(-8)^3 = p^3 + q^3 + 3(-3)(-8) \Rightarrow a^3 + b^3 = -584$.
15. Following the same procedure as the last problem, we get

$$m^3 + n^3 = (m + n)^3 - 3mn(m + n) = (-1)^3 - 3(-1)(2024) = 6071$$

Manipulating Systems of Equations

Observe that, subtracting the two equations, we get $4x + 4y = -17$, or $x + y = -\frac{17}{4}$.

16. Adding all 3 equations yields $2a + 2b + 2c = 26$, so $a + b + c = 13$.
17. Multiplying the first equation by 5 and subtracting the second yields $5(x + 7y) - (3x + 34y) = 2x + y$, so $2x + y = 5(2) - 8 = 2$.
18. This takes trial and error, but taking the first equation, subtracting 2 times the second equation, and then subtracting 3 times the third equation yields

$$9x - 5z - 2(4x - y) - 3(y - 2z) = 9x - 5z - 8x + 2y - 3y + 6z = x - y + z$$

so $x - y + z = 4 - 2(11) - 3(-7) = 3$.