Some Number Theory Facts

1. DIVISIBILITY: The following can be used to determine whether a number n is divisible by the number k (see table).

k	$k \mid n$[1] if...
3	
4	
6	
7	
8	
9	
11	

2. NUMBER OF DIVISORS: Let n be an integer with prime factorization

$$n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$$

Then the function

$$\tau(n) = (e_1 + 1)(e_2 + 1) \cdots (e_k + 1)$$

gives the <u>number of divisors</u> of n. (τ is the Greek letter *tau*.)

Explanation:

3. SUM OF DIVISORS: Consider the same n from (1). Then the function

$$\sigma(n) = (1 + p_1 + p_1^2 + \cdots + p_1^{e_1})(1 + p_2 + p_2^2 + \cdots + p_2^{e_2}) \cdots (1 + p_k + p_k^2 + \cdots + p_k^{e_k})$$

gives the <u>sum of the divisors</u> of n. (σ is the Greek letter *sigma*.)

Explanation:

4. TOTIENT FUNCTION: Consider the same n from (1). Then the <u>totient function</u>

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right)$$

gives the <u>number of positive integers less than n that are relatively prime with n.</u>

Explanation:

5. Every even number is of the form ________, where ________, and every odd number is of the form ____________, where ________.

[1]This notation means k divides n.

Practice Questions

To get a sample of many more math competition questions using some of these formulas, try the following Alcumus topics:

- **Divisors**
- **Prime Factorization**
- **Divisibility Rules**
- **Divisor Arithmetic**
- **Counting Divisors**
- **Divisibility Problem-Solving**

1. How many integers between 10 and 999 are divisible by 9 and have a 10's digit of 9?
2. Determine if 2730 is divisible by each of the following.
   (a) 2 (b) 3 (c) 4 (d) 5 (e) 6 (f) 7 (g) 9 (h) 11 (i) 12 (j) 21 (k) 42 (l) 63
3. Find the largest prime factor of 3990.
4. The number $A03B4$, where A and B are positive integers, is divisible by 33. Find all possible values of $A + B$.
5. For $n = 96$, compute $\tau(n), \sigma(n)$, and $\phi(n)$.
6. Prove the following:
   (a) The sum of an even number and an odd number is odd
   (b) The square of an odd number is an odd number
   (c) The sum of 3 consecutive even numbers must be divisible by 6
7. Determine the number of divisors of each of the following.
   (a) 24 (b) 144 (c) $8!$ (d) $a^2b^3c^4$, where a, b, c are unique primes
8. Find the sum of the divisors of 496.
9. Find the smallest positive integer with 12 factors.
10. How many numbers between 10 and 200 have an odd number of factors?
11. Prove that the square of an even number must be divisible by 4.
12. Prove that the square of an odd number must leave a remainder of 1 when divided by 4.
13. What is the largest number that the sum of 5 consecutive odd numbers is guaranteed to be divisible by? How about the sum of 6 consecutive odd numbers?
14. Take any two consecutive odd numbers. Now, multiply them and add 1. Repeat this process a few times.

    What do you think is happening here, and can you prove it?

Solutions

1. There are 12 such numbers. Let $a9b$ be a number divisible by 9. Then $a+b$ equals 0, 9, or 18. Each integer $a9b$ then corresponds to an ordered pair (a,b), and there are 12 such pairs: $(0,0),(0,9),(1,8),(2,7),(3,6),(4,5),$ $(5,4),(6,3),(7,2),(8,1),(9,0)$, and $(9,9)$.

2. (a) Yes (b) Yes (c) No (d) Yes (e) Yes (f) Yes (g) No (h) No (i) Yes (j) Yes (k) Yes (l) No

3. Since $A03B4$ is divisible by 11, $A-0+3-B+4=7+(A-B)$ must be divisible by 11. This gives $A-B=-7$ or $A-B=4$, which yields the possibilities in the table below. Since $A03B4$ is also divisible by 3, $A+B+7$ is divisible by 7, which yields $A+B=2,5,8,11,14$, or 17.

4. $3990=2(3)(5)(7)(19)$, so the largest prime factor is 19.

5. $96=2^5\cdot 3^1$, so $\tau(96)=(5+1)(1+1)=12$, $\sigma(96)=(1+2+4+8+16+32)(1+3)=252$, and $\phi(96)=96\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)=96\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)=32$.

6. Prove the following:

   (a) Let $m=2k$ and $n=2p+1$, where $k,p\in\mathbb{Z}$. Then $m+n=2k+2p+1=2(k+p)+1=2q+1$, where $q=k+p$ is an integer. Therefore, $m+n$ is odd.

   (b) Let $n=2k+1$ be any odd number. Then $n^2=(2k+1)^2=4k^2+4k+1=2(2k^2+2k)+1=2m+1$, where $m=2k^2+2k$ is an integer. Therefore, n^2 is odd.

   (c) Let p,q,r be the three even numbers. Then $q=2k$ for some integer k, which gives $p=2k-2$ and $r=2k+2$. Therefore, $p+q+r=(2k-2)+2k+(2k+2)=6k$, which is a multiple of 6.

7. Determine the number of divisors of each of the following.

   (a) $24=2^3\cdot 3$, so $\tau(24)=(3+1)(1+1)=8$

   (b) $144=2^4\cdot 3^2$, so $\tau(144)=(4+1)(2+1)=15$

   (c) $8!=2(3)(4)(5)(6)(7)(8)=2(3)(2^2)(5)(2\cdot 3)(7)(2^3)=2^6\cdot 3^2\cdot 5\cdot 7$, so $\tau(8!)=(6+1)(2+1)(1+1)(1+1)=84$

   (d) $\tau(a^2b^3c^4)=(2+1)(3+1)(4+1)=60$

8. $496=2^4\cdot 31$, so $\sigma(496)=(1+2+4+8+16)(1+31)=992$

9. Our integer can take any of the forms p^11, p^5q, p^3q^2, or pqr^2, where p,q,r are unique primes. Testing these out with small primes gives that the smallest possible value is $5(3)(2^2)=60$.

10. Based on the formula for $\tau(n)$, the only way a number can have an odd number of factors is if the power of every single one of its prime factors is even. If this is the case, then a factor of 2 can be pulled from each power of a prime and n can be rewritten as $\left(p_1^{e_1/2}p_2^{e_2/2}\cdot p_k^{e_k/2}\right)^2$. In short, n must be a perfect square, of which there are 11 between 10 and 200.

11. Let $n=2k$. Then $n^2=(2k)^2=4k^2$, which is a multiple of 4.

12. Let $n=2k+1$. Then $n^2=(2k+1)^2=4k^2+4k+1=4(k^2+k)+1$, which would leave a remainder of 1 when divided by 4.

13. Let n be the middle odd number. Then the five consecutive odds are $n-4,n-2,n,n+2$, and $n+4$. Adding these gives $5n$, so our sum must be divisible by at least 5. On the other hand, for 6 consecutive odds, let m be the median of all 6 numbers. m must be the average of the two middle odd numbers and must therefore be even, or of the form $2k$. Thus, our 6 consecutive odds are $2k-5, 2k-3, 2k-1, 2k+1, 2k+3$, and $2k+5$, which have a sum of $12k$ and hence be divisible by at least 12.

14. You should be getting perfect squares! Try representing the two consecutive odd numbers using the definition of an odd number.