Rockdale Magnet

MATH TEAM

2024-2025

Chapter 1: Number Theory

Practice 1: Number Vocabulary

Date: Tuesday, August 13

To begin our study of number theory, we have a large number of terms to define.

1. A whole number is known as an __________. The set of all integers is notated _____, so

$$\mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}$$

2. An integer d is a ________ or _______ of another integer n if there is some integer k such that ________. We can write this as $d \mid n$.
   - Example 1: $5 \mid 20$, but $5 \nmid 21$
   - Example 2: Is $24 \mid 8^3$ true or false?
3. The divisors of a number n that are not 1 or n are called __________ divisors.
4. An integer $n > 1$ is ________ if k has exactly ___ divisors.
   - 1 is not prime because it has only *one* divisor.
5. An integer $n > 1$ is ____________ if n has greater than ___ divisors.
6. The greatest common divisor (GCD) of two integers m and n is the largest integer g such that ________ and ________. We write the GCD as

$$g = \gcd(m, n)$$

7. The least common multiple (LCM) of two integers m and n is the smallest integer l such that ________ and ________. We write the LCM as

$$l = \operatorname{lcm}(m, n)$$

8. If two integers m and n have no common divisors besides 1, then m and n are called __________________ __________. Mathematically, this is equivalent to stating $\gcd(m, n) =$___.
9. The last digit of an integer is called the _______ digit (also known as the "ones" digit).
10. The Fundamental Theorem of Arithmetic states that every integer n has a unique ____________ ____________________.

Examples

1. Find all of the divisors of 24.
2. Find $\gcd(24, 54)$.
3. Which of the following are true?
   (a) $3 \mid 9$ (b) $9 \mid 3$ (c) $8 \mid 8$ (d) $5 \mid 0$ (e) $0 \mid 5$ (f) $8 \mid 8n, n \in \mathbb{Z}$ (g) $\frac{1}{4} \mid \frac{1}{2}$
4. Find $\text{lcm}(12, 16)$.
5. Determine whether the following are correct prime factorizations.
   (a) $40 = 2^3 \cdot 5$
   (b) $54 = 1 \cdot 2 \cdot 3^3$
   (c) $148 = 4 \cdot 37$
   (d) $2024 = 2^3 \cdot 253$
6. (a) Find the smallest positive integer that has exactly 5 divisors.
   (b) Find the smallest positive integer that has exactly 9 divisors.
7. Let $a = 2^3 \cdot 5^2$ and $b = 2^2 \cdot 3^2 \cdot 5$. Find $\text{lcm}(a, b)$ and $\gcd(a, b)$.

Practice 1: Number Vocabulary - SOLUTIONS

Date: Tuesday, August 13

To begin our study of number theory, we have a large number of terms to define.

1. A whole number is known as an integer. The set of all integers is notated $\mathbb{Z}$, so
$$\mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}$$
2. An integer d is a divisor or factor of another integer n if there is some integer k such that $n = dk$. We can write this as $d \mid n$.
   - Example 1: $5 \mid 20$, but $5 \nmid 21$
   - Example 2: Is $24 \mid 8^3$ true or false? False. 24 has a factor of 3, but 8^3 does not.
3. The divisors of a number n that are not 1 or n are called proper divisors.
4. An integer $n > 1$ is prime if k has exactly 2 divisors.
   - 1 is not prime because it has only *one* divisor.
5. An integer $n > 1$ is composite if n has greater than 2 divisors.
6. The greatest common divisor (GCD) of two integers m and n is the largest integer g such that $g \mid m$ and $g \mid n$. We write the GCD as
$$g = \gcd(m, n)$$
7. The least common multiple (LCM) of two integers m and n is the smallest integer l such that $m \mid l$ and $n \mid l$. We write the LCM as
$$l = \text{lcm}(m, n)$$
8. If two integers m and n have no common divisors besides 1, then m and n are called relatively prime. Mathematically, this is equivalent to stating $\gcd(m, n) = 1$.
9. The last digit of an integer is called the units digit (also known as the "ones" digit).
10. The Fundamental Theorem of Arithmetic states that every integer n has a unique prime factorization.

Examples

1. $1, 2, 3, 4, 6, 8, 12, 24$
2. 6
3. (a), (c), (d), and (f) are true
4. Find $\text{lcm}(18, 60)$.
5. (a) Yes
   (b) No; 1 is not prime
   (c) No; 4 is not prime $(4 = 2^2)$
   (d) No; 253 is not prime $(253 = 11 \cdot 23)$
6. (a) $2^4 = 16$ (1, 2, 4, 8, 16)
   (b) $2^2 \cdot 3^2 = 36$ (1, 2, 3, 4, 6, 9, 12, 18, 36)
7. $\text{lcm}(a, b) = 2^3 \cdot 3^2 \cdot 5^2 = 1,800$, $\gcd(a, b) = 2^2 \cdot 5 = 20$

Practice 2: Number Theory Pre-Test

Date: Wednesday, August 14

Name: ____________________________

1. How many prime factors does 2024 have?

   Answer: __________

2. The four digit number $abcd$ is divisible by 5 and 11. If $a, b, c,$ and d are all unique and $a \neq 0$, find the smallest possible value of $abcd$.

   Answer: __________

3. Find the smallest positive integer that is divisible by 2, 3, 4, 5, 6, 7, 8, and 9.

   (A) 840
   (B) 2,520
   (C) 5,040
   (D) 362,880

4. Let $q = 1 + 2^2 + 3^2 + 4^2 + \cdots + 119^2 + 120^2$. Find the remainder when q is divided by 12.

   (A) 2
   (B) 6
   (C) 8
   (D) 10

5. What is the last digit of $3^9 + 4^8 + 5^7 + 6^6 + 7^5 + 8^4 + 9^3$?

   Answer: __________

6. For any integer n, let S_n be the set defined as

   $$S_n := \{k \mid k < n \text{ and } \gcd(k, n) = 1\}$$

   How many elements are in S_{50}?

   (A) 20
   (B) 25
   (C) 30
   (D) 40

7. The sum of all of the divisors of a number k is given by $(1 + 3 + 9 + 27)(1 + 2 + 4 + 8)(1 + 5)$. What is k?

   (A) 40
   (B) 1,080
   (C) 1,800
   (D) 3,600

8. What is the largest prime number that divides both 2,117 and 2,263?

   (A) 29
   (B) 31
   (C) 69
   (D) 73

9. A number is a *palindrome* if it reads the same forwards as backwards. for instance, 838 and 777 are palindromes, while 102 and 551 are not. How many integers between 100 and 500 are palindromes that are divisible by 12?

   (A) 1
   (B) 2
   (C) 3
   (D) 4

10. How many proper divisors does 720 have?

    Answer: __________

11. When all of the divisors of a number n are multiplied together, the result is 12^3. What is the number?

    (A) 12
    (B) 16
    (C) 24
    (D) 36

Practice 3: Divisibility Rules and Bases

Date: Tuesday, August 20

1. There are easy criteria for determining whether a number is divisible by any one of the following.

Divisor	Rule
2	$2 = 2^1$, so look at the last ____ digit and see if it's divisible by 2.
3	Check if the ____ of the digits is divisible by 3.
4	$2 = 2^2$, so look at the last ____ digits and see if it's divisible by 4.
5	The number must end in 5 or 0.
6	The number must be divisible by both ____ and ____.
7	Take the last 2 digits and add that to 2 times the rest of the number. If this is divisible by 7, then so must the number be. Repeat if necessary.
8	$2 = 2^3$, so look at the last ____ digits and see if it's divisible by 8.
9	Check if the ____ of the digits is divisible by 9.
10	The number must end in 0.
11	Take the first digit. Subtract the 2nd digit, then add the 3rd digit, and so on in alternating fashion. If the result is divisible by 11, then so must the number be.
12	The number must be divisible by ____ and ____.

2. Our number system is in base ____. This means that every digit is a multiple of a ________________. For instance,

$$472 = 400 + 70 + 2 = 4(10^2) + 7(10^1) + 2(10^0)$$

and

$$365,124 = 3(10^5) + 6(10^4) + 5(10^3) + 1(10^2) + 2(10^1) + 4(10^0)$$

It turns out numbers can be written in *any* base. When a number, say $pqrs$, is written in base b, we write $pqrs_b$.

(a) Example: Write 29_{10} in base 2, base 3, and base 4.

(b) Example: For any number in base b, what is the largest a digit can possibly equal?

(c) Example: There are two positive integers k such that $k_{10} = a0b_7$ and $k_{10} = b0a_5$. Find both values of k.

Practice 3: Divisibility Rules and Bases - SOLUTIONS

Date: Tuesday, August 20

1. There are easy criteria for determining whether a number is divisible by any one of the following.

Divisor	Rule
2	$2 = 2^1$, so look at the last 1 digit and see if it's divisible by 2.
3	Check if the sum of the digits is divisible by 3.
4	$2 = 2^2$, so look at the last 2 digits and see if it's divisible by 4.
5	The number must end in 5 or 0.
6	The number must be divisible by both 2 and 3.
7	Take the last 2 digits and add that to 2 times the rest of the number. If this is divisible by 7, then so must the number be. Repeat if necessary. Example: $4723 \longrightarrow 2(47) + 23 = 117 \longrightarrow 2(1) + 17 = 19$ $7 \nmid 19$, so $7 \nmid 4723$.
8	$2 = 2^3$, so look at the last 3 digits and see if it's divisible by 8.
9	Check if the sum of the digits is divisible by 3.
10	The number must end in 0.
11	Take the first digit. Subtract the 2nd digit, then add the 3rd digit, and so on in alternating fashion. If the result is divisible by 11, then so must the number be.
12	The number must be divisible by 3 and 4.

2. Our number system is in base ____. This means that every digit is a multiple of a ________________. For instance,

$$472 = 400 + 70 + 2 = 4(10^2) + 7(10^1) + 2(10^0)$$

and

$$365,124 = 3(10^5) + 6(10^4) + 5(10^3) + 1(10^2) + 2(10^1) + 4(10^0)$$

It turns out numbers can be written in *any* base. When a number, say $pqrs$, is written in base b, we write $pqrs_b$.

(a)

$$\begin{aligned} 29 &= 16 + 8 + 4 + 1 = 1(2^4) + 1(2^3) + 1(2^2) + 0(2^1) + 1(2^0) = 11101_2 \\ &= 27 + 2 = 1(3^3) + 0(3^2) + 0(3^1) + 2(3^0) = 1002_3 \\ &= 16 + 3(4) + 1 = 1(4^2) + 3(4^1) + 1(4^0) = 131_4 \end{aligned}$$

(b) $b - 1$

(c) We write $k = 49a + b$ and $k = 25b + a$. Setting these equal and separating a and b yields $b = 2a$ (or $a = \frac{1}{2}b$). Since $b < 5$ and a must be an integer, b can only equal 4 or 2. This makes $k = 204_7 = 2(49) + 4 = 102$ or $k = 102_7 = 49 + 2 = 51$.

Practice 4: Divisibility and Bases Practice

Date: Wednesday, August 21

1. How many integers between 10 and 999 are divisible by 9 and have a 10's digit of 9?
2. Convert to base 10: (a) 1234_5 (b) 10102_3 (c) 1000_6 (d) 11111_2
3. In 2(d), you may have noticed that 11111_2, or five 1s, is 1 less than a power of 2 (namely, 2^5).
   (a) How is 999 related to 10^3? How is 9999 related to 10^4?
   (b) Predict the value of 333_4 in base 10 without computation. Then, check yourself.
   (c) Predict the value of 22222_3 in base 10 without computation. Then, check yourself.
   (d) Complete the conjecture: in base b, the number $(b-1)(b-1)\cdots(b-1)(b-1)_b$, with n $(b-1)$'s, is equal to ________________ in base 10.
4. Determine if 2730 is divisible by each of the following.
   (a) 2 (b) 3 (c) 4 (d) 5 (e) 6 (f) 7 (g) 9 (h) 11 (i) 12 (j) 21 (k) 42 (l) 63
5. The number $A03B4$, where A and B are positive integers, is divisible by 33. Find all possible values of $A+B$.
6. In what base b does 92_{10} equal 332_b?
7. True or false? $4 \mid 4444_5$. Explain.
8. If $52a_6 = a0a_8$, what is a?

Practice 4: Divisibility and Bases Practice - SOLUTIONS

Date: Wednesday, August 21

1. There are 12 such numbers. Let $a9b$ be a number divisible by 9. Then $a+b$ equals 0, 9, or 18. Each integer $a9b$ then corresponds to an ordered pair (a, b), and there are 12 such pairs: $(0,0)$, $(0,9)$, $(1,8)$, $(2,7)$, $(3,6)$, $(4,5)$, $(5,4)$, $(6,3)$, $(7,2)$, $(8,1)$, $(9,0)$, and $(9,9)$.

2. (a) $1(5^3) + 2(5^2) + 3(5) + 4 = 194$
   (b) $1(3^4) + 1(3^2) + 2 = 92$
   (c) $1(6^3) = 216$
   (d) $1(2^4) + 1(2^3) + 1(2^2) + 1(2) + 1 = 31$

3. (a) 999 is 1 less than 10^3 and 9999 is 1 less than 10^4.
   (b) 1 less than 4^3, so 63.
   (c) 1 less than 3^5, so 728.
   (d) $b^{n+1} - 1$

4. (a) Yes (b) Yes (c) No (d) Yes (e) Yes (f) Yes (g) No (h) No (i) Yes (j) Yes (k) Yes (l) No

5. Since $A03B4$ is divisible by 11, $A-0+3-B+4 = 7+(A-B)$ must be divisible by 11. This gives $A-B = -7$ or $A-B = 4$, which yields the possibilities in the table below. Since $A03B4$ is also divisible by 3, $A+B+7$ is divisible by 7, which yields $A+B = 2, 5, 8, 11, 14$, or 17.

A	B	$A-B$	$A+B$
1	8	−7	9
2	9	−7	11 ✓
5	1	4	6
6	2	4	8 ✓
7	3	4	10
8	4	4	12
9	5	4	14 ✓

   Checking the sums of the possibilities in the table shows that the only values of A and B satisfying both divisibility rules are $(A=2, B=9)$, $(A=6, B=2)$, and $(A=9, B=5)$. The possible sums are therefore $8, 11$, and 14.

6. We get $3b^2 + 3b + 2 = 92$, or $3b^2 + 3b - 90 = 0$. Factoring gives $3(b^2 + b - 30) = 3(b+6)(b-5)$, so $b = 5$ (since $b > 0$).

7. True, and you don't have to convert it to base 10!

8. $5(36) + 2(6) + a = 64a + a$, so $a = 3$.

Practice 5: Some Number Theory Facts

Date: Tuesday, August 27

1. EUCLIDEAN ALGORITHM: A procedure for finding the greatest common divisor of 2 numbers.
   (a) Example: Find $\gcd(301, 133)$.

   (b) Example: Find $\gcd(1080, 936)$.

2. The sum of the first n positive integers is

$$1 + 2 + 3 + \cdots + (n-1) + n =$$

   Proof:

3. The sum of the first n odd numbers is

$$1 + 3 + 5 + \cdots + (2n-1) + (2n+1) =$$

   Visual proof:

4. The sum of the first n squares is

$$1 + 4 + 9 + \cdots + (n-1)^2 + n^2 =$$

5. The sum of the first n powers of 2 is

$$1 + 4 + 8 + \cdots + 2^{n-1} + 2^n =$$

Practice

1. Find $\gcd(836, 551)$.
2. Compute $1 + 2 + 3 + \cdots + 150$.
3. Compute $101 + 102 + \cdots + 999 + 1000$.
4. A father asks his daughter to walk the dog for 20 minutes ever day and says he'll pay her $50 for the month (30 days) for doing so. The daughter proposes an alternative strategy. She says that the father should pay her 1 cent on the first day and then double how much he pays her every day (so 2 cents on day 2, 4 cents on day 3, and so on). Should the father agree?
5. Let $k = 1^2 + 2^2 + 3^2 + \cdots + 2023^2$. If d is the positive difference between the largest prime factor of k and smallest prime factor of k, what is the value of d?
6. The sum of the first k positive integers is 190 for some k. What is k?

Practice 5: Some Number Theory Facts - SOLUTIONS

Date: Tuesday, August 27

1. EUCLIDEAN ALGORITHM: A procedure for finding the greatest common divisor of 2 numbers.

   Step 1: Call the numbers a and b with $a > b$. Assume that $b \nmid a$. (If $b \mid a$, then $\gcd(a, b) = b$.)

   Step 2: Find the biggest multiple bq of b that's less than or equal to a so that you can write $a = bq + r$ (r for *remainder*!).

   Step 3: Repeat Step 2, but replace a and b with b and r, respectively.

   Step 4: If the remainder equals 0, then $\gcd(a, b)$ equals the remainder from the previous step. If the remainder does not equal 0, then repeat Step 3.

   (a) Example: Find $\gcd(301, 133)$.
   Solution:

   $$\begin{aligned} 301 &= 2(133) + 35 && (35 \neq 0\text{, so repeat with 133 and 35}) \\ 133 &= 3(35) + 28 && (28 \neq 0\text{, so repeat with 35 and 28}) \\ 35 &= 1(28) + 7 && (7 \neq 0\text{, so repeat with 28 and 7}) \\ 28 &= 4(7) + 0 \end{aligned}$$

   Therefore, $\gcd(301, 133)$ is the last non-zero remainder, so $\gcd(301, 133) = 7$.

   (b) Example: Find $\gcd(1080, 936)$.
   Solution:

   $$\begin{aligned} 1080 &= 1(936) + 144 && (144 \neq 0\text{, so repeat with 936 and 144}) \\ 936 &= 6(144) + 72 && (72 \neq 0\text{, so repeat with 144 and 72}) \\ 144 &= 2(72) + 0 \end{aligned}$$

   Therefore, $\gcd(1080, 936)$ is the last non-zero remainder, so $\gcd(1080, 936) = 72$.

2. The sum of the first n positive integers is

   $$1 + 2 + 3 + \cdots + (n-1) + n = \frac{n(n+1)}{2}$$

   Proof: For any n, let $S = 1 + 2 + 3 + \cdots + (n-1) + n$. Then we have

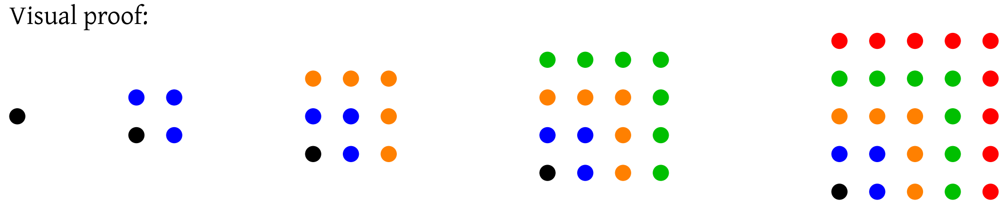
   $$\begin{array}{ccccccccccccccc} S & = & 1 & + & 2 & + & 3 & + & \cdots & + & (n-2) & + & (n-1) & + & n \\ +S & = & n & + & (n-1) & + & (n-2) & + & \cdots & + & 3 & + & 2 & + & 1 \\ \hline 2S & = & (n+1) & + & (n+1) & + & (n+1) & + & \cdots & + & (n+1) & + & (n+1) & + & (n+1) \end{array}$$

   Therefore, $2S = n(n+1)$, so $S = \frac{n(n+1)}{2}$.

3. The sum of the first n odd numbers is

   $$1 + 3 + 5 + \cdots + (2n-1) + (2n+1) = n^2$$

   Visual proof:



4. The sum of the first n squares is

$$1 + 4 + 9 + \cdots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}$$

5. The sum of the first n powers of 2 is

$$1 + 4 + 8 + \cdots + 2^{n-1} + 2^n = 2^{n+1} - 1$$

Practice

1. $\gcd(836, 551) = 19$

$$\begin{aligned} 836 &= 1(551) + 285 \\ 551 &= 1(285) + 266 \\ 285 &= 1(266) + 19 \\ 266 &= 14(19) + 0 \end{aligned}$$

2. $\frac{150(151)}{2} = 75(151) = 11,325$
3. Compute the sum from 1 to 100, then from 1 to 1000, then...
4. Unless the father has over 10 million dollars lying around, probably not.
5. $k = \frac{2023(2024)(4047)}{6} = 2023(1012)(1349)$. Now, compute the prime factorization and go from there.
6. $190 = \frac{k(k+1)}{2}$, or $k^2 + k = 380$. Factor!

Practice 6: Number Theory Practice

Date: Wednesday, August 28

1. (a) Let $a = 8$ and $b = 20$. Compute $\gcd(a, b)$.
   (b) Compute $\text{lcm}(a, b)$.
   (c) Compute $\gcd(a, b) \cdot \text{lcm}(a, b)$. What other product is this equal to?
   (d) Choose 2 other numbers and see if the same result occurs.
2. Find $\gcd(2233, 870)$.
3. Try an *algebraic* example of the Euclidean algorithm. For any positive n, find the greatest possible value of $\gcd(12n + 7, 5n + 1)$.
4. Compute $\frac{1+4+9+16+\cdots+9801+10000}{1+2+3+4+\cdots+99+100}$.
5. The number $123456789b$ is divisible by 3. What is the difference between the greatest and smallest possible values of b?
6. How many ordered pairs of integers (a, b) exist such that the five digit number $7a93b$ is divisible by 33?
7. The number 39_{10} is equivalent to 124_k. What is k?
8. Compute $8 + 24 + 40 + \cdots + 2008 + 2024$.
9. How many 2-digit positive integers are divisible by 6 and have exactly 12 factors?

Practice 6: Number Theory Practice - SOLUTIONS

1. (a) 4
   (b) 40
   (c) $160 = ab$
   (d) It should!
2. $2233 = 2(870)+493 \implies 870 = 1(493)+377 \implies 493 = 1(377)+116 \implies 377 = 3(116)+29 \implies 116 = 4(29)$. Therefore, $\gcd(2233, 870) = 29$.
3. $12n + 7 = 2(5n + 1) + (2n + 5) \implies 5n + 1 = 2(2n + 5) + (n - 9) \implies 2n + 5 = 2(n - 9) + 23$. Therefore, $\gcd(12n + 7, 5n + 1) = \gcd(n - 9, 23)$. This greatest common divisor must divide 23, so the largest possible value is 23. (This occurs, for instance, when $n = 32$ or $n = 55$.)
4. $\frac{1^2+2^2+\cdots+99^2+100^2}{1+2+\cdots+99+100} = \frac{\frac{100(101)(201)}{6}}{\frac{100(101)}{2}} = \frac{201}{6} \cdot 2 = \frac{201}{3} = 67$
5. The sum $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + b = \frac{9(10}{2} + b = 45 + b$ must be divisible by 3, so b could equal $3, 6$, or 9. The desired difference is therefore $9 - 3 = 6$.
6. Since $33 \mid n$, we also have $11 \mid n$, so $7 - a + 9 - 3 + b = 16 - (a + b)$ is divisible by 11. Therefore, $a + b = 5$. Since $3 \mid n$, we also have $7+a+9-3+b = 13+(a+b)$ must be divisible by 3. Since $a+b = 5$, we get a sum of digits of 18, which is in fact divisible by 3. Therefore, any ordered pair (a, b) of positive integers satisfying $a+b = 5$ will yield a number divisible by 33. There are 6 such ordered pairs: $(0, 5)$, $(1, 4)$, $(2, 3)$, $(3, 2)$, $(4, 1)$, and $(5, 0)$.
7. $39 = k^2 + 2k + 4 \implies k^2 + 2k - 35 = 0 \implies (k + 7)(k - 5) = 0 \implies k = 5$
8. $8 + 24 + 40 + \cdots + 2008 + 2024 = 8(1 + 3 + 5 + \cdots + 251 + 253) = 8(127^2) = 129,032$
   (Note: 253 is the 127th odd number because $253 = 2(127) - 1$).
9. The integer must be divisible by 2 and 3. Our number n must therefore be of the form $n = 2^a \cdot 3^b k$ for some k. If $k = 1$ (no other prime factors of n), then we must have $(a + 1)(b + 1) = 12$, which means (a, b) could be $(0, 11)$, $(11, 0)$, $(1, 5)$, $(5, 1)$, $(2, 3)$, or $(3, 2)$. Only $(5, 1)$ and $(3, 2)$ would yield two digit numbers (96 and 72, respectively). If $k \neq 1$, then we must have another prime factor. With the two digit requirement, we choose $k = 5^c$. With 12 divisors, we get $(a + 1)(b + 1)(c + 1) = 12$. If $c > 1$, then the resulting n will be larger than 100, so we set $c = 1$, which yields possibilities of (a, b) as $(1, 2)$ and $(2, 1)$. These give $n = 90$ and $n = 60$. If $k = 7$, then the only possibility is $a = 2$ and $b = 1$, which leads 84. Therefore, there are exactly 5 such numbers n (60, 72, 84, 90, and 96).

Practice 7: Divisor Formulas

Date: Tuesday, September 3

1. NUMBER OF DIVISORS: Let n be an integer with prime factorization

$$n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$$

   Then the function

$$\tau(n) = (e_1 + 1)(e_2 + 1) \cdots (e_k + 1)$$

   gives the number of divisors of n. (τ is the Greek letter *tau*.)

   Explanation:

2. SUM OF DIVISORS: Consider the same n from (1). Then the function

$$\sigma(n) = (1 + p_1 + p_1^2 + \cdots + p_1^{e_1})(1 + p_2 + p_2^2 + \cdots + p_2^{e_2}) \cdots (1 + p_k + p_k^2 + \cdots + p_k^{e_k})$$

   gives the sum of the divisors of n. (σ is the Greek letter *sigma*.)

   Explanation:

3. TOTIENT FUNCTION: Consider the same n from (1). Then the totient function

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right)$$

   gives the number of positive integers less than n that are relatively prime with n.

   Explanation:

4. Every even number is of the form ________, where ________, and every odd number is of the form ____________, where ________.

Examples

1. For $n = 96$, compute $\tau(n)$, $\sigma(n)$, and $\phi(n)$.
2. Prove the following:
   (a) The sum of an even number and an odd number is odd
   (b) The square of an odd number is an odd number
   (c) The sum of 3 consecutive even numbers must be divisible by 6

Practice 7: Divisor Formulas - SOLUTIONS

Date: Tuesday, September 3

1. NUMBER OF DIVISORS: Let n be an integer with prime factorization
$$n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$$
Then the function
$$\tau(n) = (e_1 + 1)(e_2 + 1) \cdots (e_k + 1)$$
gives the number of divisors of n. (τ is the Greek letter *tau*.)

   Explanation: If you wanted to *build* a factor of n, then you would end up making a number by choosing its factors from the prime factors of n. For prime p_1, you could choose p_1 to any power from 0 ($p_1^0 = 1$ is indeed a factor of n) to e_1; this means you have $e_1 + 1$ choices. For each prime, in fact, you could choose anywhere from 0 up to the power itself, which explains the +1 in each factor. Your total number of possible numbers you can construct would then be the product of the number of choices you had for each prime factor.

2. SUM OF DIVISORS: Consider the same n from (1). Then the function
$$\sigma(n) = (1 + p_1 + p_1^2 + \cdots + p_1^{e_1})(1 + p_2 + p_2^2 + \cdots + p_2^{e_2}) \cdots (1 + p_k + p_k^2 + \cdots + p_k^{e_k})$$
gives the sum of the divisors of n. (σ is the Greek letter *sigma*.)

   Explanation: Expanding this will give every possible product of powers of factors of n, just like was explained in (1).

3. TOTIENT FUNCTION: Consider the same n from (1). Then the totient function
$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right)$$
gives the number of positive integers less than n that are relatively prime with n.

   Explanation: Starting at 1, every $p_1{}^{\text{th}}$ number is divisible by p_1. Therefore, if $p_1 \mid n$, then exactly $\frac{1}{p_1}$ of the integers less than n are divisible by p_1 and hence not relatively prime with n. We must subtract this fraction of integers from n, which gives us the term $\left(1 - \frac{1}{p_1}\right)$. Continuing in this fashion with the remaining prime factors of n gives us the fraction of all positive integers less than n that have no prime factors in common, i.e. are relatively prime to n.

4. Every even number is of the form $2k$, where $k \in \mathbb{Z}$, and every odd number is of the form $2k + 1$, where $k \in \mathbb{Z}$.

Examples

1. $96 = 2^5 \cdot 3^1$, so $\tau(96) = (5 + 1)(1 + 1) = 12$, $\sigma(96) = (1 + 2 + 4 + 8 + 16 + 32)(1 + 3) = 252$, and $\phi(96) = 96\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) = 96\left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = 32$.

2. Prove the following:

   (a) Let $m = 2k$ and $n = 2p + 1$, where $k, p \in \mathbb{Z}$. Then $m + n = 2k + 2p + 1 = 2(k + p) + 1 = 2q + 1$, where $q = k + p$ is an integer. Therefore, $m + n$ is odd.

   (b) Let $n = 2k + 1$ be any odd number. Then $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2m + 1$, where $m = 2k^2 + 2k$ is an integer. Therefore, n^2 is odd.

   (c) Let p, q, r be the three even numbers. Then $q = 2k$ for some integer k, which gives $p = 2k - 2$ and $r = 2k + 2$. Therefore, $p + q + r = (2k - 2) + 2k + (2k + 2) = 6k$, which is a multiple of 6.

Practice 8: Divisor Formulas Practice

Date: Wednesday, September 4

1. Determine the number of divisors of each of the following.
   (a) 24 (b) 144 (c) $8!$ (d) $a^2b^3c^4$, where a, b, c are unique primes
2. Find the sum of the divisors of 496.
3. Find the smallest positive integer with 12 factors.
4. How many numbers between 10 and 200 have an odd number of factors?
5. Prove that the square of an even number must be divisible by 4.
6. Prove that the square of an odd number must leave a remainder of 1 when divided by 4.
7. What is the largest number that the sum of 5 consecutive odd numbers is guaranteed to be divisible by? How about the sum of 6 consecutive odd numbers?
8. Take any two consecutive odd numbers. Now, multiply them and add 1. Repeat this process a few times.

   What do you think is happening here, and can you prove it?

Practice 8: Divisor Formulas Practice - SOLUTIONS

Date: Wednesday, September 4

1. Determine the number of divisors of each of the following.

   (a) $24 = 2^3 \cdot 3$, so $\tau(24) = (3+1)(1+1) = 8$

   (b) $144 = 2^4 \cdot 3^2$, so $\tau(144) = (4+1)(2+1) = 15$

   (c) $8! = 2(3)(4)(5)(6)(7)(8) = 2(3)(2^2)(5)(2\cdot3)(7)(2^3) = 2^6\cdot3^2\cdot5\cdot7$, so $\tau(8!) = (6+1)(2+1)(1+1)(1+1) = 84$

   (d) $\tau(a^2b^3c^4) = (2+1)(3+1)(4+1) = 60$

2. $496 = 2^4 \cdot 31$, so $\sigma(496) = (1+2+4+8+16)(1+31) = 992$

3. Our integer can take any of the forms p^11, p^5q, p^3q^2, or pqr^2, where p, q, r are unique primes. Testing these out with small primes gives that the smallest possible value is $5(3)(2^2) = 60$.

4. Based on the formula for $\tau(n)$, the only way a number can have an odd number of factors is if the power of every single one of its prime factors is even. If this is the case, then a factor of 2 can be pulled from each power of a prime and n can be rewritten as $\left(p_1^{e_1/2}p_2^{e_2/2}\cdot p_k^{e_k/2}\right)^2$. In short, n must be a perfect square, of which there are 11 between 10 and 200.

5. Let $n = 2k$. Then $n^2 = (2k)^2 = 4k^2$, which is a multiple of 4.

6. Let $n = 2k+1$. Then $n^2 = (2k+1)^2 = 4k^2+4k+1 = 4(k^2+k)+1$, which would leave a remainder of 1 when divided by 4.

7. Let n be the middle odd number. Then the five consecutive odds are $n-4, n-2, n, n+2$, and $n+4$. Adding these gives $5n$, so our sum must be divisible by at least 5. On the other hand, for 6 consecutive odds, let m be the median of all 6 numbers. m must be the average of the two middle odd numbers and must therefore be even, or of the form $2k$. Thus, our 6 consecutive odds are $2k-5, 2k-3, 2k-1, 2k+1, 2k+3$, and $2k+5$, which have a sum of $12k$ and hence be divisible by at least 12.

8. You should be getting perfect squares! Try representing the two consecutive odd numbers using the definition of an odd number.

Practice 9: Modular Arithmetic

Date: Tuesday, September 10

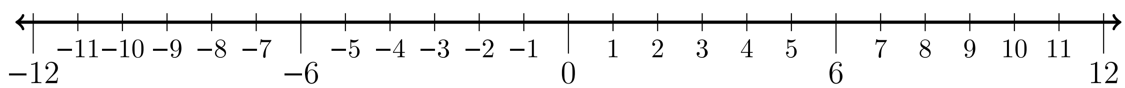
1. Let a be an integer. Then $a \pmod{n}$ gives
   - the ____________ when _____ is divided by _____
   - the __________________________ that _____ is from a multiple of _____

   The "mod" stands for *modulo*, and when two numbers are equidistant from a multiple of n (or would leave the same remainder when divided by n), we call them _______________ modulo n and use the symbol _____.

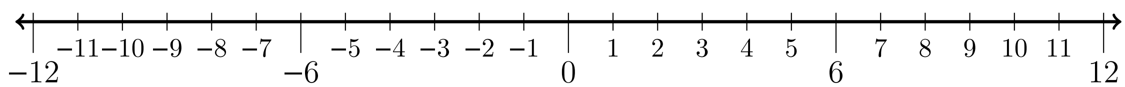
2. In modulo n, every integer is congruent to one of $\{0, 1, 2, 3, \ldots, n-1\}$, but it also congruent to infinitely other integers, including negative ones. It helps to view this on a number line.

   Examples: Suppose we're working in modulo 6.

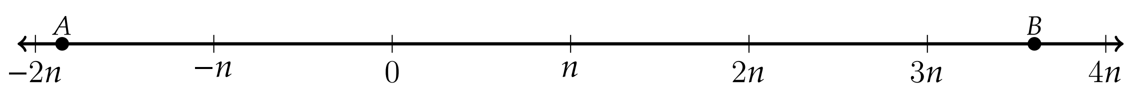
   (a) Explain why $9 \equiv 3 \equiv -3 \pmod{6}$.



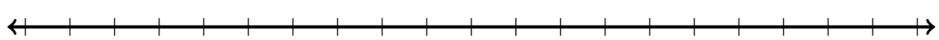
   (b) Find 2 numbers a and b, with $a < 0 < b$, such that $-7 \equiv a \equiv b \pmod{6}$.



   (c) How can you quickly find another number closer to 0 that is congruent to any number mod n? In the diagram, consider the values A and B.



   (d) Now, suppose we're in modulo 15. Find the two values of k, with $-14 < k < 14$, such that $87 \equiv k \pmod{15}$. Feel free to use the number line below to help.



   Practice

   (a) $7 \pmod{4} \equiv$

   (b) $25 \pmod{8} \equiv$

   (c) $-1 \pmod{10} \equiv$

   (d) $1694 \pmod{17} \equiv$

   (e) Find 5 numbers congruent to 3 modulo 9.

   (f) Find $k < 0$ such that $12 \equiv k \pmod{20}$.

3. Doing computations in this new arithmetic system is called modular arithmetic. There are many rules that make modular arithmetic much easier than traditional arithmetic.

   - Every number is congruent to infinitely many other numbers in mod n, so you can choose values that are convenient or small.
   - You can "mod out" - replace any number with another number it's congruent to - *before* computing any sum, difference, product, or exponentiation, and it won't affect the final value or congruence.

   Examples:

   (a) A food pantry gets a donation of 28 bags of apples. Each bag has 32 apples in it. If all of the apples are evenly distributed amongst 30 people, how many apples will be left over?

   (b) Compute $17(23)(18)(15) \pmod{12}$.

   (c) What is the remainder when 3^{13} is divided by 25?

Practice

1. Compute the 2 values closest to 0 that are congruent to the following. (This will result in 1 negative value and 1 positive value for each question).

   (a) $26 \pmod{16}$

   (b) $9 \pmod{5}$

   (c) $-15 \pmod{11}$

   (d) $-24 \pmod{7}$

   (e) $1,024 \pmod{10}$

   (f) $1,024 \pmod{100}$

2. How many integers n, where $0 \leq n \leq 1,000$, are congruent to $6 \pmod{8}$?

3. Compute $8! \pmod{23}$. (Note: $n!$ means $n(n-1)(n-2)\cdots(3)(2)(1)$.)

Practice 9: Modular Arithmetic - SOLUTIONS

Date: Tuesday, September 10

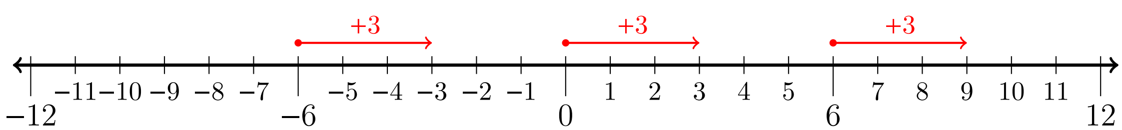
1. Let a be an integer. Then $a \pmod{n}$ gives
   - the remainder when a is divided by n
   - the signed distance that a is from a multiple of n (positive distance means *above* a multiple, negative distance means *below* a multiple)

   The "mod" stands for *modulo*, and when two numbers are equidistant from a multiple of n (or would leave the same remainder when divided by n), we call them congruent modulo n and use the symbol $\equiv$.

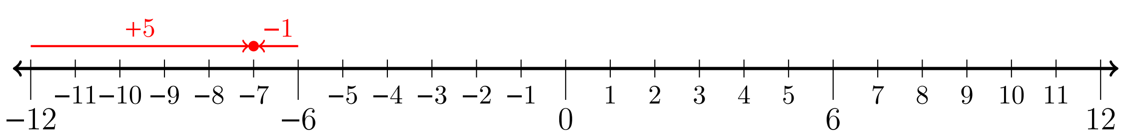
2. In modulo n, every integer is congruent to one of $\{0, 1, 2, 3, \ldots, n-1\}$, but it also congruent to infinitely other integers, including negative ones. It helps to view this on a number line.

   Examples: Suppose we're working in modulo 6.

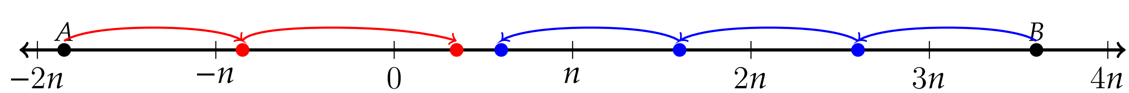
   (a) Each value is 3 greater than a multiple of 6.



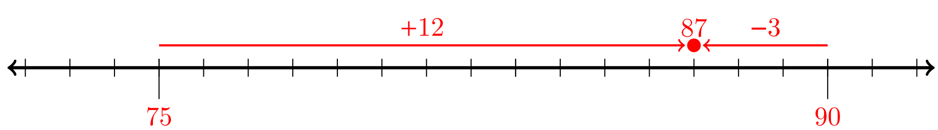
   (b) -7 is one less than a multiple of 6 and 5 greater than a multiple of 6, so $a = -1$ and $b = 5$ do the trick.



   (c)



   (d) The two values of k are 12 and -3.



   Practice

   (a) $7 \pmod{4} \equiv 3$

   (b) $25 \pmod{8} \equiv 1$

   (c) $-1 \pmod{10} \equiv 9$

   (d) $1694 \pmod{17} \equiv -6 \equiv 11$

   (e) $3 \equiv 12 \equiv 21 \equiv -6 \equiv -15 \pmod{9}$

   (f) -8

3. Examples:

   (a) $28(32) \pmod{30} \equiv -2(2) \equiv -4 \equiv 26$. There will be 26 apples left over.

   (b) $17(23)(18)(15) \pmod{12} \equiv 5(-1)(6)(3) \equiv -5(18) \equiv -5(6) \equiv -30 \equiv 6$.

(c) $3^{13} \pmod{15} \equiv (3^3)^4 \cdot 3 \equiv 27^4 \cdot 3 \equiv 2^4 \cdot 3 \equiv 16 \cdot 3 \equiv 1(3) \equiv 3$

Practice

1. (a) $26 \equiv 10 \equiv -6$
   (b) $9 \equiv 4 \equiv -1$
   (c) $-15 \equiv -4 \equiv 7$
   (d) $-24 \equiv -3 \equiv 4$
   (e) $1,024 \equiv 4 \equiv -6$
   (f) $1,024 \equiv 24 \equiv -76$

2. The smallest such integer is 6. The largest such integer is $998 = 6 + 992 = 6 + 8(124)$. Therefore, there are 125 such integers.

   There is another, more clever approach. You may notice that 1,000 is divisible by 8. Why is the answer precisely $\frac{1000}{8}$?

3. There are many ways to "mod out" here: I just chose the following.

$$8! \pmod{23} \equiv 2(3)(4)(5)(6)(7)(8) \equiv (3 \cdot 8)(4 \cdot 6)(5 \cdot 7)(2) \equiv 24 \cdot 24 \cdot 35 \cdot 2 \equiv 1 \cdot 1 \cdot 12 \cdot 2 \equiv 24 \equiv 1$$

Practice 10: Modular Arithmetic Practice

Date: Wednesday, September 11

Reminder: You can "mod out" at any point in the arithmetic process!

1. Compute $1 + 2 + 2^2 + 2^3 + \cdots + 2^{100} \pmod 8$.
2. Find a number n such that $n \equiv 5 \pmod 8$ and $n \equiv 4 \pmod 7$.
3. Compute $8! \pmod{13}$. (Recall that $n! = n(n-1)(n-2)\cdots(3)(2)(1)$ and is called "n factorial".)
4. Six waiters at a restaurant share tips each night, and over the course of a 3-day weekend, the total tips earned for the six waiters was $674, $597, and $371. If the tips are split evenly, how much is left over?
5. In traditional arithmetic, every integer has a multiplicative inverse, which is known as the *reciprocal.* For any integer n, the multiplicative inverse is $\frac{1}{n}$, and this is because

   $$n \cdot \frac{1}{n} = 1$$

   In modular arithmetic, the definition of an inverse is the same: the multiplicative inverse of an integer n is the integer n^{-1} such that

   $$n \cdot n^{-1} \equiv 1 \pmod n$$

   For instance, in modulo 10, the inverse of 3 is 7 because $3(7) \equiv 21 \equiv 1 \pmod{10}$ and the inverse of 9 is... itself!

   (a) Why is 9 its own inverse in modulo 10?

   (b) Find the inverse of the following.

   (i) 3 in modulo 8

   (ii) 5 in modulo 11

   (iii) 12 in modulo 17

   (iv) 7 in modulo 8

   (v) 6 in modulo 8

   (vi) Complete the following: in modulo n, the integer a only has a multiplicative inverse if ______________.
6. Compute $6^{800} + 7^{80} + 8^8 \pmod 8$.
7. For any number n, what is computing $n \pmod{10}$ the same thing as finding? How about $n \pmod{100}$ or $n \pmod{1000}$?
8. Compute $5^{210} \pmod{127}$.
9. Compute the hundreds digit of 314^{15}.

Practice 10: Modular Arithmetic Practice - SOLUTIONS

Date: Wednesday, September 11

Reminder: You can "mod out" at any point in the arithmetic process!

1. $1 + 2 + 2^2 \equiv 7 \pmod 8$, as every other term is a multiple of 8
2. Through inspection, you can simply compute that the smallest such integer is 53. The next such integer is 109. However, you may want to know a formal way of finding this!

   $n \equiv 4 \pmod 7$ means that n is 4 more than a multiple of 7, which gives $n = 7k + 4$ for some k. Now, since $n \equiv 5 \pmod 8$, we can substitute in $n = 7k + 4$ to get $7k + 4 \equiv 5 \pmod 8$. Subtracting 4 yields $7k \equiv 1 \pmod 8$. Now, to get rid of the 7, we multiply by its multiplicative inverse (see #5(iv)), which is 7. This gives $7(7k) \equiv 7 \pmod 8$, or $k \equiv 7 \pmod 8$ (since $49 \equiv 1 \pmod 8$). Now, we have that k is 7 more than a multiple of 8, or $k = 8m+7$ for some m. Substituting this into $n = 7k+4$ yields $n = 7(8m+7)+4 = 56m+53$. This means that every one of the desired integers is 53 more than a multiple of 56, or is equivalent to $53 \pmod{56}$. This is the set of integers $\{53, 109, 162, 215, \ldots\}$.
3. $8! \pmod{13} \equiv 8(7)(6)(5)(4)(3)(2)(1) \equiv (8 \cdot 5)(7 \cdot 3 \cdot 2)(6 \cdot 4) \equiv 40 \cdot 42 \cdot 24 \equiv 1 \cdot 3 \cdot (-2) \equiv -6 \equiv 7$
4. $674+597+371 \pmod 6 \equiv (600+74)+(600-3)+(360+11) \equiv 74+(-3)+11 \equiv (72+2)+(-3)+(12-1) \equiv 2 + (-3) + (-1) \equiv -2 \equiv 4$. There are \$6 left over.
5. (a) $9(9) \equiv 81 \equiv 1 \pmod{10}$
   (b) (i) 3
   (ii) 9
   (iii) 10
   (iv) 7
   (v) Doesn't exist!
   (vi) Complete the following: in modulo n, the integer a only has a multiplicative inverse if $\underline{\gcd(a, n) = 1}$.
6. $6^{800} + 7^{80} + 8^8 \pmod 8 \equiv (-2)^{800} + (-1)^{80} + 0 \equiv (2^4)^{200} + 1 \equiv 16^{200} + 1 \equiv 0 + 1 \equiv 1$
7. Last digit, last 2 digits, last 3 digits
8. $5^{105} \pmod{127} \equiv (5^3)^{35} \equiv 125^{35} \equiv (-2)^{35} \equiv ((-2)^7)^5 \equiv (-128)^5 \equiv (-1)^5 \equiv -1 \equiv 126$
9. $314^{15} \pmod{100} \equiv 14^{15} \equiv (14^2)^7 \cdot 14 \equiv (196)^7 \cdot 14 \equiv (-4)^7 \cdot 14 \equiv (-128 \cdot 128) \cdot 14 \equiv -28(-28)(14) \equiv 72(-2)(14^2) \equiv 144 \cdot 196 \equiv 44 \cdot (-4) \equiv -176 \equiv 24$. The hundreds digit is 2.

Practice 11: Cumulative Practice

Date: Tuesday, September 17

1. (2014 KSU) A *prim-prime* is a prime number that can be expressed as the sum of two prime numbers. What is the sum of the biggest *prim-prime* less than 100 and the smallest *prim-prime* that exists?
2. (2017 Augusta) Find the remainder when 2^{2017} is divided by 5.
3. (2024 Varsity State Tournament) Three consecutive integers are multiplied together to yield x. If x is divisible by 13, which of the following is not necessarily a divisor of x?
   (A) 6 (B) 26 (C) 39 (D) 52 (E) 78
4. The sum of the first k odd numbers, where $k > 1$, is a perfect cube. What is the smallest possible value of k?
5. (2023 Floyd County) What is the largest integer k for which $85!$ is divisible by 42^k?
6. (2024 Varsity State Tournament) Find the last two digits of 99^{2024}.
7. The first 8 terms of an arithmetic sequence add up to 208. If the first term is 5, what is the 10th term of the sequence?
8. (2024 Varsity State Tournament) If $S = 1 + 3 + 5 + \cdots + 4047$, what is the hundreds digit of S?
9. (2018 UGA Ciphering) The positive integers a and b each have exactly two prime factors: 2 and 3. If $a \nmid b$ and $b \nmid a$, what is the smallest that a can be?
10. (2019 UGA Ciphering) What is the largest positive integer k for which 3^k divides $\underbrace{999\ldots9}_{\text{2019 nines}}$?

Practice 11: Cumulative Practice - SOLUTIONS

Date: Tuesday, September 17

1. The sum of any two odd primes will be an even number, which cannot be prime. Therefore, every prim-prime must be the sum of an odd prime and the only even prime, 2. The smallest prim-prime is $2 + 3 = 5$ and the largest less than 100 is $2 + 97 = 99$, which gives a sum of 104.

2. $2^{2017} \pmod 5 \equiv (2^4)^{504} \cdot 2 \equiv 16^{504} \cdot 2 \equiv 1^{504} \cdot 2 \equiv 2$

3. Any three consecutive numbers must contain at least 1 even number and exactly 1 multiple of 3. This means that x must be divisible by $2(3) = 6, 2(13) = 26, 3(13) = 39$, and $2(3)(13) = 78$. This leaves 52, which x is not guaranteed to be divisible by, as the 3 numbers might not contain a multiple of 4. (For instance, the triplet $\{13, 14, 15\}$ gives a product that is not divisible by 4.)

4. The sum of the first k odd numbers is precisely k^2. If k^2 is also a cube, then it must actually be a *sixth* power (do you see why)? The smallest sixth power greater than 1 is $2^6 = 64 = 8^2 = 4^3$. Because 64 is 8^2, it must be the sum of the first 8 odd numbers, and hence $k = 8$.

5. We want to count how many factors of 42 exist within $85!$. Since $42 = 2(3)(7)$, this amounts to simply counting how many factors of $2(3)(7)$ we'll have. We will have far more factors of 2 and 3 than we need, so we can actually just count how many 7's there will be. There are $\lfloor \frac{85}{12} \rfloor = 12$ factors of 7 in $85!$, but $\lfloor \frac{12}{7} \rfloor = 1$ will have an additional 7 ($49 = 7^2$ has two factors of 7!). This gives a total of $7 + 1 = 8$ factors of 7, and hence 42 will be a factor 7 times, making $k = 7$.

6. The last two digits of a number are the same as that number modulo 100. We get

$$99^{2024} \pmod{100} \equiv (-1)^{2024} \equiv 1$$

   The last two digits are 01.

7. We use the formula for the sum of an arithmetic sequence $\frac{(F+L)(\#\text{ of terms})}{2}$, where F is the first term and L is the last term, to get

$$\frac{(5 + L)(8)}{2} = 208 \Rightarrow 5 + L = 52 \Rightarrow L = 47$$

   Since 52 is the 8th term of the sequence, we can solve $5 + 7d = 47$ to find the common difference of the sequence is $d = 6$. Therefore, the 10th term is $47 + 2(6) = 59$.

8. S is the sum of the first 2024 odds, so $S = 2024^2$. To get the hundreds digit, we can simply compute 2024^2 modulo 100.

$$2024^2 \pmod{100} \equiv 24^2 \equiv 576$$

   The hundreds digit is 5.

9. Since both a and b are divisible by 2 and 3, they must both be divisible by 6 but cannot divide each other. The smallest 2 such integers are 12 and 18, so $a = 12$.

10. The number is clearly divisible by $9 = 3^2$. Dividing by 9 gives us $\underbrace{111\ldots1}_{2019\text{ ones}}$. Now, the sum of digits is 2019. Since $2 + 0 + 1 + 9 = 12$ is divisible by 3 but not 9, the number $\underbrace{111\ldots1}_{2019\text{ ones}}$ is divisible by 3 but not 9. This means the original number was divisible by $9 = 3^2$ and by 3 again, for a total of 3 times. Therefore, $k = 3$.

Practice 12: Number Theory Post-Test

Date: Wednesday, September 18

Name: ____________________________

1. How many prime factors does 1914 have?

   Answer: __________

2. The three digit number abc is divisible by 5 and 11. If $a, b,$ and c are all unique and $a \neq 0$, how many possible values of a are there?

   Answer: __________

3. The least common multiple of $5, 6,$, and k is 420. What is the least possible value of k?

   (A) 28

   (B) 56

   (C) 84

   (D) 420

4. Compute $12 + 24 + 36 + 48 + \cdots 1200 \pmod 7$.

   (A) 0

   (B) 1

   (C) 3

   (D) 5

5. Compute $(98^2 + 89^2)(123456789) \pmod{10}$.

   Answer: __________

6. How many positive integers less than 100 have at least one factor in common with 100 besides 1?

   (A) 40

   (B) 50

   (C) 60

   (D) 90

7. What is the sum of all of the divisors of 180?

   (A) 258

   (B) 360

   (C) 456

   (D) 546

8. Find $\gcd(2301, 1711)$.

   (A) 59

   (B) 177

   (C) 531

   (D) 590

9. A number is a *palindrome* if it reads the same forwards as backwards. for instance, 838 and 777 are palindromes, while 102 and 551 are not. How many integers between 100 and 1000 are palindromes that have a middle digit of 7 and are divisible by 9?

   (A) 1

   (B) 2

   (C) 3

   (D) 4

10. If a and b are positive integers, how many ordered pairs (a, b) exist that satisfy $(a+1)(b+1) = 150$? (Note: It's called an *ordered* pair because the order matters, so, for instance, $(a, b) \neq (b, a)$.)

    Answer: __________

11. The fraction $\dfrac{1+3+5+\cdots+77+79}{1+2+3+\cdots+39+40}$ can be written in the simplified form $\dfrac{a}{b}$. If $a + b = k^2$, what is k?

    (A) 11

    (B) 29

    (C) 39

    (D) 40

Chapter 2: Introductory Algebra

Practice 13: Factoring & Finding Zeros

Date: Tuesday, September 24

1. To factor is to rewrite an expression as a ________.
2. To factor is to simply reverse engineer ________________.
3. Factoring well comes down to three things: knowing how to __________, knowing certain ____________, and being ____________.

   Examples: Expand the following.

   (a) $(x-2)(x+5)$

   (b) $(3x+4)(2x-1)$

   (c) $(5x+3)^2$

   (d) $(1-9y^2)(1+9y^2)$

   (e) $(x-2)(x^2+2x+4)$

4. It is generally beneficial to "pull out" or "undistribute" a greatest common divisor of multiple terms.
5. For three-term quadratics ax^2+bx+c, it helps to factor by considering the ax^2 and c terms, as these are singular products. Trial and error can then lead to a correct value of b.

   Examples: Factor the following.

   (a) $100xy-4y^2$

   (b) $x^2+9x+20$

   (c) x^2-x-30

   (d) $4x^2-5x-6$

   (e) $9y^2+12y+4$

   (f) $25-x^2$

6. Factoring is useful for many things, not least of all computing ________, or input values that will produce output values of 0. This is useful because finding any output values besides 0 is generally quite a bit more challenging. This is due to the following fact: if $ab = 0$, then either ______ or ______.

   In short, if you have a product equal to 0, then one of the factors must equal 0. You can use this to find solutions to equations quickly by turning equations into the form "function = 0".

   Example: Find a such that $5a^3 - 2a^2 + 1 = (3a - 1)^2$.

7. Factoring Identities

Name	Identity
Difference of Squares	
Difference of Cubes	
Sum of Cubes	

Practice

1. Factor $3x^2 - 10x + 8$.
2. What is the product of the 2 values of p such that $9p^2 = 73p - 8$?
3. What is the sum of the 2 values of x for which $2x^2 - 7 = (x - 1)^2$?
4. Find the prime factorization of 3599. (Hint: Can you rewrite 3599 as $x^2 - y^2$ for some x and y?)
5. Factor $(4x - 1)^2 - (3x + 2)^2$.

Practice 13: Factoring & Finding Zeros - SOLUTIONS

Date: Tuesday, September 24

1. To factor is to rewrite an expression as a product.
2. To factor is to simply reverse engineer multiplication.
3. Factoring well comes down to three things: knowing how to multiply, knowing certain identities, and being flexible.

   Examples: Expand the following.

   (a) $(x-2)(x+5) = x^2 - 2x + 5x - 10 = x^2 + 3x - 10$
   (b) $(3x+4)(2x-1) = 6x^2 - 3x + 8x - 4 = 6x^2 + 5x - 4$
   (c) $(5x+3)^2 = 25x^2 + 2(5x)(3) + 3^2 = 25x^2 + 30x + 9$
   (d) $(1-9y^2)(1+9y^2) = 1 + 9y^2 - 9y^2 - 81y^4 = 1 - 81y^4$
   (e) $(x-2)(x^2+2x+4) = x^3 + 2x^2 + 4x - 2x^2 - 4x - 8 = x^3 - 8$

4. It is generally beneficial to "pull out" or "undistribute" a greatest common divisor of multiple terms.
5. For three-term quadratics $ax^2 + bx + c$, it helps to factor by considering the ax^2 and c terms, as these are singular products. Trial and error can then lead to a correct value of b.

   Examples: Factor the following.

   (a) $100xy - 4y^2 = 4y(25x - y)$
   (b) $x^2 + 9x + 20 = (x+4)(x+5)$
   (c) $x^2 - x - 30 = (x-6)(x+5)$
   (d) $4x^2 - 5x - 6 = (4x+3)(x-2)$
   (e) $9y^2 + 12y + 4 = (3y+2)^2$
   (f) $25 - x^2 = (5+x)(5-x)$

6. Factoring is useful for many things, not least of all computing zeros, or input values that will produce output values of 0. This is useful because finding any output values besides 0 is generally quite a bit more challenging. This is due to the following fact: if $ab = 0$, then either $a = 0$ or $b = 0$.

   In short, if you have a product equal to 0, then one of the factors must equal 0. You can use this to find solutions to equations quickly by turning equations into the form "function = 0".

   Example: Find a such that $5a^3 - 2a^2 + 1 = (3a-1)^2$.

   Solution:

$$\begin{aligned} 5a^3 - 2a^2 + 1 &= 9a^2 - 6a + 1 \\ 5a^3 - 11a^2 + 6a &= 0 \\ a(5a^2 - 11a + 6) &= 0 \\ a(5a-6)(a-1) &= 0 \end{aligned}$$

   This gives $a = 0, 5a - 6 = 0$, or $a - 1 = 0$, which results in $a = 0, a = \frac{6}{5}, a = 1$.

7. Factoring Identities

Name	Identity
Difference of Squares	$x^2 - y^2 = (x + y)(x - y)$
Difference of Cubes	$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$
Sum of Cubes	$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

Practice

1. $(3x + 2)(x - 4)$
2. $9p^2 - 73p + 8 = 0 \implies (9p - 1)(p - 8) = 0 \implies p = \frac{1}{9}, p = 8$. The product of the solutions is $\frac{8}{9}$.
3. $2x^2 - 7 = x^2 - 2x + 1 \implies x^2 + 2x - 8 = 0 \implies (x + 4)(x - 2) = 0 \implies x = -4, x = 2$. The sum of the 2 values is -2.
4. $3599 = 3600 - 1 = 60^2 - 1^2 = (60 + 1)(60 - 1) = 61(59)$
5. Method 1: Expand, simplify, then factor.

$$\begin{aligned}(4x - 1)^2 - (3x + 2)^2 &= 16x^2 - 8x + 1 - 9x^2 - 12x - 4 \\ &= 7x^2 - 20x - 3 \\ &= (7x + 1)(x - 3)\end{aligned}$$

Method 2: Use difference of squares!

$$\begin{aligned}(4x - 1)^2 - (3x + 2)^2 &= ((4x - 1) + (3x + 2))((4x - 1) - (3x + 2)) \\ &= (7x + 1)(x - 3)\end{aligned}$$

Practice 14: Factoring Practice

Date: Wednesday, September 25

Recall the following factoring facts.

- To factor is to rewrite as a product. To be quick at factoring, you just need to understand how to multiply.
- When factoring a 3-term quadratic, it helps to focus on the squared term and the constant term. You can then play around with multiplying out your possible factorization to see if the linear term (the middle term) will be correct.
- Square of a binomial: $(x \pm y)^2 = x^2 \pm 2xy + y^2$
- Sum of cubes: $x^3 + y^3 = (x + y)(x^2 - xy + y^2$
- Difference of squares: $x^2 - y^2 = (x + y)(x - y)$
- Difference of cubes: $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

Practice

1. Factor the following completely, if possible.
   (a) $f(y) = y^2 - 3y - 54$
   (b) $g(x) = x^2 - 49$
   (c) $h(x) = 6x^2 - 13x + 5$
   (d) $s(x) = x^3 + x^2 - 6x$
   (e) $f(x) = 11x^2 - 11x + 22$
   (f) $g(x) = 32x - 8$
   (g) $f(x) = x^2 + 7x - 10$
   (h) $T(x) = 6 - 24x^2$
   (i) $f(a) = 9a^4 - 16$
   (j) $g(x) = 3x^2 + 5x - 4$
   (k) $h(s) = s^4 - 1$
   (l) $f(x) = -2x^2 + 8x$
   (m) $p(t) = 3t^2 - 27t - 30$
   (n) $m(x) = 2x^3 - 50x$
   (o) $T(y) = 81 - 9y^2$
   (p) $s(x) = 9x^2 + 6x + 1$
   (q) $A(x) = 6x^2 - 7x - 20$
   (r) $f(x) = 20x^2 + 27x - 14$
   (s) $t(n) = n^2 + 6n + 9$
   (t) $f(x) = x^2 + 14x - 51$

   (a)
   (b)
   (c)
   (d)
   (e)
   (f)
   (g)
   (h)
   (i)
   (j)
   (k)
   (l)
   (m)
   (n)
   (o)
   (p)
   (q)
   (r)
   (s)
   (t)

2. Find the prime factorization of 2491.
3. The sum of two numbers is 12. If the sum of the squares of the two numbers is 40, what is the product of the two numbers?
4. Solve for x: $(3x + 5)^2 - (2x - 1)^2 = x + 6$.
5. Compute $1995^2 + 10(1995) + 5^2$.
6. If $x + 2y = 8$ and $x^2 - 4y^2 = 120$, compute $2y - x$.
7. EXTENSION:
   (a) Construct a quadratic function that has solutions of 4 and 7.
   (b) What is the sum of the solutions? What is the product of the solutions?
   (c) Expand the quadratic and put it in standard form $ax^2 + bx + c$.
   (d) Where do you see the sum of the solutions? Where do you see the product of the solutions?
   (e) Let's see if you can try an application of this: find the sum of the solutions of $x^2 + 13x + 5 = 0$.

Practice 14 - Factoring Practice SOLUTIONS

Date: Wednesday, September 25

1. (a) $f(y) = (y-9)(y+6)$
   (b) $g(x) = (x+7)(x-7)$
   (c) $h(x) = (3x-5)(2x-1)$
   (d) $s(x) = x(x+3)(x-2)$
   (e) $f(x) = 11(x^2-x+2)$
   (f) $g(x) = 8(4x-1)$
   (g) $g(x) = x^2+7x-10$
   (h) $T(x) = 6(1+2x)(1-2x)$
   (i) $f(a) = (3a^2+4)(3a^2-4)$
   (j) $g(x) = 3x^2+5x-4$
   (k) $h(s) = (s^2+1)(s+1)(s-1)$
   (l) $f(x) = -2x(x-4)$
   (m) $p(t) = 3(t-10)(t+1)$
   (n) $m(x) = 2x(x+5)(x-5)$
   (o) $T(y) = 9(3+y)(3-y)$
   (p) $s(x) = (3x+1)^2$
   (q) $A(x) = (3x-4)(2x+5)$
   (r) $f(x) = (5x-2)(4x+7)$
   (s) $t(n) = (n+3)^2$
   (t) $f(x) = (x+17)(x-3)$

2. $2491 = 2500 - 9 = 50^2 - 3^3 = (50+3)(50-3) = 53(47)$

3. Let the two numbers be a and b. Then we have $a+b = 12$ and $a^2+b^2 = 40$. We can use the identity $(a+b)^2 = a^2+b^2+2ab$ to get

$$\begin{aligned}(a+b)^2 &= a^2+b^2+2ab\\ 12^2 &= 40+2ab\\ 104 &= 2ab\\ 52 &= ab\end{aligned}$$

   The product of the numbers is 52.

4. $(3x+5)^2-(2x-1)^2 = x+6 \implies 9x^2+30x+25-(4x^2-4x+1) = x+6 \implies 5x^2+34x+24 = x+6 \implies 5x^2+33x+18 = 0 \implies (5x+3)(x+6) = 0 \implies x = -\frac{3}{5}, x = -6$

5. This can be rewritten as $1995^2+2(1995)(5)+5^2$, which is just the identify $(a+b)^2 = a^2+2ab+b^2$ where $a = 1995$ and $b = 5$. Therefore, the expression equals $(1995+5)^2 = 2000^2 = 4,000,000$.

6. We use the difference of squares identity.

$$\begin{aligned}x^2-4y^2 &= 120\\ (x+2y)(x-2y) &= 120\\ 8(x-2y) &= 120\\ x-2y &= 15\\ -15 &= 2y-x\end{aligned}$$

   We get $2y-x = -15$.

7. EXTENSION:
   (a) $(x-4)(x-7)$
   (b) The sum is 11 and the product is 28.
   (c) $x^2-11x+28$
   (d) The sum is the opposite of the coefficient of x and the product is equal to the constant term.
   (e) The sum of the solutions is -13.

Practice 15: Expansion

Date: Tuesday, October 8

CANCELLED - Virtual day

1. It is useful to be able to expand powers of polynomials - especially binomials quickly. This leads to the development of what is known as the ________________________________ and ___________________________.

$$(x+y)^0 =$$

$$(x+y)^1 =$$

$$(x+y)^2 =$$

$$(x+y)^3 =$$

$$(x+y)^4 =$$

$$(x+y)^5 =$$

2. The triangle of coefficients above is called ___________________________, and each number in it is known as a _______________ _______________.

3. The binomial coefficient $\binom{n}{k}$ gives how many ways you can select k objects out of n objects.

    - ♦ It is equal to the kth entry in the nth row of Pascal's triangle, remembering there is a 0th row and a 0th entry in each row.
    - ♦ It is equivalently equal to

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

Examples:

(a) Compute $\binom{4}{3}$, $\binom{5}{2}$, and $\binom{7}{4}$.

(b) What is the coefficient of x^3 in the expansion of $(x+1)^9$?

4. We can utilize Pascal's Triangle to expand a power of any binomial by using the diagram above, only needing to remember that "x" and "y" might be terms that need ______________ around them before they can be exponentiated! In particular, in the expansion of $(x + y)^n$, each term will have the form

$$\binom{n}{k} x^k y^{n-k}$$

This makes sense because, in the expansion $(x+y)(x+y)\cdots(x+y)$, the term $\binom{n}{k} x^k y^{n-k}$ means that you're choosing k ____, the rest $(n-k)$ ____, and there are $\binom{n}{k}$ ways to choose this many ____.

Examples:

(a) Expand $(x + 2)^4$.

(b) Expand $(2x - 1)^3$.

Practice

1. If $(a - 2b)^5$ is expanded, what will the coefficient of a^2b^3 be?
2. A teacher has a bag with 7 marbles in it, each with a different color (red, orange, yellow, green, blue, purple, and black). A student is going to draw 3 marbles out without replacement and the teacher will give them 10 bonus points on a test if the student is able to draw either the red, orange, and yellow marbles together (in any order) or the green, blue, and purple together (in any order). What is the simplified probability that the student will get the bonus points?
3. What is the value of the constant term in the expansion of $(x^2 + \frac{3}{x^2})^6$?

Practice 15: Expansion - SOLUTIONS

Date: Tuesday, October 8

1. It is useful to be able to expand powers of polynomials - especially binomials quickly. This leads to the development of what is known as the binomial coefficient and Pascal's Triangle.

$$(x+y)^0 =$$

$$(x+y)^1 = x+y$$

$$(x+y)^2 = x^2+2xy+y^2$$

$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$

$$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$$

$$(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$$

2. The triangle of coefficients above is called ____________________, and each number in it is known as a ____________ ____________.

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
\iddots & & & & & & \vdots & & & & & & \ddots
\end{array}$$

3. The binomial coefficient $\binom{n}{k}$ gives how many ways you can select k objects out of n objects.

   - ♦ It is equal to the kth entry in the nth row of Pascal's triangle, remembering there is a 0th row and a 0th entry in each row.
   - ♦ It is equivalently equal to

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

   Examples:

   (a) $\binom{4}{3} = 4$, $\binom{5}{2} = 10$, $\binom{7}{4} = \frac{7(6)(5)(4)}{4(3)(2)(1)} = 7(5) = 35$

   (b) The term will be $\binom{9}{3}x^3(1)^6$, so the coefficient is simply $\binom{9}{3} = \frac{9(8)(7)}{3(2)(1)} = 3(4)(7) = 84$.

4. We can utilize Pascal's Triangle to expand a power of any binomial by using the diagram above, only needing to remember that "x" and "y" might be terms that need ____________ around them before they can be exponentiated! In particular, in the expansion of $(x+y)^n$, each term will have the form

$$\binom{n}{k}x^k y^{n-k}$$

This makes sense because, in the expansion $(x+y)(x+y)\cdots(x+y)$, the term $\binom{n}{k}x^k y^{n-k}$ means that you're choosing k $\underline{x}$'s, the rest $(n-k)$ $\underline{y}$'s, and there are $\binom{n}{k}$ ways to choose this many $\underline{x}$'s.

Examples:

(a) $(x+2)^4 = x^4 + 4x^3(2) + 6x^2(2)^2 + 4x(2)^3 + 2^4 = x^4 + 8x^3 + 24x^2 + 32x + 16$

(b) $(2x-1)^3 = (2x)^3 + 3(2x)^2(-1) + 3(2x)(-1)^2 + (-1)^3 = 8x^3 - 12x + 6x - 1$

Practice

1. The term will be $\binom{5}{2}a^2(-2b)^3 = 10a^2(-8b^3) = -80a^2b^3$, so the coefficient is -80.
2. There are 12 ways to win (for each set of 3 marbles, there are 6 possible orderings that could win; for instance, {ROY, RYO, YRO, YOR, ORY, OYR}) and $\binom{7}{3} = \frac{7(6)(5)}{3(2)} = 35$ total ways to choose 3 marbles out of 7. The probability of winning is therefore $\frac{12}{35}$.
3. The term will be constant only if the powers of x in the numerator and denominator are equal. If we look at the entire expansion (inefficient, but helpful to see what's going on), we get

$$(x^2)^6 + 6(x^2)^5\left(\frac{3}{x^2}\right) + 15(x^2)^4\left(\frac{3}{x^2}\right)^2 + 20(x^2)^3\left(\frac{3}{x^2}\right)^3 + 15(x^2)^2\left(\frac{3}{x^2}\right)^4 + 6(x^2)\left(\frac{3}{x^2}\right)^5 + \left(\frac{3}{x^2}\right)^6$$

If we ignore the coefficients for a moment, the powers of x in this expression will simplify into

$$x^{12} + \frac{x^{10}}{x^2} + \frac{x^8}{x^4} + \frac{x^6}{x^6} + \frac{x^4}{x^8} + \frac{x^2}{x^{10}} + \frac{1}{x^{12}}$$

The middle term will be a constant. Therefore, the constant term is the coefficient of this term, or $20(3^3) = 540$.

$$\begin{aligned}
\sin^2\left(\frac{\pi}{24}\right)\cos^2\left(\frac{\pi}{24}\right) &= \left(\sin\left(\frac{\pi}{24}\right)\cos\left(\frac{\pi}{24}\right)\right)^2 \\
&= \left(\frac{1}{2}\sin\left(\frac{\pi}{12}\right)\right)^2 \\
&= \left(\frac{1}{2}\sin\left(\frac{1}{2}\left(\frac{\pi}{6}\right)\right)\right)^2 \\
&= \left(\frac{1}{2}\sqrt{\frac{1-\sqrt{3}/2}{2}}\right)^2 \\
&= \frac{2-\sqrt{3}}{16}
\end{aligned}$$

Practice 16: Practice

Date: Wednesday, October 9

CANCELLED - Virtual Day

Practice 17: Math League #1

Date: Tuesday, October 15

Practice 18: AMC Prep

Date: Tuesday, October 22

The AMC A will be on Wednesday, November 6 during Math Team practice. Be prepared to stay until 4:45 or 5:00 pm! **You will need a laptop!**

You will take the AMC 10 if you are in 9th or 10th grade and the AMC 12 if you are in 11th or 12th grade. You will be given hard copies of practice problems during Math Team practice this week and next week, but you can also find all of the previous AMCs and solutions at `https://artofproblemsolving.com/wiki/index.php/AMC_Problems_and_Solutions`.

Both the AMC 10 and AMC 12 are 25-question, 75-minute multiple choice tests. You get 6 points for every correct answer, 1.5 points for each blank answer, and 0 points for incorrect answers. Calculators are prohibited.

Practice 19: AMC Prep

Date: Wednesday, October 23

The AMC A will be on Wednesday, November 6 during Math Team practice. Be prepared to stay until 4:45 or 5:00 pm! **You will need a laptop!**

You will take the AMC 10 if you are in 9th or 10th grade and the AMC 12 if you are in 11th or 12th grade. You will be given hard copies of practice problems during Math Team practice this week and next week, but you can also find all of the previous AMCs and solutions at `https://artofproblemsolving.com/wiki/index.php/AMC_Problems_and_Solutions`.

Both the AMC 10 and AMC 12 are 25-question, 75-minute multiple choice tests. You get 6 points for every correct answer, 1.5 points for each blank answer, and 0 points for incorrect answers. Calculators are prohibited.

Practice 20: AMC Prep

Date: Tuesday, October 29

The AMC A will be on Wednesday, November 6 during Math Team practice. Be prepared to stay until 4:45 or 5:00 pm! **You will need a laptop!**

You will take the AMC 10 if you are in 9th or 10th grade and the AMC 12 if you are in 11th or 12th grade. You will be given hard copies of practice problems during Math Team practice this week and next week, but you can also find all of the previous AMCs and solutions at `https://artofproblemsolving.com/wiki/index.php/AMC_Problems_and_Solutions`.

Both the AMC 10 and AMC 12 are 25-question, 75-minute multiple choice tests. You get 6 points for every correct answer, 1.5 points for each blank answer, and 0 points for incorrect answers. Calculators are prohibited.

Practice 21: Kennesaw State Contest

Date: Wednesday, October 30

The Kennesaw State University Math Contest (Part I) will be administered this day during Math Team practice. Be prepared to stay until 5:00 pm!

The KSU contest is a 25-question, 90-minute multiple choice test. You get 6 points for every correct answer, 2 points for each blank answer, and 0 points for incorrect answers. Calculators are prohibited.

Practice 22: AMC A!!

Date: Wednesday, November 6

Today is the AMC A! Be prepared to stay until 4:45 or 5:00 pm, and bring your laptop!

Practice 23: AMC B!!

Date: Tuesday, November 12

Today is the AMC B! Be prepared to stay until 4:45 or 5:00 pm, and bring your laptop!

Practice 24: Math League #2!

Date: Wednesday, November 13

Practice 25: Vièta's Relations

Date: Tuesday, November 19

Today, we'll be studying something known as Viète's relations. These describe the relationship between the ________ and ________________ of a polynomial.

THE QUADRATIC CASE

Let $f(x) = (x - p)(x - q)$ be a quadratic function.

1. What are the zeros of f?
2. Expand $f(x)$ and rewrite it in the form $x^2 + bx + c$.
3. How is the coefficient b related to the zeros of f?
4. How is the coefficient c related to the zeros of f?

This relationship will always hold for a quadratic $f(x) = (x - p)(x - q)$. In fact, it also holds for a quadratic $f(x) = a(x - p)(x - q)$ if you first divide everything by ____.

VIÈTA'S RELATION #1

If p and q satisfy $ax^2 + bx + c = 0$, then

$$p + q =$$

$$pq =$$

THE CUBIC CASE

Let $g(x) = (x - p)(x - q)(x - r)$ be a cubic function.

1. What are the zeros of f?
2. Expand $f(x)$ and rewrite it in the form $x^3 + bx^2 + cx + d$.
3. How is the coefficient b related to the zeros of g?
4. How is the coefficient c related to the zeros of g?
5. How is the coefficient d related to the zeros of g?

VIÈTA'S RELATION #2

If p, q, and r satisfy $ax^2 + bx + c = 0$, then

$$p + q + r =$$

$$pq + pr + qr =$$

$$pqr =$$

Examples

1. Find the sum and product of the roots of $f(x) = 3x^2 + 12x - 5$.

2. If $x = 4$ and $x = 3$ are two of the solutions of $x^3 + bx^2 + 9x = 2$, what is other solution?

Practice

1. Let x and y satisfy $a^2 + 1012a = 2024$. Compute $\frac{1}{x} + \frac{1}{y}$.

2. Let a, b, and c be the zeros of $f(x) = 2x^3 - 4x^2 + 5x - 3$. Compute $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

Practice 25: Vièta's Relations - SOLUTIONS

Date: Tuesday, November 19

Today, we'll be studying something known as Viète's relations. These describe the relationship between the zeros and coefficients of a polynomial.

THE QUADRATIC CASE

Let $f(x) = (x-p)(x-q)$ be a quadratic function.

1. What are the zeros of f?

$$x = p, x = q$$

2. Expand $f(x)$ and rewrite it in the form $x^2 + bx + c$.

$$\begin{aligned}(x-p)(x-q) &= x^2 - px - qx + pq \\ &= x^2 + (-p-q)x + pq \\ &= x^2 + (-(p+q))x + pq\end{aligned}$$

3. How is the coefficient b related to the zeros of f?

b is the opposite of the sum of the zeros.

4. How is the coefficient c related to the zeros of f?

c is the product of the zeros.

This relationship will always hold for a quadratic $f(x) = (x-p)(x-q)$. In fact, it also holds for a quadratic $f(x) = a(x-p)(x-q)$ if you first divide everything by a.

VIÈTA'S RELATION #1

If p and q satisfy $ax^2 + bx + c = 0$, then

$$\begin{aligned}p + q &= -\frac{b}{a} \\ pq &= \frac{c}{a}\end{aligned}$$

THE CUBIC CASE

Let $g(x) = (x-p)(x-q)(x-r)$ be a cubic function.

1. What are the zeros of f?

$$x = p, x = q, x = r$$

2. Expand $f(x)$ and rewrite it in the form $x^3 + bx^2 + cx + d$.

$$\begin{aligned}(x-p)(x-q)(x-r) &= (x^2 - px - qx + pq)(x-r) \\ &= x^3 - rx^2 - px^2 + prx - qx^2 + qrx + pqx - pqr \\ &= x^3 - (p+r+q)x^2 + (pq+pr+qr)x - pqr\end{aligned}$$

3. How is the coefficient b related to the zeros of g?

b is the opposite of the sum of the zeros.

4. How is the coefficient c related to the zeros of g?

   c is the sum of the product of each *pair* of zeros (the "pairwise products").

5. How is the coefficient d related to the zeros of g?

   d is the opposite of the product of the zeros.

VIÈTA'S RELATION #2

If p, q, and r satisfy $ax^2 + bx + c = 0$, then

$$p + q + r = -\frac{b}{a}$$

$$pq + pr + qr = \frac{c}{a}$$

$$pqr = -\frac{d}{a}$$

TIP

Starting at the nth coefficient that isn't the leading coefficient, where b is the "first" coefficient, Viète's relations say

the nth coefficient is equal to the pairwise products of n roots at a time, times $(-1)^n$ (alternate signs, starting with a negative)

Examples

1. Find the sum and product of the roots of $f(x) = 3x^2 + 12x - 5$.

   The sum is $-\frac{12}{3} = 4$ and the product is $-\frac{5}{3}$.

2. If $x = 4$ and $x = 3$ are two of the solutions of $x^3 + bx^2 + 9x = 2$, what is other solution?

   9 is the sum of pairwise products, so if we let a be the other root, we get $4a + 3a + 4(3) = 9$, or $7a = -3$. Therefore, the other root is $a = x = -\frac{3}{7}$.

Practice

1. Let x and y satisfy $a^2 + 1012a = 2024$. Compute $\frac{1}{x} + \frac{1}{y}$.

$$a^2 + 1012a = 2024 \implies a^2 + 1012a - 2024 = 0 \quad \text{(set equal to 0)}$$

$$\frac{1}{x} + \frac{1}{y} = \frac{y}{xy} + \frac{x}{y} \quad \text{(common denominator)}$$

$$= \frac{x + y}{xy} \quad \text{(simplify)}$$

$$= \frac{-1012}{-2024} \quad \text{(using Viète's)}$$

$$= \frac{1}{2}$$

2. Let a, b, and c be the zeros of $f(x) = 2x^3 - 4x^2 + 5x - 3$. Compute $\frac{1}{a} + \frac{1}{b} + \frac{1}{a}$.

   This is very similar to the last problem. We find a common denominator and get that the desired expression is equivalent to $\frac{ab+ac+bc}{abc}$, which, by Viète's, is $\frac{5/2}{-3/2} = -\frac{5}{3}$.

Practice 26: Vièta's Relations Practice

Date: Wednesday, November 20

1. Find the sum of the roots of $f(x) = \frac{2024}{2025}x^2 - 2024x + 1$.
2. One of the roots of the equation $x^3 - 69x^2 + 1146x = 2024$ is $x = 2$. Find the sum and the product of the other two roots.
3. Let a and b be the two roots of $x^2 - 25x + 8$. Find $(a + 1)(b + 1)$.
4. Let x and y be the solutions of $a^2 - 12a + 14 = 0$. Compute $x^2 + y^2$.
   (Hint: Start by expanding the expression $(x + y)^2$.)
5. Let $f(x) = x^3 - 5x^2 + 2x - 7$. If the roots of g are p, q, and r, compute $p^2qr + pq^2r + pqr^2$.
6. (AMC 10A 2003) What is the sum of the reciprocals of the solutions of $\frac{2003}{2004}x + 1 + \frac{1}{x} = 0$?
   (Hint: Turn it into a quadratic!)
7. (AMC 10A 2006) Let a and b be the roots of the equations $x^2 - mx + 2 = 0$. Suppose that $a + \frac{1}{b}$ and $b + \frac{1}{a}$ are the roots of the equation $x^2 - px + q = 0$. What is q?

Practice 26: Vièta's Relations Practice - SOLUTIONS

Date: Wednesday, November 20

1. $-\frac{-2024}{2024/2025} = 2025$
2. If a and b are the other roots, then $2 + a + b = 69$ and $2ab = 2024$, so $a + b = 67$ and $ab = 1012$.
3. $(a+1)(b+1) = ab + (a+b) + 1 = 8 + 25 + 1 = 34$
4. $(x+y)^2 = x^2 + y^2 + 2xy \Rightarrow 12^2 = x^2 + y^2 + 2(14) \Rightarrow x^2 + y^2 = 116$
5. $p^2qr + pq^2r + pqr^2 = p(pqr) + q(pqr) + r(pqr) = pqr(p+q+r) = 7(5) = 35$
6. $\frac{2003}{2004}x + 1 + \frac{1}{x} = 0 \Rightarrow x(\frac{2003}{2004}x + 1 + \frac{1}{x} = 0) \Rightarrow \frac{2003}{2004}x^2 + x + 1 = 0$. If the roots are a and b, then the sum of the reciprocals is $\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} = \frac{-1/(2003/2004)}{1/(2003/2004)} = -1$.
7. $q = \left(a + \frac{1}{b}\right)\left(b + \frac{1}{a}\right) = ab + \frac{1}{ab} + 2 = 2 + \frac{1}{2} + 2 = \frac{9}{2}$

Practice 27: Simon's Favorite Factoring Trick

Date: Tuesday, December 3

When you first learn about solving quadratic equations, one strategy you learn is completing the square. This makes use of the identity $(x+a)^2 =$ ____________________ to rewrite certain quadratic expressions as squares. For instance,

$$x^2 + 6x \qquad \text{equals} \qquad x^2 + 6x + \_\_\_\_ - \_\_\_\_ = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

where we have added 9 to "complete" the square $(x+3)^2$.

When you complete a square, you can compensate for the additional term by either _______________ it from the same expression or _______________ it to the other side of an equation.

In case you don't have much experience with completing the square, try a couple!

1. Rewrite $f(x) = x^2 - 8x$ by completing the square.

2. Solve $x^2 + 12x = -5$ by completing the square.

While completing the square is taught in typical math classes, there is a very similar tool that is not: **Simon's Favorite Factoring Trick**.

Simon's Favorite Factoring Trick "completes" expressions so that they may become _______________ even if they aren't perfect squares. Let's try some examples.

Examples

1. (a) Expand $(a+1)(b+2)$.
   (b) Factor the expression from (a) using grouping.
   (c) Suppose you had the expression $ab + 2a + b$. How could you "complete" this expression so that it's factorable?
2. How could you "complete" the expression $xy + 2x + 3y$ to become factorable? What would the factors be?
3. How could you "complete"' the expression $ab + bc + ac$ to become factorable? What would the factors be?

Let's try an actual problem using SFFT.

4. If p and q are positive integers such that $pq + 2p + 2q = 2024$, what is the largest possible value of p?

Practice

1. A rectangle has width w and height h. Both the width and height of the rectangle are increased by 2, and the resulting rectangle has twice the area of the original rectangle. How many possibilities are there for the ordered pair (w, h)?

2. The product of 2 distinct positive integers a and b is equal to 24 more than their sum. How many ordered pairs (a, b) satisfy this description?

Practice 27: Simon's Favorite Factoring Trick - SOLUTIONS

Date: Tuesday, December 3

When you first learn about solving quadratic equations, one strategy you learn is completing the square. This makes use of the identity $(x+a)^2 = x^2 + 2ax + a^2$ to rewrite certain quadratic expressions as squares. For instance,

$x^2 + 6x$ equals $x^2 + 6x + \underline{9} - \underline{9} = \underline{(x+3)^2 - 9}$

where we have added 9 to "complete" the square $(x+3)^2$.

When you complete a square, you can compensate for the additional term by either subtracting it from the same expression or adding it to the other side of an equation.

In case you don't have much experience with completing the square, try a couple!

1. Rewrite $f(x) = x^2 - 8x$ by completing the square.
   Solution: $f(x) = x^2 - 8x + 16 - 16 = (x-4)^2 - 16$
2. Solve $x^2 + 12x = -5$ by completing the square.
   Solution: $x^2 + 12x + 36 = -5 + 36 \implies (x+6)^2 = 31 \implies x + 6 = \pm\sqrt{31} \implies x = -6 \pm \sqrt{31}$

While completing the square is taught in typical math classes, there is a very similar tool that is not: **Simon's Favorite Factoring Trick**.

Simon's Favorite Factoring Trick "completes" expressions so that they may become factorable even if they aren't perfect squares. Let's try some examples.

Examples

1. (a) Expand $(a+1)(b+2)$.
   Solution: $ab + 2a + b + 2$
   (b) Factor the expression from (a) using grouping.
   Solution: $a(b+2) + 1(b+2) = (a+1)(b+2)$
   (c) Suppose you had the expression $ab + 2a + b$. How could you "complete" this expression so that it's factorable?
   Solution: Add 2!
2. How could you "complete" the expression $xy + 2x + 3y$ to become factorable? What would the factors be?
   Solution: We want $x(y+2) + 3y$ to be factorable, which would require another $y + 2$ term. This would give us $x(y+2) + 3(y+2)$, which would require adding $3(2) = 6$. If we added the 6, then $xy + 2x + 3y$ would become $xy + 2x + 3y + 6 = (x+3)(y+2)$.
3. How could you "complete"' the expression $ab + bc + ac$ to become factorable? What would the factors be?
   Solution: If we group the first two terms, we get $b(a+c) + ac$, but this leads nowhere. If we instead

Let's try an actual problem using SFFT.

4. If p and q are positive integers such that $pq + 2p + 2q = 2024$, what is the largest possible value of p?
   Solution: $pq + 2p + 2q = 2024 \implies p(q+2) + 2(q+2) = 2024 + 4 \implies (p+2)(q+2) = 2028$. If we want p to be large, then we want $q + 2$ to be as small as possible. Since q must be positive, we can't have $q = 0$. If we test $q = 1$, we get $(p+2) \cdot 3 = 2028$, which gives $p + 2 = 676$. Therefore, the largest possible value of p is 674.

Practice

1. A rectangle has width w and height h. Both the width and height of the rectangle are increased by 2, and the resulting rectangle has twice the area of the original rectangle. How many possibilities are there for the ordered pair (w, h)?
   Solution: The new area is given by both $(w + 2)(h + 2)$ and $2wh$. Setting these equal gives

$$\begin{aligned}(w+2)(h+2) &= 2wh \\ wh + 2w + 2h + 4 &= 2wh \\ 4 &= wh - 2w - 2h \\ 4 + 4 &= wh - 2w - 2h + 4 \qquad \text{(Use SFFT)} \\ 8 &= (w-2)(h-2)\end{aligned}$$

   The factorizations of 8 are $(1, 8)$, $(8, 1)$, $(2, 4)$, $(4, 2)$, $(-2, 4)$, and $(-4, 2)$. Only the first four of these would yield positive values of w and h, so there are 4 possible ordered pairs (w, h). (They are $(3, 10)$, $(10, 3)$, $(4, 6)$, and $(6, 4)$.)

2. The product of 2 distinct positive integers a and b is equal to 24 more than their sum. How many ordered pairs (a, b) satisfy this description?
   Solution: $ab = a + b + 24 \Rightarrow ab - a - b = 24 \Rightarrow ab - a - b + 1 = 25 \Rightarrow (a-1)(b-1) = 25$. The ordered factorizations of 25 are $(5, 5)$, $(-5, -5)$, $(1, 25)$, $(25, 1)$, $(-1, -25)$, and $(-25, -1)$. This gives 6 ordered pairs of a and b, but two of these - $(5, 5)$ and $(-5, -5)$ - would not yield *distinct* values of a and b. Furthermore, the pairs $(-1, -25)$ and $(-25, -1)$ would not yield positive values of both a and b. Therefore, there are 2 possible ordered pairs (a, b). (They are $(2, 26)$ and $(26, 2)$.)

Practice 28: Expansion

Date: Wednesday, December 4

REMINDER: Math League #3 is next Tuesday!

1. It is useful to be able to expand powers of polynomials - especially binomials quickly. This leads to the development of what is known as the ______________________________ and _________________________.

$$(x+y)^0 =$$

$$(x+y)^1 =$$

$$(x+y)^2 =$$

$$(x+y)^3 =$$

$$(x+y)^4 =$$

$$(x+y)^5 =$$

2. The triangle of coefficients above is called _________________________, and each number in it is known as a ______________ ______________.

3. The binomial coefficient $\binom{n}{k}$ gives how many ways you can select k objects out of n objects.

   - It is equal to the kth entry in the nth row of Pascal's triangle, remembering there is a 0th row and a 0th entry in each row.
   - It is equivalently equal to

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

   Examples:

   (a) Compute $\binom{4}{3}$, $\binom{5}{2}$, and $\binom{7}{4}$.

   (b) What is the coefficient of x^3 in the expansion of $(x+1)^9$?

4. We can utilize Pascal's Triangle to expand a power of any binomial by using the diagram above, only needing to remember that "x" and "y" might be terms that need ______________ around them before they can be exponentiated! In particular, in the expansion of $(x + y)^n$, each term will have the form

$$\binom{n}{k}x^k y^{n-k}$$

This makes sense because, in the expansion $(x+y)(x+y)\cdots(x+y)$, the term $\binom{n}{k}x^k y^{n-k}$ means that you're choosing k ____, the rest $(n - k)$ ____, and there are $\binom{n}{k}$ ways to choose this many ____.

Examples:

(a) Expand $(x + 2)^4$.

(b) Expand $(2x - 1)^3$.

Practice

1. If $(a - 2b)^5$ is expanded, what will the coefficient of a^2b^3 be?
2. A teacher has a bag with 7 marbles in it, each with a different color (red, orange, yellow, green, blue, purple, and black). A student is going to draw 3 marbles out without replacement and the teacher will give them 10 bonus points on a test if the student is able to draw either the red, orange, and yellow marbles together (in any order) or the green, blue, and purple together (in any order). What is the simplified probability that the student will get the bonus points?
3. What is the value of the constant term in the expansion of $(x^2 + \frac{3}{x^2})^6$?

Practice 28: Expansion - SOLUTIONS

Date: Wednesday, December 4

1. It is useful to be able to expand powers of polynomials - especially binomials quickly. This leads to the development of what is known as the binomial coefficient and Pascal's Triangle.

$$(x+y)^0 =$$

$$(x+y)^1 = x+y$$

$$(x+y)^2 = x^2+2xy+y^2$$

$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$

$$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$$

$$(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$$

2. The triangle of coefficients above is called ______________________, and each number in it is known as a ____________ ____________.

$$\begin{array}{ccccccccccccc} & & & & & & 1 & & & & & & \\ & & & & & 1 & & 1 & & & & & \\ & & & & 1 & & 2 & & 1 & & & & \\ & & & 1 & & 3 & & 3 & & 1 & & & \\ & & 1 & & 4 & & 6 & & 4 & & 1 & & \\ & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\ 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\ \iddots & & & & & & \vdots & & & & & & \ddots \end{array}$$

3. The binomial coefficient $\binom{n}{k}$ gives how many ways you can select k objects out of n objects.

   - It is equal to the kth entry in the nth row of Pascal's triangle, remembering there is a 0th row and a 0th entry in each row.
   - It is equivalently equal to

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

   Examples:

   (a) $\binom{4}{3} = 4$, $\binom{5}{2} = 10$, $\binom{7}{4} = \frac{7(6)(5)(4)}{4(3)(2)(1)} = 7(5) = 35$

   (b) The term will be $\binom{9}{3}x^3(1)^6$, so the coefficient is simply $\binom{9}{3} = \frac{9(8)(7)}{3(2)(1)} = 3(4)(7) = 84$.

4. We can utilize Pascal's Triangle to expand a power of any binomial by using the diagram above, only needing to remember that "x" and "y" might be terms that need ____________ around them before they can be exponentiated! In particular, in the expansion of $(x+y)^n$, each term will have the form

$$\binom{n}{k}x^k y^{n-k}$$

This makes sense because, in the expansion $(x+y)(x+y)\cdots(x+y)$, the term $\binom{n}{k}x^k y^{n-k}$ means that you're choosing k $\underline{x}$'s, the rest $(n-k)$ $\underline{y}$'s, and there are $\binom{n}{k}$ ways to choose this many $\underline{x}$'s.

Examples:

(a) $(x+2)^4 = x^4 + 4x^3(2) + 6x^2(2)^2 + 4x(2)^3 + 2^4 = x^4 + 8x^3 + 24x^2 + 32x + 16$

(b) $(2x-1)^3 = (2x)^3 + 3(2x)^2(-1) + 3(2x)(-1)^2 + (-1)^3 = 8x^3 - 12x + 6x - 1$

Practice

1. The term will be $\binom{5}{2}a^2(-2b)^3 = 10a^2(-8b^3) = -80a^2b^3$, so the coefficient is -80.

2. There are 12 ways to win (for each set of 3 marbles, there are 6 possible orderings that could win; for instance, {ROY, RYO, YRO, YOR, ORY, OYR}) and $\binom{7}{3} = \frac{7(6)(5)}{3(2)} = 35$ total ways to choose 3 marbles out of 7. The probability of winning is therefore $\frac{12}{35}$.

3. The term will be constant only if the powers of x in the numerator and denominator are equal. If we look at the entire expansion (inefficient, but helpful to see what's going on), we get

$$(x^2)^6 + 6(x^2)^5\left(\frac{3}{x^2}\right) + 15(x^2)^4\left(\frac{3}{x^2}\right)^2 + 20(x^2)^3\left(\frac{3}{x^2}\right)^3 + 15(x^2)^2\left(\frac{3}{x^2}\right)^4 + 6(x^2)\left(\frac{3}{x^2}\right)^5 + \left(\frac{3}{x^2}\right)^6$$

If we ignore the coefficients for a moment, the powers of x in this expression will simplify into

$$x^{12} + \frac{x^{10}}{x^2} + \frac{x^8}{x^4} + \frac{x^6}{x^6} + \frac{x^4}{x^8} + \frac{x^2}{x^{10}} + \frac{1}{x^{12}}$$

The middle term will be a constant. Therefore, the constant term is the coefficient of this term, or $20(3^3) = 540$.

Practice 29: Math League #3

Date: Tuesday, December 10

Practice 30: Practice

Date: Wednesday, December 11

1. What is the constant term in the expansion of $\left(x^2 + \frac{3}{x^2}\right)^4$?
2. How many ordered pairs of positive integers (x, y) satisfy the equation $xy + 3x + 2y = 25$?

   (A) 0

   (B) 1

   (C) 2

   (D) 4
3. The sum of two numbers is 12 and the sum of the squares of the two numbers is 86. What is the product of the two numbers?
4. What is the sum of the solutions of the equation $3x^2 - 9x + 5 = 0$?

   (A) -9

   (B) -3

   (C) 3

   (D) 9
5. Suppose that

$$\begin{aligned} 5x + 2y &= 4 \\ 3x + y &= 7 \\ z - x &= 11 \end{aligned}$$

   What is $x + y + z$?
6. The sum of the reciprocals of the solutions of $x^2 - 12x = 14$ can be written in the form m/n, where m and n are relatively prime. What is $m + n$?
7. Write a simplified and factored expression for $(x + 3)^3 + (x - 3)^3$.
8. What is the sum of the coefficients of $(4 - 5x)^7$?
9. If $x + \frac{1}{x} = 7$, what is $x^3 + \frac{1}{x^3}$?

Practice 30: Practice - SOLUTIONS

Date: Wednesday, December 11

1. The constant term will be the middle term: $\binom{4}{2}(x^2)^2\left(\frac{3}{x^2}\right)^2 = \frac{6 \cdot 9x^4}{x^4} = 54$

2. We use Simon's.

$$\begin{aligned} xy + 3x + 2y &= 25 \\ x(y+3) + 2(y+?) &= 25 \\ x(y+3) + 2(y+3) &= 25 + 6 \\ (x+2)(y+3) &= 31 \end{aligned}$$

   The only factors of 31 are 1 and 31, but if $x + 2 = 1$ or $y + 3 = 1$, then x or y would be negative. Therefore, there are no solutions (x, y).

3. Call the numbers x and y. Then

$$\begin{aligned} (x+y)^2 &= x^2 + y^2 + 2xy \\ (12)^2 &= 86 + 2xy \\ 58 &= 2xy \\ 29 &= xy \end{aligned}$$

   The product of the two numbers is 29.

4. This is simply $-\frac{-9}{3} = 3$ by Vièta's relations.

5. Whenever confronted with a problem like this, where you're asked for a combination of multiple unknowns, you almost never want to actually try to compute each unknown: instead, try to cleverly rewrite things. In this case, subtracting the second equation from the first equation and then adding the third equation results in $x + y + z = 4 - 7 + 11 = 8$.

6. Call the roots a and b. Then

$$\begin{aligned} \frac{1}{a} + \frac{1}{b} &= \frac{a+b}{ab} && \text{(common denominator)} \\ &= \frac{12}{14} && \text{(Vièta)} \\ &= \frac{6}{7} \end{aligned}$$

   Therefore, $m + n = 6 + 7 = 13$.

7. We can expand both completely, but we can also think ahead a bit. Each cube will be 4 terms that have identical coefficients, only the 2nd and 4th terms of $(x-3)^3$ will be negative. Therefore, the 2nd and 4th terms of $(x+3)^3$ and $(x-3)^3$ will cancel out, so we'll just be left with the 1st and 3rd terms of both cubics. Since they're identical, we can just compute the 1st and 3rd terms of the first cubic and double it. Therefore, $(x+3)^3 + (x-3)^3 = 2(x^3 + 3(x)(3^2)) = 2x^3 + 54x$.

8. As you'll see next week, there is a very quick solution to this: simply plug in 1 to any polynomial - regardless of the form its in - and the output will equal the sum of the coefficient of the polynomial in standard form (can you see why?). Therefore, the sum of coefficients of $(4-5x)^7$ is $(4-5(1))^7 = -1$.

9. If we cube both sides of the first equation, we can rearrange to isolate $x^3 + \frac{1}{x^3}$.

$$\begin{aligned}
\left(x + \frac{1}{x}\right)^3 &= 7^3 \\
x^3 + 3x^2\left(\frac{1}{x}\right) + 3x\left(\frac{1}{x}\right)^2 + \left(\frac{1}{x}\right)^3 &= 343 && \text{(expand)} \\
x^3 + 3x + \frac{3}{x} + \frac{1}{x^3} &= 343 && \text{(simplify)} \\
x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right) &= 343 && \text{(rearrange \& factor)} \\
x^3 + \frac{1}{x^3} + 21 &= 343 && \text{(substitute } x + \tfrac{1}{x} = 7\text{)} \\
x^3 + \frac{1}{x^3} &= 322
\end{aligned}$$

Practice 32: PARTY!

Date: Wednesday, December 18

Practice 33: Polynomial Tricks

Date: Tuesday, January 7

This lesson is dedicated to a few tricks related to polynomials and Vièta's relations.

Powers of $x + \frac{1}{x}$

Expansions of powers of $\left(x + \frac{1}{x}\right)$ are quite common in math competitions.

1. Examining the structure of powers of $\left(x + \frac{1}{x}\right)$
   (a) Expand $\left(x + \frac{1}{x}\right)^2$.
   (b) Expand $\left(x + \frac{1}{x}\right)^3$. Then, factor out a 3 from the two terms with a coefficient of 3.
   (c) Expand $\left(x + \frac{1}{x}\right)^4$. Then, factor out a coefficient from the two terms with the same coefficient.

Now for some problems.

2. If $x + \frac{1}{x} = 7$, what is $x^2 + \frac{1}{x^2}$?
3. Suppose $a + \frac{1}{a} = 4$. Compute $a^3 + \frac{1}{a^3}$.
4. If $x + \frac{1}{x} = 5$, what is $x^4 + \frac{1}{x^4}$?
5. (Extension) If $x^3 + \frac{1}{x^3} = 8$, what is $x^5 + \frac{1}{x^5}$?

Sums of Coefficients of Polynomials

6. Examining structure
   (a) Expand $f(x) = (x + 1)^2$ and find the sum of the coefficients of f.
   (b) Evaluate $f(1)$.
   (c) Expand $g(x) = (x + 2)^3$ and find the sum of the coefficients of g.
   (d) Evaluate $g(1)$.
   (e) What do you notice from (a)-(d)?

Now for some problems.

7. Find the sum of the coefficients of $(2x - 3)^8$.

8. Find the sum of the coefficients of $(x-3)^5$.
9. Find the sum of the coefficients of $(x+2y)^4$.

Using Vièta's relations with cubics

10. Examining structure of $(a+b)^3$
    (a) Expand $(a+b)^3$.
    (b) Factor out $3ab$ from the middle terms and put the first and fourth terms next to each other. .
    (c) If a and b were the roots of a quadratic, do you see the sum and product of these roots in your expression in (b)?

Now for some problems.

11. Suppose p and q are the roots of $f(x) = x^2 + 8x - 3$. Compute $p^3 + q^3$.
12. Suppose m and n are the solutions of $x^2 + x + 2024 = 0$. Compute $m^3 + n^3$.

Manipulating Systems of Equations

13. Example
    (a) Many systems of equations ask you to compute something involving *multiple* terms. In these cases, it is often wise not to compute the individual terms, but rather to manipulate the system. For instance, suppose you were asked to compute $x + y$ with the system

$$\begin{aligned} 9x - 4y &= 5 \\ 5x - 9y &= 22 \end{aligned}$$

    You may be tempted to use elimination to solve for x, then substitute x to solve for y, and then to finally add x and y. Instead, try *subtracting* the two equations. What do you get, and how can you finally solve to find $x + y$ without ever finding the value of x or y?

Now for some problems.

14. If $a + b = 5$, $b + c = 8$, and $a + c = 13$, what is $a + b + c$?
15. If $x + 7y = 2$ and $3x + 34y = 8$, what is $2x + y$?
16. Find $x - y + z$ if

$$\begin{aligned} 9x - 5z &= 4 \\ 4x - y &= 11 \\ y - 2z &= -7 \end{aligned}$$

Practice 33: Polynomial Tricks - SOLUTIONS

Date: Tuesday, January 7

This lesson is dedicated to a few tricks related to polynomials and Vièta's relations.

Powers of $x + \frac{1}{x}$

Expansions of powers of $\left(x + \frac{1}{x}\right)$ are quite common in math competitions.

1. Examining the structure of powers of $\left(x + \frac{1}{x}\right)$
   (a) Expand $\left(x + \frac{1}{x}\right)^2$. **Answer:** $\mathbf{x^2 + \frac{1}{x^2} + 2x\left(\frac{1}{x}\right) = x^2 + \frac{1}{x^2} + 2}$
   (b) Expand $\left(x + \frac{1}{x}\right)^3$. Then, factor out a 3 from the two terms with a coefficient of 3.
   Answer: $\mathbf{x^3 + 3x^2\left(\frac{1}{x}\right) + 3x\left(\frac{1}{x}\right)^2 + \left(\frac{1}{x}\right)^3 = x^3 + \frac{1}{x^3} + 3x + \frac{3}{x} = x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right)}$
   (c) Expand $\left(x + \frac{1}{x}\right)^4$. Then, factor out a coefficient from the two terms with the same coefficient.
   Answer (less work shown): $\mathbf{x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4} = x^4 + \frac{1}{x^4} + 6 + 4\left(x^2 + \frac{1}{x^2}\right)}$

Now for some problems.

2. If $x + \frac{1}{x} = 7$, what is $x^2 + \frac{1}{x^2}$?
   Answer: From 1(a), $\mathbf{x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2 = 7^2 - 2 = 47}$
3. Suppose $a + \frac{1}{a} = 4$. Compute $a^3 + \frac{1}{a^3}$.
   Answer: From 1(b), $\mathbf{x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right) = 4^3 - 3(4) = 52}$
4. If $x + \frac{1}{x} = 5$, what is $x^4 + \frac{1}{x^4}$?
   Answer: From 1(c), $\mathbf{x^4 + \frac{1}{x^4} = \left(x + \frac{1}{x}\right)^4 - 6 - 4\left(x^2 + \frac{1}{x^2}\right)}$**. We can compute** $\mathbf{x^2 + \frac{1}{x^2} = 5^2 - 2 = 23}$**, so** $\mathbf{x^4 + \frac{1}{x^4} = 5^4 - 6 - 4(23) = 527}$
5. (Extension) If $x + \frac{1}{x} = 6$, what is $x^5 + \frac{1}{x^5}$?
   Answer: Expanding $\mathbf{\left(x + \frac{1}{x}\right)^5}$ **gives** $\mathbf{x^5 + \frac{1}{x^5} + 5\left(x^3 + \frac{1}{x^3}\right) + 10\left(x + \frac{1}{x}\right)}$**. Following a similar procedure to #3 gives** $\mathbf{x^3 + \frac{1}{x^3} = 198}$**. Therefore,**

$$\mathbf{x^5 + \frac{1}{x^5} = \left(x + \frac{1}{x}\right)^5 - 5\left(x^3 + \frac{1}{x^3}\right) - 10\left(x + \frac{1}{x}\right) = 6^5 - 5(198) - 10(6) = 6726}$$

.

Sums of Coefficients of Polynomials

6. Examining structure
   (a) Expand $f(x) = (x + 1)^2$ and find the sum of the coefficients of f.
   Answer: $\mathbf{f(x) = x^2 + 2x + 1}$**, with sum of coefficients equal to 4.**
   (b) Evaluate $f(1)$. **Answer:** $\mathbf{f(1) = (1 + 1)^2 = 4}$
   (c) Expand $g(x) = (x + 2)^3$ and find the sum of the coefficients of g.
   Answer: $\mathbf{g(x) = x^3 + 3x^2(2) + 3x(2)^2 + 2^3 = x^3 + 6x^2 + 12x + 8}$**, with sum of coefficients equal to 27.**
   (d) Evaluate $g(1)$.
   Answer: $\mathbf{g(1) = (1 + 2)^3 = 27}$
   (e) What do you notice from (a)-(d)?
   Answer: The sum of coefficients of a polynomial $\mathbf{p(x)}$ **equals** $\mathbf{p(1)}$**.**

Now for some problems.

7. Find the sum of the coefficients of $(2x-3)^8$.
   Answer: $\mathbf{(2(1)-3)^8 = (-1)^8 = 1}$

8. Find the sum of the coefficients of $(x-3)^5$.
   Answer: $\mathbf{(1-3)^5 = (-2)^5 = -32}$

9. Find the sum of the coefficients of $(x+2y)^4$.
   Answer: $\mathbf{(1+2(1))^4 = 3^4 = 81}$

Using Vièta's relations with cubics

10. Examining structure of $(a+b)^3$

    (a) Expand $(a+b)^3$.
    Answer: $\boldsymbol{a^3 + 3a^2b + 3ab^2 + b^3}$

    (b) Factor out $3ab$ from the middle terms and put the first and fourth terms next to each other.
    Answer: $\boldsymbol{a^3 + b^3 + 3ab(a+b)}$

    (c) If a and b were the roots of a quadratic, do you see the sum and product of these roots in your expression in (b)?
    Answer: The final term has both the sum and product of roots.

Now for some problems.

11. Suppose p and q are the roots of $f(x) = x^2 + 8x - 3$. Compute $p^3 + q^3$.
    Answer: We know from Vièta's relations that $\boldsymbol{p+q=-8}$ and $\boldsymbol{pq=-3}$. Since $\boldsymbol{(p+q)^3 = p^3+q^3+3pq(p+q)}$, we can solve for $\boldsymbol{p^3+q^3}$: $\boldsymbol{(-8)^3 = p^3+q^3+3(-3)(-8) \Rightarrow a^3+b^3 = -584}$.

12. Suppose m and n are the solutions of $x^2 + x + 2024 = 0$. Compute $m^3 + n^3$.
    Answer: Following the same procedure as the last problem, we get

$$\boldsymbol{m^3+n^3 = (m+n)^3 - 3mn(m+n) = (-1)^3 - 3(-1)(2024) = 6071}$$

Manipulating Systems of Equations

13. Example

    (a) Many systems of equations ask you to compute something involving *multiple* terms. In these cases, it is often wise not to compute the individual terms, but rather to manipulate the system. For instance, suppose you were asked to compute $x+y$ with the system

$$9x - 4y = 5$$
$$5x - 9y = 22$$

    You may be tempted to use elimination to solve for x, then substitute x to solve for y, and then to finally add x and y. Instead, try *subtracting* the two equations. What do you get, and how can you finally solve to find $x+y$ without ever finding the value of x or y?

Now for some problems.

14. If $a+b=5$, $b+c=8$, and $a+c=13$, what is $a+b+c$?
    Answer: Adding all 3 equations yields $\boldsymbol{2a+2b+2c=26}$, so $\boldsymbol{a+b+c=13}$.

15. If $x+7y=2$ and $3x+34y=8$, what is $2x+y$?
    Answer: Multiplying the first equation by 5 and subtracting the second yields $\boldsymbol{5(x+7y)-(3x+34y)=2x+y}$, so $\boldsymbol{2x+y=5(2)-8=2}$.

16. Find $x - y + z$ if

$$\begin{aligned} 9x - 5z &= 4 \\ 4x - y &= 11 \\ y - 2z &= -7 \end{aligned}$$

Answer: This takes trial and error, but taking the first equation, subtracting 2 times the second equation, and then subtracting 3 times the third equation yields

$$9x - 5z - 2(4x - y) - 3(y - 2z) = 9x - 5z - 8x + 2y - 3y + 6z = x - y + z$$

so $x - y + z = 4 - 2(11) - 3(-7) = 3$.

Chapter 2: Counting

Practice 34: Factorials & Permutations

Date: Wednesday, January 8

Definition: The expression $n!$ is called n ______________ and is defined by

$$n! = n(n-1)(n-2)\cdots 3(2)(1)$$

NOTE: By convention, $0! = 1$.

Practice

1. Compute $n!$ for $n \in \{1, 2, 3, 4, 5, 6, 7\}$.

2. If $9! = 362,880$, what is $10!$?

3. Compute $\dfrac{2025!}{2023!}$.

4. Compute $\dfrac{101! - 100!}{99!}$.

Factorials are incredibly useful for counting permutations and combinations.

Definition: A permutation of n objects is a reordering of those objects where the order of the objects ________ matter. A combination of n objects is simply a set of n objects where the order __________ matter.

Examples

1. Suppose you have 4 books - call them A, B, C, and D - to arrange on a shelf.

   (A) List all possible permutations of the books.

   (B) How many permutations are there, and what does this have to do with the factorial function?

2. Suppose you have 7 books - call them A, B, C, D, E, F, and G - and you are going to select 4 to place on a shelf. How many possible orderings of books on the shelf are there?

One way to answer the previous question is using a formula for permutations. If you have n objects and you want to select r of them, where the order of the r objects matters, then there are

$$_nP_r = \frac{n!}{(n-r)!}$$

possibilities.

For instance, to answer the last question, we computed

$$_7P_4 = \frac{7!}{(7-4)!} = \frac{7!}{3!} = \frac{7(6)(5)(4)(3)(2)(1)}{3(2)(1)} = 7(6)(5)(4)$$

NOTE: It's not worth memorizing this formula - you likely won't use it in practice, and it's more important that you know where it comes from and what it means!

Practice

5. 10 runners are entering a race. Only 3 of these runners can win gold, silver, and bronze medals. How many different ways can the 3 medals be awarded?

6. Let any string of 5 unique characters be called a *5-string*. For instance, *ABRQT* and *RTQBA* are unique 5-strings, while *AACLM* is not a 5-string. How many 5-strings are there using characters from the English alphabet?

7. How many 5-strings are there that start with a vowel and end with a Z?

8. 7 people are going to sit down in a row of 7 chairs. How many ways can the 7 people be seated?

9. 7 people are going to sit at a round table. If rotations of the table are considered identical, how many ways can the 7 people be seated?

10. 7 people are going to sit at a round table, but two of the people are married to each other and would like to sit next to each other. How many ways can the 7 people be seated that satisfy the married couple's request?

11. 7 people are going to sit at a round table, but two of the people are mortal enemies and cannot sit next to each other. How many ways can the 7 people be seated that will keep the enemies from sitting next to one another?

Practice 34: Factorials & Permutations - SOLUTIONS

Date: Wednesday, January 8

Definition: The expression $n!$ is called n factorial and is defined by

$$n! = n(n-1)(n-2)\cdots 3(2)(1)$$

NOTE: By convention, 0! = 1.

Practice

1. Compute $n!$ for $n \in \{1, 2, 3, 4, 5, 6, 7\}$.
   Answer: $\mathbf{1! = 1, 2! = 2, 3! = 6, 4! = 24, 5! = 120, 6! = 720, 7! = 5040}$

2. If $9! = 362,880$, what is $10!$?
   Answer: $\mathbf{10! = 10 \cdot 9! = 10(362880) = 3628800}$

3. Compute $\frac{2025!}{2023!}$.
   Answer: $\mathbf{\frac{2025 \cdot 2024 \cdot 2023!}{2023!} = 2025(2024) = 4098600}$

4. Compute $\frac{101! - 100!}{99!}$.
   Answer: $\mathbf{\frac{101!}{99!} - \frac{100!}{99!} = 101(100) - 100 = 100(101-1) = 100(100) = 10000}$

Factorials are incredibly useful for counting permutations and combinations.

Definition: A permutation of n objects is a reordering of those objects where the order of the objects does matter. A combination of n objects is simply a set of n objects where the order does not matter.

Examples

1. Suppose you have 4 books - call them A, B, C, and D - to arrange on a shelf.

   (A) List all possible permutations of the books.
   Answer:

ABCD	BACD	CABD	DABC
ABDC	BADC	CADB	DACB
ACBD	BCAD	CBAD	DBAC
ACDB	BCAD	CBDA	DBCA
ADBC	BDAC	CDAB	DCAB
ADCB	BDCA	CDBA	DCBA

   (B) How many permutations are there, and what does this have to do with the factorial function?
   Answer: There are 24 permutations, which is precisely 4!.

2. Suppose you have 7 books - call them A, B, C, D, E, F, and G - and you are going to select 4 to place on a shelf. How many possible orderings of books on the shelf are there?
   Answer: $\mathbf{7! = 5040}$

One way to answer the previous question is using a formula for permutations. If you have n objects and you want to select r of them, where the order of the r objects matters, then there are

$${}_nP_r = \frac{n!}{(n-r)!}$$

possibilities.

For instance, to answer the last question, we computed

$$_7P_4 = \frac{7!}{(7-4)!} = \frac{7!}{3!} = \frac{7(6)(5)(4)(3)(2)(1)}{3(2)(1)} = 7(6)(5)(4)$$

NOTE: It's not worth memorizing this formula - you likely won't use it in practice, and it's more important that you know where it comes from and what it means!

Practice

5. 10 runners are entering a race. Only 3 of these runners can win gold, silver, and bronze medals. How many different ways can the 3 medals be awarded?
   Answer: $10 \cdot 9 \cdot 8 = 720$

6. Let any string of 5 unique characters be called a *5-string*. For instance, *ABRQT* and *RTQBA* are unique 5-strings, while *AACLM* is not a 5-string. How many 5-strings are there using characters from the English alphabet?
   Answer: $26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 = 7893600$

7. How many 5-strings are there that start with a vowel and end with a Z?
   Answer: There are 5 choices for the first letter, 24 choices for the second letter (as it can't be the first letter or a Z), 23 choices for the third letter, 22 choices for the fourth letter, and 1 choice for the last letter, so there are $5 \cdot 24 \cdot 23 \cdot 22 \cdot 1 = 60720$ such strings.

8. 7 people are going to sit down in a row of 7 chairs. How many ways can the 7 people be seated?
   Answer: $7! = 5040$

9. 7 people are going to sit at a round table. If rotations of the table are considered identical, how many ways can the 7 people be seated?
   Answer: It's still 7!permutations, only each arrangement is repeated 7 times (for 7 equivalent rotations), so there are $\frac{7!}{7} = 6! = 720$ unique arrangements.

10. 7 people are going to sit at a round table, but two of the people are married to each other and would like to sit next to each other. How many ways can the 7 people be seated that satisfy the married couple's request?
    Answer: Treat the two seats the married couple are at as one seat. Then, there are $\frac{6!}{6} = 5! = 120$ unique arrangements of 6 seats. For each of these arrangements, there are 2 ways the married couple can be seated (who is on the left and who is on the right), so there are $2 \cdot 120 = 240$ seatings.

11. 7 people are going to sit at a round table, but two of the people are mortal enemies and cannot sit next to each other. How many ways can the 7 people be seated that will keep the enemies from sitting next to one another?
    Answer: From #9, there are 720 arrangements of 7 people at a table, and from #10, there are 240 ways that 2 specific people can be seated next to each other. Therefore, there are $720 - 240 = 480$ ways in which 2 specific people are *not* seated next to each other.

Practice 35: Math League #4

Date: Tuesday, January 14

Practice 36: Combinations

Date: Wednesday, January 15

Previously, we looked at how many ways you can *permute* r objects out of n objects, which led to the formula

$$_nP_r = \frac{n!}{(n-r)!}$$

If we simply want to *select* r objects from n objects **without considering the order**, then the formula above would overcount every permutation of the r objects. For instance, we computed that, if there are 7 books and you want to place 4 of them on a shelf, there are $\frac{7!}{3!} = 7(6)(5)(4)$ ways to do this. However, if we just want to select 4 books, then the formula above overcounts every permutation of the 4 books. We showed that there are $4!$ ways to arrange these books, so we need to divide our result above by $4!$ to get

$$\frac{\frac{7!}{3!}}{4!} = \frac{7!}{3!4!} = \frac{7(6)(5)(4)}{4(3)(2)(1)} = 70$$

This leads to a formula for the number of **combinations** of selecting r objects from n objects:

$$_nC_r = \binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

The notation $_nC_r$ is rarely used. Instead, you will frequently see $\binom{n}{r}$, which is called **the binomial coefficient** and is said as "***n* choose *r***".

The numerator seems abstract, but the key to note is that there are **exactly *r* terms in it**. For example, if we wanted to compute $\binom{11}{5}$, we would write 5 consecutive decreasing integers starting at 11 in the numerator and put $5!$ in the denominator:

$$\binom{11}{5} = \frac{11(10)(9)(8)(7)}{5(4)(3)(2)(1)}$$

Simplifying quotients gives this value as 462.

Practice

1. Compute $\binom{7}{2}$, $\binom{8}{3}$, and $\binom{10}{4}$.

2. Every row in Pascal's Triangle, shown below, is actually a binomial coefficient.

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&3&&3&&1& \\ 1&&4&&6&&4&&1 \end{array}$$

   (a) Compute $\binom{4}{0}$, $\binom{4}{1}$, $\binom{4}{2}$, $\binom{4}{3}$, and $\binom{4}{4}$. Which row of Pascal's Triangle do these values correspond to?

   (b) If the row 1-4-6-4-1 is the "fourth" row, what is the row with just a 1 in it?

3. Compute $\binom{9}{3}$ and $\binom{9}{6}$. Can you explain in terms of combinations of objects why these are equal?

4. Explain why $\binom{n}{r} = \binom{n}{n-r}$ for all $n, r \in \mathbb{N}$ with $r \leq n$.

5. Suppose there are 10 students in a certain school club.

   (a) If 3 students are to be randomly chosen to be the president, vice president, and secretary, respectively, how many ways can the leadership positions be chosen?

   (b) If 3 students are to be randomly chosen to simply be "officers," in how many ways can the leadership positions be chosen?

   (c) Explain the difference between (a) and (b).

6. Let S be the set $S := \{1, 2, 3, \ldots, 19\}$. Randomly select 2 unique numbers from S.

   (a) What is the probability their product will be odd?

   (b) What is the probability their sum will be even?

7. One known identity with binomial coefficients is $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$.

   (a) Test the identity out with a couple of small values of n and r.

   (b) Can you explain the identity in terms of Pascal's Triangle?

   (c) Explain the identity in terms of counting. (HInt: Think of selecting $r + 1$ officers from a club of $n + 1$ students and whether or not one particular student is an officer or not.)

8. A drawer has 8 identical blue socks and 6 identical red socks in it. If 4 socks are randomly selected from the drawer, what is the probability that one pair of blue socks and one pair of red socks are selected?

9. Refer back to the previous scenario. If 4 socks are randomly selected from the drawer, what is the probability that 2 pairs of matching socks are selected?

Practice 36: Combinations - SOLUTIONS

Date: Wednesday, January 15

Previously, we looked at how many ways you can *permute* r objects out of n objects, which led to the formula

$$_nP_r = \frac{n!}{(n-r)!}$$

If we simply want to *select* r objects from n objects **without considering the order**, then the formula above would overcount every permutation of the r objects. For instance, we computed that, if there are 7 books and you want to place 4 of them on a shelf, there are $\frac{7!}{3!} = 7(6)(5)(4)$ ways to do this. However, if we just want to select 4 books, then the formula above overcounts every permutation of the 4 books. We showed that there are $4!$ ways to arrange these books, so we need to divide our result above by $4!$ to get

$$\frac{\frac{7!}{3!}}{4!} = \frac{7!}{3!4!} = \frac{7(6)(5)(4)}{4(3)(2)(1)} = 70$$

This leads to a formula for the number of **combinations** of selecting r objects from n objects:

$$_nC_r = \binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

The notation $_nC_r$ is rarely used. Instead, you will frequently see $\binom{n}{r}$, which is called **the binomial coefficient** and is said as "***n* choose *r***".

The numerator seems abstract, but the key to note is that there are **exactly *r* terms in it**. For example, if we wanted to compute $\binom{11}{5}$, we would write 5 consecutive decreasing integers starting at 11 in the numerator and put $5!$ in the denominator:

$$\binom{11}{5} = \frac{11(10)(9)(8)(7)}{5(4)(3)(2)(1)}$$

Simplifying quotients gives this value as 462.

Practice

1. Compute $\binom{7}{2}$, $\binom{8}{3}$, and $\binom{10}{4}$.
   Answer: $\binom{7}{2} = \frac{7(6)}{2} = 21$, $\binom{8}{3} = \frac{8(7)(6)}{3(2)(1)} = 56$, $\binom{10}{4} = \frac{10(9)(8)(7)}{4(3)(2)(1)} = 210$

2. Every row in Pascal's Triangle, shown below, is actually a binomial coefficient.

   1
   1 1
   1 2 1
   1 3 3 1
   1 4 6 4 1

   (a) Compute $\binom{4}{0}$, $\binom{4}{1}$, $\binom{4}{2}$, $\binom{4}{3}$, and $\binom{4}{4}$. Which row of Pascal's Triangle do these values correspond to?
   Answer: $\binom{4}{0} = 1$, $\binom{4}{1} = 4$, $\binom{4}{2} = 6$, $\binom{4}{3} = 4$, $\binom{4}{4} = 1$

   (b) If the row 1-4-6-4-1 is the "fourth" row, what is the row with just a 1 in it?
   Answer: The 0th row

3. Compute $\binom{9}{3}$ and $\binom{9}{6}$. Can you explain in terms of combinations of objects why these are equal?
   Answer: $\binom{9}{3} = \frac{9(8)(7)}{3(2)(1)} = 84$, $\binom{9}{6} = \frac{9(8)(7)(6)(5)(4)}{6(5)(4)(3)(2)(1)} = \frac{9(8)(7)}{3(2)(1)} = 84$. **Choosing 3 objects out of 9 objects is the same as *not choosing* $9-3=6$ objects.**

4. Explain why $\binom{n}{r} = \binom{n}{n-r}$ for all $n, r \in \mathbb{N}$ with $r \leq n$.
Answer: Choosing r objects from n objects is the same as *not choosing* $n - r$ objects from n objects.

5. Suppose there are 10 students in a certain school club.

   (a) If 3 students are to be randomly chosen to be the president, vice president, and secretary, respectively, how many ways can the leadership positions be chosen?
   Answer: $10(9)(8) = 720$

   (b) If 3 students are to be randomly chosen to simply be "officers," in how many ways can the leadership positions be chosen?
   Answer: $\binom{10}{3} = \frac{10(9)(8)}{3(2)(1)} = 120$

   (c) Explain the difference between (a) and (b).
   Answer: Order matters in (a), but not in (b).

6. Let S be the set $S := \{1, 2, 3, \ldots, 19\}$. Randomly select 2 unique numbers from S.

   (a) What is the probability their product will be odd?
   Answer: To get an odd product, 2 odd numbers must be selected. There are 10 odd numbers to choose from, so there are $\binom{10}{2} = \frac{10(9)}{2(1)} = 45$ ways to select the 2 odd numbers. There are $\binom{19}{2} = \frac{19(18)}{2(1)} = 171$ ways to select 2 numbers overall, so the probability is $\frac{45}{171} = \frac{5}{19}$.

   (b) What is the probability their sum will be even?
   Answer: To get an even sum, two odd numbers or two even numbers must be selected. There are 45 ways to get 2 odd numbers and $\binom{9}{2} = \frac{9(8)}{2(1)} = 36$ ways to get 2 even numbers, so the probability is $\frac{45+36}{171} = \frac{9}{19}$.

7. One known identity with binomial coefficients is $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$.

   (a) Test the identity out with a couple of small values of n and r.
   Example answer: For $n = 6$ and $r = 2$, we get $\binom{6}{2} + \binom{6}{3} = 15 + 20 = 35 = \binom{7}{3}$.

   (b) Can you explain the identity in terms of Pascal's Triangle?
   Answer: Each entry in the triangle is the sum of the entries directly above it.

   (c) Explain the identity in terms of counting. (HInt: Think of selecting $r + 1$ officers from a club of $n + 1$ students and whether or not one particular student is an officer or not.)
   Answer: Consider a particular student X. If X is an officer, then there are n members left to choose from and r officers left to choose for a total of $\binom{n}{r}$ possibilities. If X is not an officer, then there are n members left to choose from and $r + 1$officers left to choose for a total of $\binom{n}{r+1}$ possibilities. Since the $\binom{n+1}{r+1}$ ways to choose $r + 1$ officers can only occur if X either is or is not an officer, $\binom{n+1}{r+1}$ is equivalent to $\binom{n}{r} + \binom{n}{r+1}$.

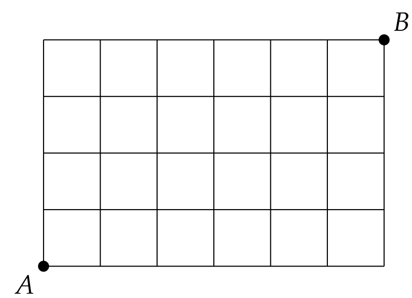
8. A drawer has 8 identical blue socks and 6 identical red socks in it. If 4 socks are randomly selected from the drawer, what is the probability that one pair of blue socks and one pair of red socks are selected?
Answer: There are $\binom{8}{2} = 28$ ways to get 2 blue socks, $\binom{6}{2} = 15$ ways to get 2 red socks, and $\binom{14}{4} = \frac{14(13)(12)(11)}{4(3)(2)(1)} = 7(13)(11)$ total ways to get 4 socks. Therefore, the probability is $\frac{28(15)}{7(13)(11)} = \frac{4(15)}{13(11)} = \frac{60}{143}$.

9. Refer back to the previous scenario. If 4 socks are randomly selected from the drawer, what is the probability that 2 pairs of matching socks are selected?
Answer: We could get 2 pairs of blue socks, 2 pairs of red socks, or one pair of each. We already computed there are 15(28) ways to get one pair of each. There are $\binom{8}{4} = 70$ ways to get 4 blue socks (2 blue pairs) and $\binom{6}{4} = 15$ ways to get 4 red socks (2 red pairs), so the probability is $\frac{15(28)+70+15}{7(13)(11)} = \frac{505}{1001}$.

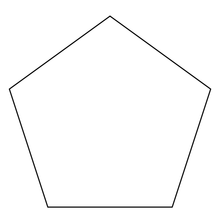
Practice 37: Counting Practice

Date: Tuesday, January 21

1. If you rearrange the letters in the word SANDWICH, how many different "words" can be formed? (Any sequence of letters counts as a "word.")
2. If you rearrange the letters in the word SEQUOIAS, how many different "words" can be formed? (Hint: Don't forget that the order of the two S's doesn't matter.)
3. If you rearrange the letters in the word IMPOSSIBILITY, how many different "words" can be formed?
4. Compute $\binom{12}{3}$ and $\binom{15}{6}$.
5. Compute $\binom{2024}{1991} - \binom{2024}{33}$.
6. Eight runners run a race. One runner, Steve, either finishes second or third. How many different finishes to the race can there be?
7. Mr. Hornbeck recently went to a wedding where he was seated at a table of 8 people. He was seated next to his wife and one of her friends. The other 5 people did not know each other. In how many ways could the table have been arranged if Mr. Hornbeck, his wife, and her friend had to be seated together?
8. A math team consists of 6 seniors, 8 juniors, 5 sophomores, and 9 freshmen. If 5 members are to be randomly selected to serve as officers, then the probability that the officers are two seniors and three juniors can be written as $\frac{m}{n}$, where m and n are relatively prime. Compute $m + n$.
9. Consider the rectangular grid below.
   (a) Draw 2 different paths from A to B using only moves directly up or directly to the right.
   (b) Count the number of total moves in each of your two paths.
   (c) Count the number of "up" moves in each of your two paths.
   (d) How many total paths are there from A to B using only moves directly up or directly to the right?



10. Fifteen people are at a party. Every person shakes hands with every other person, but no one can shake hands with themselves. How many handshakes occur? (Hint: How many people does it take to form a handshake, and does order matter?)
11. Consider the pentagon below.



   (a) Draw all of the diagonals of the pentagon. How many are there?
   (b) One way to count the diagonals is as follows: A diagonal consists of 2 vertices, where order doesn't matter, and there are 5 vertices. Therefore, there are $\binom{5}{2} = 10$ diagonals. What's wrong with this argument?
   (c) Can you find a way to use the number of vertices to arrive at your answer from (a)?
12. How many diagonals does a hexagon have? How about an octagon? How about an n-gon (polygon with n sides)?

Practice 37: Counting Practice - SOLUTIONS

Date: Tuesday, January 21

1. $8! = \mathbf{40,320}$

2. The answer would be the same as before, but each permutation involving two S's is double counted, as the order of the S's doesn't matter. Therefore, we must divide the answer in #1 by 2, giving us an answer of $8!/2 = \mathbf{20,160}$.

3. There are $13!$ ways to rearrange the letters, but we will overcount the permutations of the 4 I's and the 2 S's. Therefore, there are $\frac{13!}{4!2!} = \mathbf{129,729,600}$ possible "words."

4. $\binom{12}{3} = \frac{12(11)(10)}{3(2)(1)} = \mathbf{220}$, $\binom{15}{6} = \frac{15(14)(13)(12)(11)(10)}{6(5)(4)(3)(2)(1)} = \mathbf{5005}$

5. Since $1991 + 33 = 2024$, $\binom{2024}{1991} = \binom{2024}{33}$. Therefore, the difference is $\mathbf{0}$.

6. If Steve finishes second, then there are $7! = 5040$ ways the other runners can finish. If Steve finishes third, there are also 5040 ways. Therefore, there are $2(5040) = \mathbf{10080}$ possible finishes to the race.

7. Pretend the seats Mr. Hornbeck, his wife, and her friend are in are one seat. Then there are 6 total seats, and there are $\frac{6!}{6} = 5! = 120$ arrangements. For each of these arrangements, there are $3! = 6$ ways that Mr. Hornbeck, his wife, and her friend can be arranged amongst themselves. Therefore, there are **720** possible seatings.

8. There are $\binom{6}{2} = 15$ ways to pick the seniors and $\binom{5}{3} = 10$ ways to pick the juniors. There are $\binom{6+8+5+9}{5} = \binom{28}{5}$ ways to choose the five officers. Therefore, the probability is

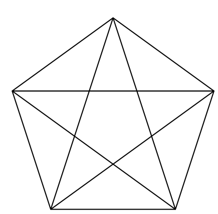
$$\frac{15(10)}{\binom{28}{5}} = \frac{15(10)}{\frac{28(27)(26)(25)(24)}{5(4)(3)(2)(1)}} = \frac{15(10)}{28(9)(26)(5)(3)} = \frac{5}{14(9)(26)} = \frac{5}{3276}$$

   Thus $m = 5$ and $n = 3276$, so $m + n = \mathbf{3281}$.

9. (b) 10 total moves
   (c) 4 up moves
   (d) Each path is simply a different choice of 4 up moves among 10 total moves, so there are $\binom{10}{4} = \mathbf{210}$ total paths.

10. Each handshake that occurs is simply a selection of 2 people, so $\binom{15}{2} = \mathbf{105}$ handshakes occur.

11.



   (a) There are $\mathbf{5}$ diagonals.
   (b) This includes pairs of adjacent vertices.
   (c) Each vertex can be connected to any other vertex except for itself and the 2 adjacent vertices. Therefore, each vertex can form a diagonal with 2 other vertices. This leads to $\frac{5}{2}$ diagonals, only we have double counted (since the order of the vertices doesn't matter). There are thus $\frac{5(2)}{2} = 5$ diagonals.

12. A hexagon has $\frac{6(6-3)}{2} = \mathbf{9}$ diagonals; an octagon has $\frac{8(8-3)}{2} = \mathbf{20}$ diagonals; an n-gon has $\boldsymbol{\frac{n(n-3)}{2}}$ diagonals.

Practice 38: Mixed Competition Practice

Date: Wednesday, January 22

1. (Augusta 2017 #11) What is the value of $\frac{13! - 12!}{11!}$?
   (A) 121 (B) 156 (C) 144 (D) 132 (E) 169

2. (Augusta 2017 #18) Find the remainder when 2^{2017} is divided by 5.
   (A) 0 (B) 1 (C) 2 (D) 3 (E) 4

3. (Augusta 2017 #19) Find the probability that a randomly picked integer x from the set $\{1, 2, \ldots, 7^{10000}\}$ will satisfy $\gcd(x, 7^{10000}) = 1$.
   (A) $\frac{1}{7}$ (B) $\frac{7^{10000} - 1}{7^{10000}}$ (C) $\frac{7^{10000} - 10000}{7^{10000}}$ (D) $\frac{6}{7}$ (E) None of the above

4. (2006 AMC 12B #7) Mr. and Mrs. Lopez have two children. When they get into their family car, two people sit in the front, and the other two sit in the back. Either Mr. Lopez or Mrs. Lopez must sit in the driver's seat. How many seating arrangements are possible?
   (A) 4 (B) 12 (C) 16 (D) 24 (E) 48

5. (2006 AMC 12B #8) How many even three-digit integers have the property that their digits, read left to right, are in strictly increasing order?
   (A) 21 (B) 34 (C) 51 (D) 72 (E) 150

6. (2020 AMC 10A #6) How many 4-digit positive integers (that is, integers between 1000 and 9999, inclusive) having only even digits are divisible by 5?
   (A) 80 (B) 100 (C) 125 (D) 200 (E) 500

7. (2002 AMC 10B #9) Using the letters A, M, O, S, and U, we can form five-letters "words". If these "words" are arranged in alphabetical order, then the "word" $USAMO$ occupies position
   (A) 112 (B) 113 (C) 114 (D) 115 (E) 116

8. (1990 AHSME #16) At one of George Washington's parties, each man shook hands with everyone except his spouse, and no handshakes took place between women. If 13 married couples attended, how many handshakes were there among these 26 people?
   (A) 78 (B) 185 (C) 234 (D) 312 (E) 325

9. (2003 AMC 12B #19) Let S be the set of permutations of the sequence $1, 2, 3, 4, 5$ for which the first term is not 1. A permutation is chosen randomly from S. The probability that the second term is 2, in lowest terms, is a/b. What is $a + b$?
   (A) 5 (B) 6 (C) 11 (D) 16 (E) 19

10. (2020 AMC 10A #15) A positive integer divisor of $12!$ is chosen at random. The probability that the divisor chosen is a perfect square can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?
    (A) 3 (B) 5 (C) 12 (D) 18 (E) 23

Practice 38: Mixed Competition Practice - SOLUTIONS

Date: Wednesday, January 22

1. $\dfrac{13! - 12!}{11!} = \dfrac{13!}{11!} - \dfrac{12!}{11!} = 13(12) - 12 = 12(13 - 1) = 144$. **The answer is (C).**

2. $2^{2017} \pmod 5 \equiv 2 \cdot (2^2)^{1008} \equiv 2 \cdot (-1)^{1008} \equiv 2 \cdot 1 \equiv 2$. **The answer is (C).**

3. The only numbers that have a common factor with 7^{10000} are multiples of 7. Therefore, for any x that is not a multiple of 7, $\gcd(x, 7^{10000}) = 1$. Precisely $\frac{6}{7}$ of the integers in the set are not multiples of 7. **The answer is (D).**

4. If Mr. Lopez sits in the driver's seat, the other 3 seats can be filled in $3! = 6$ ways. There are also 6 ways the seats can be filled if Mrs. Lopez sits in the driver's seat. This gives a total of 12 arrangements. **The answer is (B).**

5. The last digit can be either 4, 6, or 8 (do you see why it can't be 2?). If the last digit is 4, there are $\binom{3}{2} = 3$ choices for the first two digits, each pair of which corresponds to a specific order (1-2, 1-3, 2-3). Similarly, if the last digit is 6, there are $\binom{5}{2} = 10$ possibilities; if the last digit is 8, there are $\binom{7}{2} = 21$ possibilities. This gives a total of $3 + 10 + 21 = 34$ such integers. **The answer is (B).**

6. There are 4 choices for the first digit (2, 4, 6, 8), 5 choices for each of the second and third digit (0, 2, 4, 6, 8), and 1 choice for the last digit (0). This gives a total of $4(5)(5) = 100$ such integers. **The answer is (B).**

7. There are $4(4!) = 96$ words that do not start with U, all of which occur before $USAMO$. There are $3!$ words that start with each of UA, UM, and UO, which give another 18 words that occur before $USAMO$. In fact, $USAMO$ is the first word that starts with US, so it must sit in position $96 + 18 + 1 = 115$. **The answer is (D).**

8. Among the men, there will be $\binom{13}{2} = 78$ handshakes, and each man will shake hands with 12 women for another $13(12) = 156$ handshakes. This is a total of $78 + 156 = 234$ handshakes. **The answer is (C).**

9. There are $5! = 120$ total permutations of 1 through 5 and $4! = 24$ of these start with 1, so $120 - 24 = 96$ do not start with 1. Of these, $3 \cdot 3!$ have a 2 in the second position (the first digit can't be 1 or 2, so 3 choices, and there are $3!$ ways to arrange the last 3 digits). This gives a probability of $\frac{3 \cdot 3!}{4 \cdot 4!} = \frac{3}{16}$, so $a + b = 3 + 16 = 19$. **The answer is (E).**

10. $12!$ factors into $2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$. Any perfect square factor must have an even exponent. In this case, to construct a perfect square factor, we can choose from an even number of 2's, an even number of 3's, and an even number of 5's. We have 6 choices for the number of 2's (0, 2, 4, 6, 8, 10), 3 choices for the number of 3's (0, 2, 4), and 2 choices for the number of 5's (0, 2). This gives $6 \cdot 3 \cdot 2$ perfect square factors. (To see an example of why this is valid, suppose we used 6 2's, 0 3's, and 2 5's. Our factor would be $2^6 \cdot 3^0 \cdot 5^2 = (2^3 \cdot 5)^2 = 40^2$.) There are $(10 + 1)(5 + 1)(2 + 1)(1 + 1)(1 + 1) = 11 \cdot 6 \cdot 3 \cdot 2 \cdot 2$ total factors of 12!, so the probability is $\frac{6 \cdot 3 \cdot 2}{11 \cdot 6 \cdot 3 \cdot 2 \cdot 2} = \frac{1}{22}$, giving $m + n = 1 + 22 = 23$. **The answer is (E).**

Chapter 3: Geometry

Practice 41: Introductory Overview

Date: Tuesday, February 4

The most useful strategies for elementary geometry problems are...

1. If you want an unknown quantity (length, angle, area, etc.), **give it a name** (like x, A, θ, etc.).
2. Use as few "names" as possible: always try to name express quantities in terms of other quantities.
3. You often need to draw other segments to make triangles, squares, etc.
4. You can often compare areas of shapes by finding identical shapes using folds, rotations, etc.

The most useful facts for elementary geometry problems are ones that enable you to relate quantities together, i.e. to write equations. The most frequently used of these facts are...

1. The angles in a triangle sum to $180°$.
2. Pythagorean Theorem: If a right triangle has side lengths a, b, c (where $a \leq b < c$), then $a^2 + b^2 = c^2$.
3. Special Right Triangles: In a 30-60-90 triangle (half of an equilateral triangle) with sides $h, m,$ and s (for hypotenuse, medium, and short), the following relationships hold:

   $$h = 2s \qquad\qquad m = \sqrt{3}s$$

   These can be rearranged to find other relationships when necessary.

   In a 45-45-90 (half of a square) with sides h and ℓ (for hypotenuse and leg), the following relationships hold:

   $$h = \ell\sqrt{2} \qquad\qquad \ell = \frac{h}{\sqrt{2}} = h \cdot \frac{\sqrt{2}}{2}$$

4. Law of Cosines: If a triangle ΔABC has side lengths $a, b,$ and c across from sides $\overline{BC}, \overline{AC},$ and $\overline{AB}$, respectively, then

   $$c^2 = a^2 + b^2 - 2ab\cos(\angle C)$$

5. Triangle Similarity: If two or more pairs of corresponding angles in two triangles are congruent, then the triangles are similar and their side lengths are proportional, or in a *constant ratio*. If $\Delta ABC \sim \Delta DEF$, then

   $$\frac{AB}{DE} = \frac{BC}{DE} = \frac{AC}{DF}$$

   A number of other proportions hold as well, such as

   $$\frac{AB}{BC} = \frac{DE}{EF} \qquad \text{and} \qquad \frac{AC}{BC} = \frac{DF}{EF}$$

Beyond this, there are many other facts and formulas that are helpful to know. A few of these include...

1. The circumference of a circle, or the length around the circle, is $C = \pi d = 2\pi r$, where d and r are the diameter and radius of the circle, respectively.
2. The area of a circle is $A = \pi r^2$, where r is the radius.
3. The area of a triangle ABC, sometimes written $[ABC]$, can be computed many ways, including

$$\begin{aligned}[ABC] &= \frac{1}{2}bh && \text{(where } b \text{ and } h \text{ are perpendicular)}\\ &= \frac{1}{2}ab\sin\angle C && \text{(where } \angle C \text{ is between } a \text{ and } b\text{)}\\ &= \sqrt{s(s-a)(s-b)(s-c)} && \text{(Heron's Formula: where } s = \tfrac{a+b+c}{2}\text{)}\\ &= \frac{s^2\sqrt{3}}{4} && \textbf{(if the triangle is equilateral)}\end{aligned}$$

If the triangle is given in the coordinate plane with vertices with coordinates (x_1, y_1), (x_2, y_2), and (x_3, y_3), then the area can be computed using

$$[ABC] = \frac{1}{2}\left\|\begin{matrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{matrix}\right\| = \frac{1}{2}\left|(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)\right|$$

(half the abs. value of the determinant of a matrix where 2 sides are expressed as vectors)

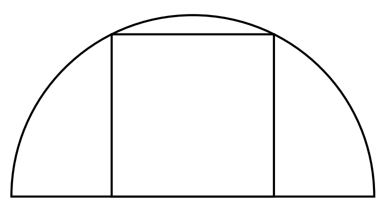
$$= I + \frac{B}{2} - 1$$

(Pick's Theorem: where I = lattice points inside ABC and B = lattice points on the edge of ABC)

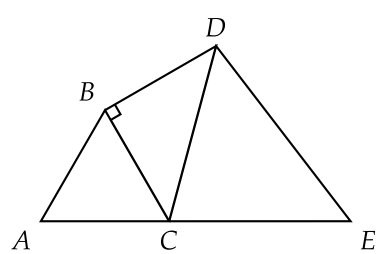
There is much, much more to learn than this, but with these strategies and facts alone, you can solve many problems!

Practice

1. A square is inscribed in a semicircle with radius 6. Find the area of the square.



2. Triangle ABC has $m\angle B = 90°$ and $BC = 8$. Let BD be the altitude from B to $\overline{AC}$. If $DC = 6$, find AD.
3. If $m\angle E = 52.5°$, $m\angle ACB = 60°$, $CE = CD$, $AC = BC$, and $CE = 8$, find the area of ΔABC.

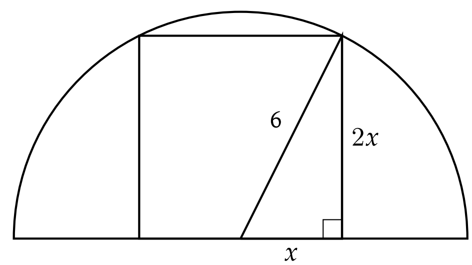


4. Three circles, each with radius 2, are externally tangent to one another. If the area of the region enclosed by the 3 circles can be written as $a\sqrt{b} - c\pi$, where c is not the square of any prime, compute $a + b + c$.
5. Find the area of the triangle with vertices $(3, 5)$, $(7, -1)$, and $(8, 2)$ in three different ways.

Practice 41: Introductory Overview - SOLUTIONS

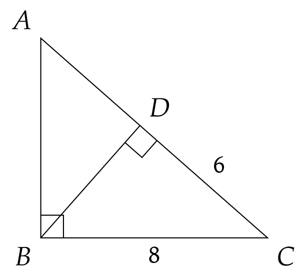
Date: Tuesday, February 4

1. Connect the center of the circle to one corner of the square and let the side length of the square be $2x$. Then a right triangle with side lengths x, $2x$, and 6.

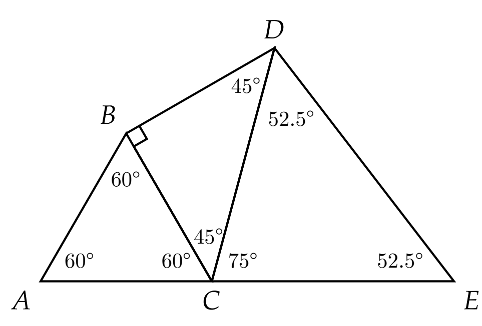


We can now use the Pythagorean Theorem to get $x^2 + (2x)^2 = 6^2$, or $x^2 = \frac{36}{5}$. Rather than solving for x, we want the area of the square: the square will have area $(2x)^2 = 4x^2$, so we simply compute $4x^2 = 4\left(\frac{36}{5}\right) = \mathbf{\frac{144}{5}}$.

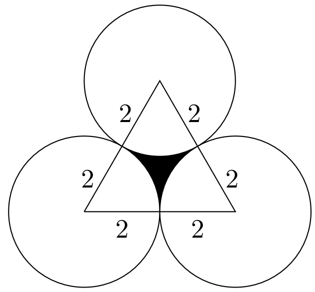
2. Triangle ΔDCB is similar to ΔBCA, so $\frac{DC}{BC} = \frac{BC}{AC}$, or $\frac{6}{8} = \frac{8}{AC}$. This gives $AC = \frac{32}{3}$, so $DC = \frac{32}{3} - 6 = \mathbf{\frac{14}{3}}$.



3. Since $CD = CE$, $m\angle D = m\angle E = 52.5°$, so $m\angle C = 180 - 105 = 75$. Then, we can compute $m\angle BCD = 180 - (60 + 75) = 45°$. This ensures ΔBDC is a 45-45-90 triangle, so $BC = \frac{8}{\sqrt{2}} = 4\sqrt{2}$. Finally, since $AB = BC$, we know $m\angle A = m\angle B$. Given that $m\angle BCA = 60$, it must be the case that $m\angle A = m\angle B = 60°$. Therefore, ΔABC is equilateral with side length $4\sqrt{2}$ and its area is $[ABC] = \frac{(4\sqrt{2})^2\sqrt{3}}{4} = \mathbf{8\sqrt{3}}$.



4. Connecting the centers of the circle creates an equilateral triangle with side length 4.



We can find the enclosed area by subtracting the area of the 3 circular sectors from the area of the triangle.

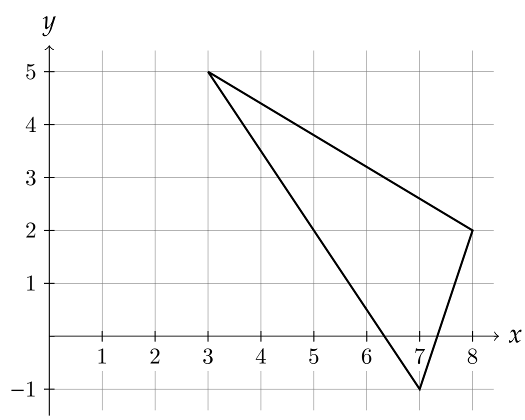
$$\begin{aligned}\text{Enclosed area} &= \text{Triangle area} - 3(\text{Sector area}) \\ &= \frac{4^2\sqrt{3}}{4} - 3\left(\frac{2^2\pi}{6}\right) \\ &= 4\sqrt{3} - 2\pi\end{aligned}$$

Thus $a = 4, b = 3, c = 2$, and $\boldsymbol{a + b + c = 9}$.

5. Way 1: Using $(3, 5)$ as our "origin," we can express two sides of the triangle as the vectors $\begin{pmatrix} 7-3 \\ -1-5 \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$ and $\begin{pmatrix} 8-3 \\ 2-5 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$. Therefore, the area is

$$\frac{1}{2}\left\|\begin{matrix} 4 & 5 \\ -6 & -3 \end{matrix}\right\| = \frac{1}{2}|-12-(-30)| = \mathbf{9}$$

Way 2: We draw the triangle in the plane.



Now, we use Pick's Theorem. There are $I = 8$ lattice points in the interior of the triangle and 4 lattice points on its boundary, giving us an area of

$$I + \frac{B}{2} - 1 = 8 + \frac{4}{2} - 1 = \mathbf{9}$$

Practice 41: Geometry Practice

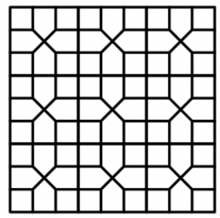
Go to `https://artofproblemsolving.com/`, click **RESOURCES**, and then select **ALCUMUS**.

You may need to make an account. You can press **Change Focus** to switch to a different topic.

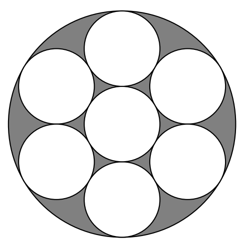
Practice 48: Geometry Competition Practice

Date: Wednesday, March 5

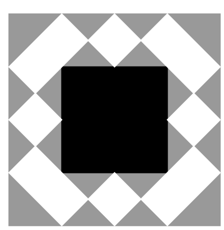
1. (2001 AMC 12 #10) The plane is tiled by congruent squares and congruent pentagons as indicated. The percent of the plane that is enclosed by the pentagons is closest to
(A) 50 (B) 52 (C) 54 (D) 56 (E) 58



2. (2002 AMC 12A #5) Each of the small circles in the figure has radius one. The innermost circle is tangent to the six circles that surround it, and each of those circles is tangent to the large circle and to its small-circle neighbors. Find the area of the shaded region.
(A) π (B) 1.5π (C) 2π (D) 3π (E) 3.5π



3. (2002 AMC 12A #8) Betsy designed a flag using gray triangles, small white squares, and a black center square, as shown. Let G be the total area of the gray triangles, W the total area of the white squares, and B the area of the black square. Which of the following is correct?
(A) $G = W$ (B) $W = B$ (C) $G = B$ (D) $3G = 2B$ (E) $2B = W$

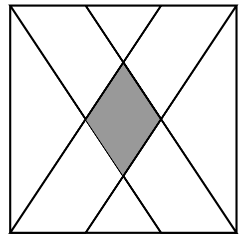


4. (AMC 2003 12A #7) How many non-congruent triangles with perimeter 7 have integer side lengths? (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

5. (GA Math League Feb. 2024) When the lengths of the diagonals of a rhombus are squared and added, the sum is 100. What is the perimeter of the rhombus?

6. (2002 AMC 12B #5) Let v, w, x, y, and z be the degree measures of the five angles of a pentagon. Suppose that $v < w < x < y < z$ and v, w, x, y, and z form an arithmetic sequence. Find the value of x.
(A) 72 (B) 84 (C) 90 (D) 108 (E) 120

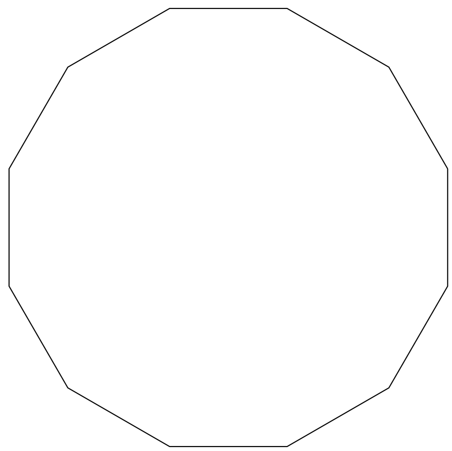
7. (GA Math League Nov. 2022) Each vertex of a square is connected to one of the trisection points of a side of the square not containing that vertex, as shown. If the area of the square is 360, what is the area of the shaded region?



8. A square and an equilateral triangle have the same perimeter. Let A be the area of the circle circumscribed about the square and B the area of the circle circumscribed around the triangle. Find A/B.

   (A) $\frac{9}{16}$ (B) $\frac{3}{4}$ (C) $\frac{27}{32}$ (D) $\frac{3\sqrt{6}}{8}$ (E) E

9. (GA Math League Dec. 2022) Inside a regular dodecagon of area 360, there are 3 unshaded equilateral triangles and 3 unshaded pentagons, as shown. Each pentagon has two $90°$ angles, two $150°$ angles, and one $60°$ angle. What is the area of the entire shaded region?

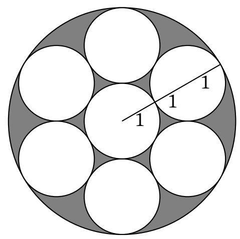


Practice 48: Geometry Practice - SOLUTIONS

Date: Wednesday, March 5

1. The diagram is made of 9 congruent squares. In one of these squares, you can connect the edges around the "X" in the middle to form a square. Then, the 4 pentagons form 4 squares (top middle, middle left, middle right, bottom middle) on the edges and 1 square in the middle (the "X" square). There are 4 squares left over. Therefore, the pentagons form $\frac{5}{9} = 0.\overline{5} \approx 0.56$, or 56%, of the plane. **The answer is (D).**

2. Draw a radius from the middle circle through the intersection of the circle to the upper right.



The outer circle has a radius of 3 and an area of 9π. There are 6 inner circles each with area π. The shaded area is therefore $9\pi - 6\pi = 3\pi$. **The answer is (D).**

3. The entire diagram can be tiled in terms of triangles the same size as the gray ones. There are 24 gray triangles, 16 black triangles, and 24 white triangles. Therefore, $G = W$. **The answer is (A).**

4.

5.

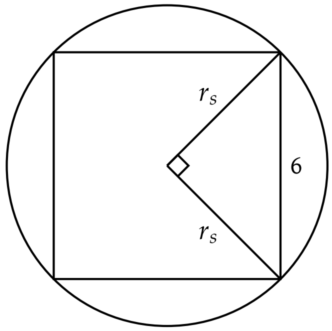
6. A pentagon can be divided into 3 triangles such that the sum of angles of the pentagon is the sum of angles of the 3 triangles. This makes $v + w + x + y + z = 3(180) = 540$. Now, since the angles form an arithmetic sequence, there is a common difference d such that the sequence can be written

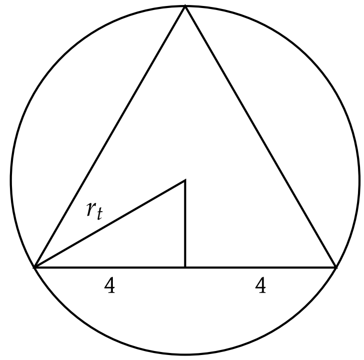
$$x - 2d, x - d, x, x + d, x + 2d$$

so that $v = x - 2d, w = x - d, y = x + d$, and $z = 2d$. Summing these yields $5x = 540$, or $x = 108$. **The answer is (C).**

7.

8. We could use variables, but it's simplest to use a number. Say the perimeter of the square and equilateral triangle is 24. Then the side lengths are 6 and 8, respectively.





We can use properties of isosceles and equilateral right triangles to compute $r_s = \frac{6}{\sqrt{2}} = 3\sqrt{2}$ and $r_t = 4 \cdot \frac{2\sqrt{3}}{3} = \frac{8\sqrt{3}}{3}$. Therefore, the ratio of the area of the square's circle to the area of the triangle's circle is

$$\frac{(3\sqrt{2})^2\pi}{\left(\frac{8\sqrt{3}}{3}\right)^2\pi} = \frac{27}{32}$$

The answer is (C).

9.

10.