The *Pell companion numbers* are the sequence $1, 3, 7, 17, 41, \ldots$ that is recursively defined by

$$a_n = 2a_{n-1} + a_{n-2}$$

Assuming $a_n = c^n$[1], we can write

$$\begin{aligned} c^n &= 2c^{n-1} + c^{n-2} \\ c^2 - 2c - 1 &= 0 \\ c &= \frac{2 \pm 2\sqrt{2}}{2} \\ &= 1 \pm \sqrt{2} \end{aligned}$$

This gives two possible values of c, call them $c_1 = 1+\sqrt{2}$ and $c_2 = 1-\sqrt{2}$, so $a_n = \lambda_1(1+\sqrt{2})^n + \lambda_2(1-\sqrt{2})^n$.

Using the initial conditions $a_1 = 1$ and $a_2 = 3$ yield

$$1 = \lambda_1(1+\sqrt{2}) + \lambda_2(1-\sqrt{2}) \tag{1}$$
$$3 = \lambda_1(1+\sqrt{2})^2 + \lambda_2(1-\sqrt{2})^2$$

Solving yields

$$\begin{aligned} 1 &= \lambda_1 + \lambda_2 + \sqrt{2}(\lambda_1 - \lambda_2) \\ 3 &= 3(\lambda_1 + \lambda_2) + 2\sqrt{2}(\lambda_1 - \lambda_2) \\ \Rightarrow 1 &= \lambda_1 + \lambda_2 \\ \Rightarrow \lambda_2 &= 1 - \lambda_1 \end{aligned}$$

Substituting this into (1) yields

$$\begin{aligned} 1 &= \lambda_1 + 1 - \lambda_1 + \sqrt{2}(\lambda_1 - (1 - \lambda_1)) \\ 1 &= 1 - \sqrt{2} + 2\sqrt{2}\lambda_1 \\ \lambda_1 &= \frac{1}{2} \end{aligned}$$

Substituting this into (2) yields

$$\lambda_2 = 1 - \frac{1}{2} = \frac{1}{2}$$

Therefore $a_n = \left(\frac{1}{2}\right)(1+\sqrt{2})^n + \left(\frac{1}{2}\right)(1-\sqrt{2})^n = \frac{1}{2}\left((1+\sqrt{2})^n + (1-\sqrt{2})^n\right)$.

The *Pell numbers* are the sequence $1, 2, 5, 12, 29, \ldots$ that is recursively defined by

$$b_n = 2b_{n-1} + b_{n-2}$$

Using the initial conditions $b_1 = 1$ and $b_2 = 2$, we can similarly find that

$$b_n = \frac{\sqrt{2}}{4}(1+\sqrt{2})^n - \frac{\sqrt{2}}{4}(1-\sqrt{2})^n = \frac{\sqrt{2}}{4}\left((1+\sqrt{2})^n - (1-\sqrt{2})^n\right)$$

The main feature of the Pell numbers and the Pell companion numbers is that, as n increases, the ratio $\frac{a_n}{b_n}$ approaches $\sqrt{2}$, giving rational estimates of the irrational $\sqrt{2}$. In particular,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \sqrt{2}$$

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[1] This method for finding a closed form for a sequence defined by a linear recursion is covered in **Section 10.3: Linear Recurrences** of the *Intermediate Counting and Probability* book published by Art of Problem Solving.

THEOREM Let n be a positive integer. Then

$$(\sqrt{2}-1)^n = \begin{cases} b_n\sqrt{2}-a_n & n \text{ is odd} \\ a_n - b_n\sqrt{2} & n \text{ is even} \end{cases}$$

PROOF

We will prove using the two cases.

Case 1: n is odd
Then we have

$$\begin{aligned} b_n\sqrt{2}-a_n &= \sqrt{2}\left(\frac{\sqrt{2}}{4}\left((1+\sqrt{2})^n-(1-\sqrt{2})^n\right)\right) - \frac{1}{2}\left((1+\sqrt{2})^n+(1-\sqrt{2})^n\right) \\ &= \frac{1}{2}\left(\left((1+\sqrt{2})^n-(1-\sqrt{2})^n\right)\right) - \frac{1}{2}\left((1+\sqrt{2})^n+(1-\sqrt{2})^n\right) \\ &= -(1-\sqrt{2})^n \\ &= -(-1)^n(\sqrt{2}-1)^n \\ &= -(-1)(\sqrt{2}-1)^n && \text{(since } n \text{ is odd)} \\ &= (\sqrt{2}-1)^n \end{aligned}$$

Case 2: n is even
Then we have

$$\begin{aligned} a_n-b_n\sqrt{2} &= \frac{1}{2}\left((1+\sqrt{2})^n+(1-\sqrt{2})^n\right) - \sqrt{2}\left(\frac{\sqrt{2}}{4}\left((1+\sqrt{2})^n-(1-\sqrt{2})^n\right)\right) \\ &= \frac{1}{2}\left((1+\sqrt{2})^n+(1-\sqrt{2})^n\right) - \frac{1}{2}\left((1+\sqrt{2})^n-(1-\sqrt{2})^n\right) \\ &= (1-\sqrt{2})^n \\ &= (-1)^n(\sqrt{2}-1)^n \\ &= (\sqrt{2}-1)^n && \text{(since } n \text{ is even)} \end{aligned}$$

The proof is complete.

In practice, this gives a much faster method for computing powers of $(\sqrt{2}-1)^n$.

Example: Compute $(\sqrt{2}-1)^{11}$.

Solution: We compute the Pell and companion Pell numbers up to the 11th terms using the recursive definition.

$$\begin{aligned} a_n &: 1, 3, 7, 17, 41, 99, 239, 577, 1393, 3363, 8119 \\ b_n &: 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741 \end{aligned}$$

By the Theorem, $(\sqrt{2}-1)^n = b_{11}\sqrt{2}-a_{11} = \mathbf{5741\sqrt{2}-8119}$.