Differentiation, dimension and the duality of continuity

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Abstract

I propose a link between orders of differentiation and dimension, before discussing what I see as dual aspects of continuity.

1. Linear differentials

The derivative of a univariate function f at a point x is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The denominator h has a power equal to 1 and the difference $f(x+h)-f(x)$ is approximated at a linear rate (for differentiable functions). In multivariate calculus, this is what yields linear differentials. The question is why linearity in differentiation? Why this precise speed of convergence?

This is not a matter of conventions since most functions we work with would not be differentiable at a different rate (especially higher). For example,

$$\frac{f(x+h) - f(x)}{h^\alpha}$$

converges to 0 when $\alpha < 1$ if f is differentiable at x, whereas it diverges when $\alpha > 1$ if the canonical derivative is non-null. Most elementary functions are differentiable at the canonical rate (power functions, polynomials, the exponential, the logarithm, trigonometric functions...), and even composing those functions with powers does not affect this property. Those functions are not necessarily elementary because solutions of elementary differential equations - what makes use of the above definition - but because they can be defined using only the basic operations of addition, multiplication, exponents, limits, real and imaginary parts, basic functional equations... In any case, even curves that we naturally consider to be smooth are differentiable at this same rate, and first-order approximations are the reason behind Hooke's law in physics [27].

This also extends beyond the first order: the next order must be quadratic. This is what provides Taylor approximations with polynomials, i.e integer powers. Provided enough regularity, we cannot consider intermediate speeds for approximations.

2. Hausdorff dimension

The underlying reason is linked to space dimensions: curves are one-dimensional objects. One may have indeed noticed the similarity between the 0-∞ disjunction above when $\alpha < 1$ and $\alpha > 1$, and the definition of the Hausdorff dimension (or that of the packing dimension [11, Chapter 3]). We recall the definition of the Hausdorff dimension.

Definition 1 (Diameter)**.** Let $(\mathrm{X}, \mathtt{d})$ be a metric space. The *diameter* function is defined for subsets $\mathrm{S} \subset \mathrm{X}$ by

$$\mathtt{diam}(\mathrm{S}) = \sup_{x,y \in \mathrm{S}} \mathtt{d}(x, y),$$

with the convention that $\mathtt{diam}(\emptyset) = 0$.

Definition 2 (Hausdorff measure)**.** Let $d \geqslant 0$. We consider the outer-measure defined by

$$\mathtt{H}^d(\mathrm{S}) := \lim_{\varepsilon \to 0} \inf_{\substack{\mathrm{S} \subset \cup \mathrm{C}_i \\ \mathtt{diam}(\mathrm{C}_i) \leqslant \varepsilon}} \sum_{i=1}^{+\infty} \mathtt{diam}^d(\mathrm{C}_i).$$

For integers d, the Hausdroff measure is proportional to the Lebesgue measure [2, 20].

One may verify that if $\mathtt{H}^d(\mathrm{S}) \in (0, +\infty)$ for some d, then $\mathtt{H}^p(\mathrm{S}) = 0$ for all $p > d$ and $\mathtt{H}^p(\mathrm{S}) = +\infty$ for all $p < d$ (notice that the summand for p is upper-bounded by $\varepsilon^{p-d}\mathtt{diam}^d(\mathrm{C}_i)$ for that). Intuitively, the volume becomes 0 in a higher dimensional setting and it explodes if not enough dimension. Note however that $\mathtt{H}^d(\mathrm{S})$ is not necessarily finite non-null for some $d \geqslant 0$ (see [11, Exercise 4.9] and [12], or [32]).

Definition 3 (Hausdorff dimension)**.** We define

$$\mathtt{dim}_{\mathtt{H}}\mathrm{S} := \mathtt{sup}\{d \geqslant 0 \ : \ \mathtt{H}^d(\mathrm{S}) = +\infty\} = \mathtt{inf}\{d \geqslant 0 \ : \ \mathtt{H}^d(\mathrm{S}) = 0\} \in [0; +\infty].$$

The Hausdorff dimension of a line is 1, that of a (filled) square and that of a disk is 2, and that of a (filled) cube and that of a ball is 3. Any positive value can be attained [11, Example 11.3, Corollary 7.4] (or [15], [9, Chapter 5 (translation)], [5]).

As a remark, the notion of dimension can also be defined with box-counting [11, Chapter 3].

3. Differentiation and dimension

The previous two sections exhibit similar 0-∞ disjunctions. This intuition can be formalized by the following results. The first one is well-known [11, Exercise 2.7], [12], and [11, Corollary 11.2].

Result 4. *Let $f\colon [0;1] \to \mathbb{R}$ be a Lipschitz continuous function. Then, its graph has Hausdorff and box-counting dimensions equal to 1, where*

$$\mathrm{Graph}(f) := \{(x, f(x)) \ : \ x \in [0;1]\}.$$

As a remark, Lipschitz continuous functions over $\mathbb{R}^d$ are differentiable almost everywhere by Rademacher's theorem (see [30, Theorem 10.8] for instance). An extension with more general exponents is the following result (see [11, Corollary 11.2] for instance).

Result 5. *Let $d \in [1;2[$ and $f\colon [0;1] \to \mathbb{R}$ satisfying the following conditions:*

(i) Hölder condition: $\exists \kappa > 0$ such that $|f(x) - f(y)| \leqslant \kappa |x-y|^{2-d}$ for all $x, y \in [0;1]$,
(ii) reverse Hölder condition: $\exists \kappa' > 0$ such that for all $x \in [0;1]$ and $\delta > 0$ there exists y verifying $|x-y| < \delta$ and $|f(x) - f(y)| \geqslant \kappa' \delta^{2-d}$.

Then, the graph of f has box-counting dimension d.

One may have noticed that in Section 1 the value is 0 when $\alpha < 1$, and that in Section 2 it is when $p > 1$. As highlighted by Result 5, the two quantities are linked through the reflection $x \mapsto 2 - x$ (with fixed point 1). With two variables, the relevant transformation is $x \mapsto 3 - x$ [34, Lemma 4.2].

Result 5 has no direct converse as shown by the Cantor function, which is optimally $\log(2)/\log(3)$-Hölder continuous [8, Proposition 10.1] with a graph of Hausdorff dimension 1 (see [6] and [3, Proposition 2.6.2]). See also [33, Lemma 3.2, Theorem 4.1] with Koch's curve.

4. Duality of continuity

It appears with differentiation that enough regularity yields discreteness: for the order of differentiation, i.e the powers of Taylor's approximation, and for the dimension of the graph.

This was already clear with fractal dimensions: objects regular enough have integer dimensions. A continuum of dimensions arises when considering objects such as fractals, that lack some regularity properties. This highlights a dual aspect of continuity when considering the smoothness of the curve and its Hausdorff dimension: regularity for the curve and discreteness of its dimension, or a continuum of dimensions for curves with a lack of regularity. Continuity cannot be present in both aspects.

Such a consideration is even older if we consider the Jordan curve theorem. .

Definitions 6. A *topological curve* is the image of continuous function $\varphi\colon \mathrm{I} \to \mathrm{X}$ with I an interval of $\mathbb{R}$ and X a topological space.

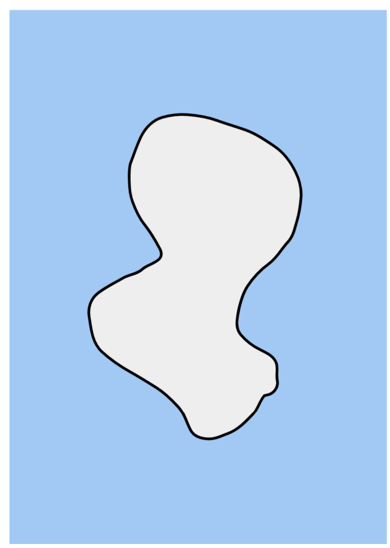
A curve is said to be *closed* if $I = [a;b]$ and $\varphi(a) = \varphi(b)$ where $a \leqslant b \in \mathbb{R}$.

A curve is said to be *simple* if φ is injective on $[a;b[\neq \emptyset$.

A *Jordan curve* is a simple closed curve.

Result 7 (Jordan curve theorem)**.** *Let $\mathrm{C} \subset \mathbb{R}^2$ be a Jordan curve.*

Then, $\mathbb{R}^2 \setminus \mathrm{C}$ has exactly two connected components, one of which is bounded (and the other not), C being the boundary of each of those components.



See [19, 13, 29, 4] for the proof, and [14, Section 2.B] or [7, Chapter II] in higher dimensions. Regarding converses, see [21, Theorem 12, p.518], [26], [21, Theorem 5', p.513], [16].

By studying continuity from an inside perspective (the curve), or from an outside perspective (two distinct areas that are not path-connected), the Jordan curve theorem illustrates how continuity isolates distinct mathematical objects.

Similarly, the mathematical branch of topology distinguishes objects depending on whether we can obtain one from a continuous deformation of the other.

5. Fractals

I conclude with a word on fractals. They were initially discovered by looking for counterexamples to wrong intuitive ideas of calculus [25, 1, 18, 24], historically referred to by some as "mathematical monsters" [28]. But is there a "wrong" side of the duality of continuity?

First, when walking down the street along human buildings, road signs and cars, all made of straight lines and smooth curves, one may notice that river networks, tree branches, clouds, mountains and coastlines all involve fractal structures [22, 23]. Fractals do exist in the real world. Even us, humans, are concerned since our circulatory, respiratory and nervous systems are all fractal [10, 17, 31].

Second, irregularity can be useful. Without friction we might not be able to grasp objects nor to walk. Many fractals, like Koch's snowflake for example, also illustrate how a surface can have a finite area but an infinite perimeter. In higher dimension they can maximize the contact surface of a given volume. Quite convenient in lungs for example, or any other interface. Space-filling curves such as Peano's curve operate similarly, folding a line of infinite length into a bounded surface. Fractals can then be viewed as solutions to optimization problems. This could perhaps be justified in non-evolutionary bodies such as clouds or coastlines by a variational approach (or by the fact that the real world handles an almost infinite number of influences, see for instance the Brownian motion).

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