

Task-02: Quantum Computing – Linear Algebra Foundations

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Independence, Basis, and Dimension

What does it mean for vectors to be independent? How does the idea of independence help us describe subspaces like the nullspace?

Linear independence:

Suppose A is an $m \times n$ matrix with $m < n$ (so $A\vec{x} = \vec{b}$ has more unknowns than equations). A has at least one free variable, so there are nonzero solutions to $A\vec{x} = 0$. A combination of the columns is zero, so the columns of this A are dependent.

We say vectors $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ are linearly independent (or just independent) if:

$$c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_n\vec{x}_n = \vec{0}$$

only when $c_1 = c_2 = \dots = c_n = 0$.

When those vectors are the columns of A , the only solution to $A\vec{x} = 0$ is $\vec{x} = 0$.

Two vectors are independent if they do not lie on the same line. Three vectors are independent if they do not lie in the same plane.

Thinking of $A\vec{x}$ as a linear combination of the column vectors of A , we see that the column vectors of A are independent exactly when the nullspace of A contains only the zero vector.

If the columns of A are independent then all columns are pivot columns, the rank of A is n , and there are no free variables. If the columns of A are dependent then the rank of A is less than n and there are free variables.

Spanning a space

Vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ span a space when the space consists of all combinations of those vectors. For example, the column vectors of A span the column space of A .

If vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ span a space S , then S is the smallest space containing those vectors.

Basis and Dimension

A basis for a vector space is a sequence of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_d$ with two properties:

- $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_d$ are independent.
- $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_d$ span the vector space.

The basis of a space tells us everything we need to know about that space.

Example: R^3

One basis for R^3 is:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

These are independent because:

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow c_1 = c_2 = c_3 = 0$$

These vectors span R^3 .

As discussed at the start of Lecture 10, the vectors:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

do not form a basis for R^3 because these are the column vectors of a matrix that has two identical rows. The three vectors are not linearly independent.

In general, n vectors in R^n form a basis if they are the column vectors of an invertible matrix.

Linear Operators and Matrices

Introduction

In both linear algebra and quantum computing, a **linear operator** is a rule or transformation that maps one vector to another in a vector space while preserving vector addition and scalar multiplication. In finite-dimensional vector spaces, linear operators are represented by **matrices**. Understanding linear operators is crucial in quantum computing, as quantum gates are implemented through unitary linear operators.

Definition of a Linear Operator

Let V and W be vector spaces over a field F (usually R or C). A function $T : V \rightarrow W$ is a **linear operator** if for all $\vec{u}, \vec{v} \in V$ and scalars $c \in F$, the following properties hold:

1. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ (Additivity)
2. $T(c\vec{v}) = cT(\vec{v})$ (Homogeneity)

If $V = W$, then T is called a **linear operator on V** .

Matrix Representation

Let $T : F^n \rightarrow F^m$ be a linear operator. There exists an $m \times n$ matrix A such that for any vector $\vec{x} \in F^n$,

$$T(\vec{x}) = A\vec{x}$$

The entries of matrix A are determined by how T acts on the standard basis vectors of F^n . For example, if

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

then the j -th column of A is $T(\vec{e}_j)$.

Example 1: Rotation Operator

Consider the linear operator $R : R^2 \rightarrow R^2$ that rotates any vector by an angle θ . Its matrix representation is:

$$R(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

This operator is linear and preserves lengths (i.e., it's orthogonal). In quantum computing, rotations are used in gates like $R_x(\theta), R_y(\theta), R_z(\theta)$.

Example 2: Quantum Gate as Linear Operator

The Pauli-X (NOT) gate acts on a qubit and flips $|0\rangle$ and $|1\rangle$:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad X|0\rangle = |1\rangle, \quad X|1\rangle = |0\rangle$$

This is a linear operator acting on a 2D complex Hilbert space, and satisfies:

$$X(\alpha|0\rangle + \beta|1\rangle) = \alpha|1\rangle + \beta|0\rangle$$

Key Properties of Linear Operators (Matrices)

Let A and B be linear operators (matrices), and $\vec{v}, \vec{w}$ be vectors.

- **Linearity:** $A(c_1\vec{v} + c_2\vec{w}) = c_1A\vec{v} + c_2A\vec{w}$
- **Matrix Composition:** If $A \in R^{m \times n}, B \in R^{n \times p}$, then $AB \in R^{m \times p}$
- **Identity Operator:** $I_n\vec{v} = \vec{v}$ for all $\vec{v} \in R^n$
- **Inverse Operator:** If A is invertible, then $A^{-1}A = I$
- **Nullspace:** $N(A) = \{\vec{x} \in F^n : A\vec{x} = \vec{0}\}$
- **Range (Image):** $\text{Im}(A) = \{A\vec{x} : \vec{x} \in F^n\}$

Change of Basis

Let A represent a linear operator in basis B , and let P be the change-of-basis matrix from B to the standard basis. Then:

$$A_{\text{standard}} = PAP^{-1}$$

This similarity transformation changes the matrix form of a linear operator while preserving its eigenvalues.

Linear Operators in Quantum Computing

In quantum mechanics:

- States are vectors in a Hilbert space $\mathcal{H}$
- Observables are represented by **Hermitian** linear operators
- Quantum gates are represented by **unitary** operators U such that $U^\dagger U = I$
- Measurement projects a state onto an eigenspace of an operator

Spectral Theorem

If A is a Hermitian linear operator on a finite-dimensional complex Hilbert space, then:

$$A = \sum_i \lambda_i v_i$$

where λ_i are real eigenvalues and $|v_i\rangle$ are orthonormal eigenvectors.

This allows operators to be "diagonalized" — crucial in quantum mechanics for understanding observables and measurements.

The Pauli Matrices

In quantum mechanics and quantum computing, the **Pauli matrices** are a set of three 2×2 complex Hermitian and unitary matrices that serve as the generators of the Lie algebra $su(2)$. They are central to describing the spin of spin- $\frac{1}{2}$ particles, single-qubit operations, and the algebra of quantum gates.

Definition

The three Pauli matrices are denoted by σ_x , σ_y , and σ_z , or alternatively X , Y , and Z :

$$\sigma_x = X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

These matrices act on 2-dimensional complex vector spaces, such as qubit states $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Properties

Each Pauli matrix has the following important properties:

- **Hermitian:** $\sigma_i^\dagger = \sigma_i$ (all eigenvalues are real).
- **Unitary:** $\sigma_i^\dagger \sigma_i = I$ (preserve norm of states).
- **Traceless:** $\text{Tr}(\sigma_i) = 0$.
- **Determinant:** $\det(\sigma_i) = -1$.
- **Eigenvalues:** For each i , eigenvalues are ± 1 .
- **Involutory:** $\sigma_i^2 = I$.

Commutation and Anticommutation

The Pauli matrices obey specific algebraic rules:

- **Commutator:**

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$$

where ϵ_{ijk} is the Levi-Civita symbol.

- **Anticommutator:**

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}I$$

where δ_{ij} is the Kronecker delta.

Algebra Summary

$$\begin{aligned}\sigma_x \sigma_y &= i\sigma_z, \\ \sigma_y \sigma_z &= i\sigma_x, \\ \sigma_z \sigma_x &= i\sigma_y, \\ \sigma_x^2 &= \sigma_y^2 = \sigma_z^2 = I\end{aligned}$$

Matrix Action on Basis States

$$\begin{aligned}X|0\rangle &= |1\rangle, & X|1\rangle &= |0\rangle \\ Y|0\rangle &= i|1\rangle, & Y|1\rangle &= -i|0\rangle \\ Z|0\rangle &= |0\rangle, & Z|1\rangle &= -|1\rangle\end{aligned}$$

Pauli Vector and Operator Form

The **Pauli vector** is:

$$\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$$

For any real 3D unit vector $\vec{n} = (n_x, n_y, n_z)$, the operator:

$$\vec{n} \cdot \vec{\sigma} = n_x \sigma_x + n_y \sigma_y + n_z \sigma_z$$

is also Hermitian and unitary. It represents a rotation axis in Bloch sphere geometry and defines a general qubit observable.

Application in Quantum Computing

- The Pauli matrices are the basic single-qubit quantum gates.
- They generate more complex gates via exponentiation: $e^{-i\theta\sigma_i}$ represents a rotation.
- They serve as basis operators for decomposing any 2×2 Hermitian operator.
- Measurement in the Pauli-Z basis is standard in most quantum circuits.
- Used in error detection and correction: Pauli group is foundational in stabilizer codes.

Bloch Sphere Interpretation

Every single-qubit pure state can be represented as:

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right) |0\rangle + e^{i\phi} \sin\left(\frac{\theta}{2}\right) |1\rangle$$

The Pauli matrices correspond to rotations around the x , y , and z axes of the Bloch sphere:

$$R_i(\theta) = e^{-i\frac{\theta}{2}\sigma_i}$$

Inner Products

In linear algebra and quantum computing, the **inner product** is a way to define angles, lengths, and orthogonality in a vector space. It is essential in defining quantum state overlaps, probabilities, and operator properties.

Definition

Let $\mathcal{H}$ be a complex vector space (Hilbert space). The inner product of two vectors $|\psi\rangle, |\phi\rangle \in \mathcal{H}$ is a complex number defined as:

$$\langle\phi|\psi\rangle$$

In coordinate form, if:

$$|\phi\rangle = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{bmatrix}, \quad |\psi\rangle = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{bmatrix}$$

then:

$$\langle\phi|\psi\rangle = \sum_{i=1}^n \phi_i^* \psi_i$$

Properties of Inner Product

Let $|\phi\rangle, |\psi\rangle, |\chi\rangle \in \mathcal{H}$, and $a \in \mathbb{C}$. Then:

- Conjugate symmetry: $\langle\phi|\psi\rangle = \overline{\langle\psi|\phi\rangle}$
- Linearity in second argument: $\langle\phi|a\psi + \chi\rangle = a\langle\phi|\psi\rangle + \langle\phi|\chi\rangle$
- Positive-definite: $\langle\psi|\psi\rangle \geq 0$, and $= 0 \iff |\psi\rangle = 0$

Norm and Orthogonality

The **norm** (length) of a vector is defined by:

$$\| |\psi\rangle \| = \sqrt{\langle\psi|\psi\rangle}$$

Two vectors are said to be **orthogonal** if:

$$\langle\phi|\psi\rangle = 0$$

Quantum Mechanics Interpretation

In quantum mechanics, the inner product $\langle\phi|\psi\rangle$ represents the **amplitude** of measuring state $|\psi\rangle$ in the state $|\phi\rangle$. The probability is:

$$P = |\langle\phi|\psi\rangle|^2$$

The inner product determines interference, state distinguishability, and expectation values.

Eigenvectors and Eigenvalues

In linear algebra, **eigenvectors** and **eigenvalues** describe how a linear transformation (matrix/operator) acts on certain special vectors.

In quantum computing, **observable quantities**, **measurement outcomes**, and **energy levels** are modeled by eigenvalues of Hermitian operators.

Definition

Let A be a linear operator (matrix) on vector space V . A non-zero vector $\vec{v}$ is an **eigenvector** of A if:

$$A\vec{v} = \lambda\vec{v}$$

where $\lambda \in \mathbb{C}$ is the associated **eigenvalue**.

Finding Eigenvalues and Eigenvectors

To find eigenvalues, solve the characteristic equation:

$$\det(A - \lambda I) = 0$$

Each solution λ is an eigenvalue. To find the eigenvectors associated with λ , solve:

$$(A - \lambda I)\vec{v} = 0$$

Quantum Measurement Interpretation

In quantum mechanics:

- Observables are Hermitian operators $\hat{O}$.
- Measurement of $\hat{O}$ on state $|\psi\rangle$ yields eigenvalue λ_i with probability:

$$P(\lambda_i) = |\langle v_i | \psi \rangle|^2$$

- The state collapses to $|v_i\rangle$ after measurement.

Example

Let:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

Then the eigenvalues are $\lambda = 2, 3$ and the eigenvectors are:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This matrix represents a quantum observable with two distinct measurement outcomes.

Adjoint and Hermitian Operators

Adjoint of a Matrix or Operator

Let A be a complex matrix or linear operator. Its **adjoint**, denoted $A^\dagger$, is the conjugate transpose:

$$A^\dagger = (\overline{A})^T$$

For example, if:

$$A = \begin{bmatrix} 1+i & 2 \\ -3i & 4 \end{bmatrix}, \quad A^\dagger = \begin{bmatrix} 1-i & 3i \\ 2 & 4 \end{bmatrix}$$

The adjoint satisfies:

$$\langle \phi | A \psi \rangle = \langle A^\dagger \phi | \psi \rangle$$

for all $|\phi\rangle, |\psi\rangle$ in the Hilbert space.

Definition: Hermitian Operators

An operator H is **Hermitian** (or self-adjoint) if:

$$H^\dagger = H$$

In matrix form, this means $H = H^\dagger$, i.e., the matrix equals its own conjugate transpose.

Key Properties of Hermitian Operators

- All eigenvalues of a Hermitian operator are real.
- Eigenvectors corresponding to distinct eigenvalues are orthogonal.
- Hermitian operators are diagonalizable via a unitary transformation.
- They represent **observable** quantities in quantum mechanics.

Example:

Let

$$H = \begin{bmatrix} 2 & i \\ -i & 3 \end{bmatrix}$$

Then:

$$H^\dagger = \begin{bmatrix} 2 & -i \\ i & 3 \end{bmatrix} = H \Rightarrow \text{Hermitian}$$

This matrix has real eigenvalues and can represent a quantum observable.

Tensor Products

Tensor products allow us to describe combined quantum systems. If a single qubit lives in a 2D Hilbert space $\mathcal{H}_2$, then two qubits live in the 4D space $\mathcal{H}_2 \otimes \mathcal{H}_2 = \mathcal{H}_4$.

Tensor Product of Vectors

Let:

$$|a\rangle = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad |b\rangle = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Then their tensor product $|a\rangle \otimes |b\rangle$ is:

$$|a\rangle \otimes |b\rangle = \begin{bmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{bmatrix} \in C^4$$

This forms a basis for two-qubit states such as $|00\rangle, |01\rangle, |10\rangle, |11\rangle$.

Tensor Product of Matrices

If $A \in C^{m \times n}$, $B \in C^{p \times q}$, then:

$$A \otimes B \in C^{mp \times nq}$$

The product is:

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots \\ a_{21}B & a_{22}B & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

Example: Pauli Matrices on 2-Qubit Systems

Let $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then:

$$X \otimes I = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

This applies an X gate to the first qubit and identity to the second.

Properties of Tensor Product

Let A, B, C, D be linear operators:

- $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$
- $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$
- $(A \otimes B)^T = A^T \otimes B^T$
- $\det(A \otimes B) = (\det A)^n (\det B)^m$, for $A \in C^{m \times m}, B \in C^{n \times n}$

Tensor Product and Entanglement

The tensor product structure enables the existence of **entangled states**, which cannot be written as a product of two separate qubit states.

Example:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

This is an entangled Bell state.

Operator Functions

In linear algebra and quantum computing, we often use **functions of operators** — especially functions like exponential, sine, or cosine — where the input is not a number, but a matrix (or operator). These are called **operator functions**.

The most important of these is the **matrix exponential**, written as:

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

Here: - A is a matrix or a linear operator, - I is the identity matrix, - This series works exactly like the exponential function for real numbers, but using matrix multiplication instead.

Why do we need this? Because in quantum mechanics, many operations on quantum states involve exponentials of operators.

For example, in quantum mechanics, the evolution of a quantum state is governed by the **time evolution operator**:

$$U(t) = e^{-iHt/\hbar}$$

where: - H is the Hamiltonian (energy operator), - $\hbar$ is the reduced Planck's constant, - $U(t)$ is a **unitary** operator (preserves probability), - The quantum state at time t is:

$$|\psi(t)\rangle = U(t) |\psi(0)\rangle = e^{-iHt/\hbar} |\psi(0)\rangle$$

This means that the future state of the system is found by applying an operator exponential to the initial state.

How do we calculate functions of operators? If the matrix A is diagonalizable, i.e., we can write:

$$A = PDP^{-1}$$

where D is a diagonal matrix and P is an invertible matrix of eigenvectors, then:

$$f(A) = Pf(D)P^{-1}$$

and $f(D)$ is computed by just applying f to the diagonal entries of D . This makes things much easier.

Example: Rotation operator

Let

$$A = \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}$$

Then,

$$e^A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

which is a rotation matrix in 2D space.

In Quantum Computing: Operator exponentials are used to define quantum gates. For example:

$$R_x(\theta) = e^{-i\theta\sigma_x/2}$$

is a rotation gate around the X-axis of the Bloch sphere. Here, σ_x is the Pauli-X matrix:

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

So, operator functions like exponentials are essential tools for time evolution, gate construction, and solving Schrödinger's equation.

The Commutator and Anti-Commutator

Operators in quantum mechanics don't always behave the way normal numbers do. For example, in arithmetic, $ab = ba$. But in quantum mechanics, applying operator A and then operator B may not be the same as doing B and then A . This is captured using the ****commutator****.

The ****commutator**** of two operators A and B is defined as:

$$[A, B] = AB - BA$$

If $[A, B] = 0$, we say that A and B **commute**, meaning their order does not matter.

If $[A, B] \neq 0$, the operators **do not commute**, and the order in which they are applied affects the outcome.

Why is this important? Because in quantum mechanics, non-commuting operators correspond to physical quantities that cannot be precisely measured at the same time. For example, the position $\hat{x}$ and momentum $\hat{p}$ operators satisfy:

$$[\hat{x}, \hat{p}] = i\hbar$$

This leads directly to the Heisenberg uncertainty principle:

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$$

So the commutator tells us about fundamental limits of precision in measurements.

Commutators of Pauli matrices:

Let $\sigma_x, \sigma_y, \sigma_z$ be the Pauli matrices. Then:

$$[\sigma_x, \sigma_y] = 2i\sigma_z, \quad [\sigma_y, \sigma_z] = 2i\sigma_x, \quad [\sigma_z, \sigma_x] = 2i\sigma_y$$

These relations are important in spin physics and quantum gates.

The ****anti-commutator**** of two operators is:

$$\{A, B\} = AB + BA$$

This is simply the sum of AB and BA . The anti-commutator is used in places where symmetry is more important than difference.

For example, in quantum systems involving fermions (particles like electrons), the creation and annihilation operators obey anti-commutation relations. Also, in quantum error correction, anti-commutators help analyze logical operator structure.

Anti-commutators of Pauli matrices:

$$\{\sigma_x, \sigma_x\} = 2I, \quad \{\sigma_x, \sigma_y\} = 0$$

This means: - Each Pauli matrix squared gives the identity (up to a factor), - Different Pauli matrices anti-commute (their anti-commutator is zero).

- The **commutator** tells us if operators interfere with each other.
- The **anti-commutator** tells us about symmetries and conservation.

Understanding these helps us know whether two quantum operations (like gates or measurements) can be done in any order, and whether they are fundamentally linked by quantum uncertainty.

The Polar and Singular Value Decompositions

In linear algebra, decomposition techniques help us understand the structure of a matrix by breaking it into simpler components. Two powerful decompositions are the **Polar Decomposition** and the **Singular Value Decomposition (SVD)**. Both are very important in quantum computing, quantum mechanics, and numerical analysis.

Singular Value Decomposition (SVD)

Let A be any $m \times n$ complex matrix. Then the **Singular Value Decomposition** of A is:

$$A = U\Sigma V^\dagger$$

where:

- U is an $m \times m$ unitary matrix,
- V is an $n \times n$ unitary matrix,
- Σ is an $m \times n$ diagonal matrix with non-negative real numbers on the diagonal.

The diagonal entries of Σ are called the **singular values** of A , and they are always non-negative real numbers, usually written as:

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_r \end{bmatrix}$$

Here, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, and $r = \text{rank}(A)$.

Geometric Meaning: SVD tells us that any linear transformation A can be broken into three parts: - Rotate using V , - Stretch along principal axes using Σ , - Rotate using U .

In Quantum Computing: - SVD is used in quantum state tomography and matrix product states. - It also plays a role in quantum algorithms like the HHL algorithm (for solving linear systems).

Polar Decomposition

The **Polar Decomposition** of a square matrix A expresses A as the product of a unitary matrix and a positive semi-definite matrix.

Let A be an $n \times n$ complex matrix. Then:

$$A = UP$$

where: - U is a unitary matrix ($U^\dagger U = I$), - P is a positive semi-definite Hermitian matrix.

Alternatively, we can also write:

$$A = PU'$$

where U' is another unitary matrix.

How to compute it: If $A = U\Sigma V^\dagger$ is the SVD of A , then:

$$P = V\Sigma V^\dagger, \quad U = A(A^\dagger A)^{-1/2}$$

Here, P contains the stretching part, and U handles the rotation and phase change.

Analogy: Polar decomposition is to matrices what writing a complex number z as $z = re^{i\theta}$ is in the complex plane: - r corresponds to P , - $e^{i\theta}$ corresponds to U .

In Quantum Computing: - Used to decompose quantum channels and noisy operations into unitary and non-unitary parts. - Helps separate the "pure quantum" part of an operation from the "lossy or decoherent" part.

Summary

- **SVD** breaks a matrix into two rotations and a scaling.
- **Polar decomposition** separates a matrix into a unitary and a Hermitian positive semi-definite part.
- Both decompositions are crucial in quantum computing for understanding operations, channels, and approximations.