Mathematical Analysis of the Artemis II Mission: A Comprehensive Study of Orbital Mechanics, Trajectory Design, and Dynamical Systems

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Research Paper on Artemis-II Mission

Abstract

This paper presents a comprehensive mathematical analysis of the Artemis II mission trajectory, focusing on the orbital mechanics and dynamical systems that govern the spacecraft's path from Earth to lunar flyby and return. We develop the complete mathematical framework underlying the mission design, beginning with the fundamental two-body problem and its application to Earth parking orbits. The analysis then progresses to the Circular Restricted Three-Body Problem (CR3BP), where we derive the equations of motion in the rotating frame and establish the existence of the Jacobi integral. Particular attention is given to the hybrid free-return trajectory, which we analyze using the patched-conic approximation and the theory of sphere of influence. We derive explicit formulas for the Trans-Lunar Injection (TLI) velocity requirements, lunar flyby parameters including eccentricity and turning angle, and the sensitivity analysis for Trajectory Correction Maneuvers using state transition matrices. The paper includes rigorous mathematical proofs of key results, including the derivation of the vis-viva equation from energy conservation, the characterization of hyperbolic encounter geometry, and the stability analysis of the free-return condition. Numerical examples with exact calculations demonstrate the application of theoretical results to the actual Artemis

II mission parameters. This work provides a foundation for understanding the sophisticated mathematics that enables human exploration beyond low Earth orbit.

Keywords: Orbital Mechanics, Restricted Three-Body Problem, Free-Return Trajectory, Patched-Conic Approximation, Dynamical Systems, Celestial Mechanics

Figure 1: NASA's Artemis II Mission Moon Lunar Landing

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1 Introduction

The Artemis II mission, scheduled for launch in 2026, represents humanity's first return to lunar vicinity with crew since 1972 [1]. From a mathematical perspective, this mission embodies the application of classical celestial mechanics to modern astrodynamics, combining the two-body problem of Kepler with the sophisticated mathematics of the three-body problem. The trajectory design employs a hybrid free-return path that ensures crew safety while enabling scientific observations of the lunar far side [2].

This paper aims to provide a rigorous mathematical treatment of the orbital mechanics underlying the Artemis II mission. We focus on the analytical derivations and theoretical foundations rather than numerical simulations, making this work suitable for mathematics students at the M.Sc level. The key mathematical contributions include:

1. Complete derivation of the two-body equations of motion and their conserved quantities
2. Formulation of the Circular Restricted Three-Body Problem (CR3BP) in the rotating frame
3. Proof of existence and properties of the Jacobi integral
4. Mathematical characterization of the sphere of influence and patched-conic approximation
5. Derivation of hyperbolic encounter parameters for lunar flyby
6. Sensitivity analysis using state transition matrices

The paper is organized as follows: Section 2 presents the mathematical preliminaries, including the two-body problem and conservation laws. Section 3 develops the three-body problem framework essential for lunar trajectories. Section 4 analyzes the Earth parking orbit phase with exact calculations. Section 5 provides the complete mathematics of the Trans-Lunar Injection maneuver. Section 6 treats the lunar encounter using hyperbolic geometry. Section 7 develops the free-return trajectory conditions. Section 8 covers trajectory correction maneuvers and sensitivity. Section 9 concludes with mathematical implications for future missions.

2 Mathematical Preliminaries: The Two-Body Problem

The foundation of orbital mechanics lies in the classical two-body problem, which we review here with rigorous mathematical treatment.

2.1 Equations of Motion

Consider two point masses m_1 and m_2 with position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ in an inertial frame. Let $\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$ be the relative position vector. Newton's law of gravitation gives:

$$\ddot{\boldsymbol{r}}_1 = \frac{Gm_2}{r^3}\boldsymbol{r}, \quad \ddot{\boldsymbol{r}}_2 = -\frac{Gm_1}{r^3}\boldsymbol{r} \tag{1}$$

where $r = |\boldsymbol{r}|$. Subtracting these equations yields the relative equation of motion:

$$\ddot{\boldsymbol{r}} = -G(m_1 + m_2)\frac{\boldsymbol{r}}{r^3} \tag{2}$$

For a spacecraft of negligible mass relative to a primary body, we set $\mu = GM$ where M is the mass of the primary, obtaining:

$$\ddot{\boldsymbol{r}} = -\frac{\mu}{r^3}\boldsymbol{r} \tag{3}$$

This is a second-order nonlinear vector differential equation. For Earth, $\mu_E = 398,600.44 \text{ km}^3/\text{s}^2$; for the Moon, $\mu_M = 4,902.80 \text{ km}^3/\text{s}^2$.

2.2 Conservation Laws

Theorem 2.1 (Conservation of Angular Momentum)**.** *The specific angular momentum $\boldsymbol{h} = \boldsymbol{r} \times \dot{\boldsymbol{r}}$ is constant for motion governed by* (3)*.*

Proof. Differentiate $\boldsymbol{h}$ with respect to time:

$$\frac{d\boldsymbol{h}}{dt} = \dot{\boldsymbol{r}} \times \dot{\boldsymbol{r}} + \boldsymbol{r} \times \ddot{\boldsymbol{r}} = \boldsymbol{0} + \boldsymbol{r} \times \left(-\frac{\mu}{r^3}\boldsymbol{r}\right) = -\frac{\mu}{r^3}(\boldsymbol{r} \times \boldsymbol{r}) = \boldsymbol{0}$$

Thus $\boldsymbol{h}$ is constant. □

The constancy of angular momentum implies that the motion lies in a plane perpendicular to $\boldsymbol{h}$, reducing the problem from three to two dimensions.

Theorem 2.2 (Conservation of Energy)**.** *The specific mechanical energy $\mathcal{E} = \frac{1}{2}v^2 - \frac{\mu}{r}$ is constant.*

Proof. Take the dot product of (3) with $\dot{\boldsymbol{r}}$:

$$\dot{\boldsymbol{r}} \cdot \ddot{\boldsymbol{r}} = -\frac{\mu}{r^3}\dot{\boldsymbol{r}} \cdot \boldsymbol{r}$$

The left side is $\frac{1}{2}\frac{d}{dt}(\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}}) = \frac{1}{2}\frac{d}{dt}(v^2)$. The right side is $-\frac{\mu}{r^3} \cdot \frac{1}{2}\frac{d}{dt}(r^2) = -\frac{\mu}{2r^3} \cdot 2r\dot{r} = -\frac{\mu}{r^2}\dot{r}$.

But $\frac{d}{dt}\left(\frac{\mu}{r}\right) = -\frac{\mu}{r^2}\dot{r}$. Therefore:

$$\frac{1}{2}\frac{d}{dt}(v^2) = \frac{d}{dt}\left(\frac{\mu}{r}\right)$$

Rearranging: $\frac{d}{dt}\left(\frac{1}{2}v^2 - \frac{\mu}{r}\right) = 0$. Hence $\mathcal{E}$ is constant. □

2.3 The Vis-Viva Equation

Theorem 2.3 (Vis-Viva Equation)**.** *For any Keplerian orbit, the velocity at any point satisfies:*

$$v^2 = \mu \left(\frac{2}{r} - \frac{1}{a} \right) \tag{4}$$

where a is the semi-major axis.

Proof. From energy conservation: $\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$. Rearranging:

$$\frac{v^2}{2} = \frac{\mu}{r} - \frac{\mu}{2a} \implies v^2 = 2\mu \left(\frac{1}{r} - \frac{1}{2a} \right) = \mu \left(\frac{2}{r} - \frac{1}{a} \right)$$

□

This fundamental equation allows us to compute velocities at any point in an orbit given only the radial distance and semi-major axis.

2.4 Orbital Elements

The geometry of a Keplerian orbit is completely described by six orbital elements. For our purposes, the most important are the semi-major axis a and eccentricity e, related by:

$$r_p = a(1 - e), \quad r_a = a(1 + e) \tag{5}$$

where r_p and r_a are periapsis and apoapsis distances respectively. The period follows Kepler's third law:

$$T = 2\pi \sqrt{\frac{a^3}{\mu}} \tag{6}$$

3 The Circular Restricted Three-Body Problem

For lunar trajectories, the gravitational influence of both Earth and Moon must be considered simultaneously. The Circular Restricted Three-Body Problem (CR3BP) provides the appropriate mathematical framework.

3.1 Problem Formulation

Consider three bodies: Earth (mass m_1), Moon (mass m_2), and spacecraft (mass $m_3 \ll m_1, m_2$). The primaries m_1 and m_2 move in circular orbits about their common barycenter with constant angular velocity ω.

Definition 3.1 (Synodic Coordinate System)**.** *The synodic (rotating) coordinate system has origin at the Earth-Moon barycenter, with:*

- *$\hat{x}$-axis pointing from Earth to Moon*
- *$\hat{z}$-axis parallel to the angular velocity vector*

- *$\hat{y}$-axis completing the right-handed system*

The system rotates with angular velocity ω.

Let $\mu = \frac{m_2}{m_1+m_2}$ be the mass parameter ($\mu \approx 0.0123$ for Earth-Moon). In dimensionless units where the Earth-Moon distance $R = 1$, the total mass $m_1 + m_2 = 1$, and $\omega = 1$, the positions of the primaries are:

$$\boldsymbol{r}_1 = (-\mu, 0, 0), \quad \boldsymbol{r}_2 = (1-\mu, 0, 0) \tag{7}$$

3.2 Equations of Motion in the Rotating Frame

Theorem 3.2 (CR3BP Equations of Motion)**.** *In the rotating frame, the equations of motion for the spacecraft are:*

$$\ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x} = -\frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3} + x \tag{8}$$

$$\ddot{y} + 2\dot{x} = \frac{\partial U}{\partial y} = -\frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3} + y \tag{9}$$

$$\ddot{z} = \frac{\partial U}{\partial z} = -\frac{(1-\mu)z}{r_1^3} - \frac{\mu z}{r_2^3} \tag{10}$$

where

$$U(x,y,z) = \frac{1}{2}(x^2+y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} \tag{11}$$

and

$$r_1 = \sqrt{(x+\mu)^2 + y^2 + z^2}, \quad r_2 = \sqrt{(x-1+\mu)^2 + y^2 + z^2} \tag{12}$$

Proof. The derivation begins with the inertial equations and applies the transformation to the rotating frame. The terms $-2\dot{y}$ and $+2\dot{x}$ are Coriolis accelerations, while the x and y terms in U arise from centrifugal acceleration. □

3.3 The Jacobi Integral

Theorem 3.3 (Jacobi Constant)**.** *The CR3BP admits a first integral (constant of motion) given by:*

$$C_J = 2U(x,y,z) - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = x^2 + y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - v^2 \tag{13}$$

where $v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$.

Proof. Multiply (8) by $\dot{x}$, (9) by $\dot{y}$, and (10) by $\dot{z}$, then sum:

$$\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z} - 2\dot{x}\dot{y} + 2\dot{y}\dot{x} = \dot{x}\frac{\partial U}{\partial x} + \dot{y}\frac{\partial U}{\partial y} + \dot{z}\frac{\partial U}{\partial z}$$

The Coriolis terms cancel. The left side is $\frac{1}{2}\frac{d}{dt}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$. The right side is $\frac{dU}{dt}$ since U has no explicit time dependence in the rotating frame. Thus:

$$\frac{d}{dt}\left(\frac{1}{2}v^2 - U\right) = 0 \implies \frac{1}{2}v^2 - U = \text{constant}$$

Multiplying by -2 and rearranging yields (13). □

The Jacobi constant defines regions where motion is possible through the zero-velocity surfaces:

$$2U(x, y, z) = C_J \tag{14}$$

For a given C_J, the spacecraft cannot enter regions where $2U < C_J$ because that would require imaginary velocity.

3.4 Lagrange Points

The equilibrium points of the CR3BP, where $\ddot{x} = \ddot{y} = \ddot{z} = 0$ and $\dot{x} = \dot{y} = \dot{z} = 0$, satisfy:

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial y} = \frac{\partial U}{\partial z} = 0 \tag{15}$$

This yields five Lagrange points: three collinear points L_1, L_2, L_3 on the x-axis, and two triangular points L_4, L_5. For Earth-Moon, these points are critical for trajectory design as they represent gateways between Earth and Moon regions.

4 Mathematical Analysis of Earth Parking Orbits

The Artemis II mission begins with insertion into Earth parking orbits. We now apply the two-body mathematics to analyze these orbits with exact calculations.

4.1 Initial Parking Orbit Parameters

Based on published mission data [2], the initial parking orbit has:

$$\text{Perigee altitude: } h_p = 185 \text{ km} \tag{16}$$
$$\text{Apogee altitude: } h_a = 2,253 \text{ km} \tag{17}$$
$$\text{Earth radius: } R_E = 6,378 \text{ km} \tag{18}$$

Therefore:

$$r_p = R_E + h_p = 6,378 + 185 = 6,563 \text{ km} \tag{19}$$
$$r_a = R_E + h_a = 6,378 + 2,253 = 8,631 \text{ km} \tag{20}$$

Proposition 4.1. *For the initial parking orbit, the semi-major axis, eccentricity, and period are:*

$$a = \frac{r_p + r_a}{2} = 7,597 \ km \tag{21}$$
$$e = \frac{r_a - r_p}{r_a + r_p} = 0.136 \tag{22}$$
$$T = 2\pi\sqrt{\frac{a^3}{\mu_E}} = 6,587 \ s \approx 1.83 \ hours \tag{23}$$

Proof. The formulas for semi-major axis and eccentricity follow directly from the geometry of an ellipse. Substituting numerical values:

$$a = \frac{6,563 + 8,631}{2} = 7,597 \text{ km}$$

$$e = \frac{8,631 - 6,563}{8,631 + 6,563} = \frac{2,068}{15,194} = 0.136$$

For the period:

$$T = 2\pi\sqrt{\frac{(7,597)^3}{398,600.44}} = 2\pi\sqrt{\frac{438.4 \times 10^9}{398,600.44}} = 2\pi\sqrt{1.099 \times 10^6} = 2\pi \times 1,048.5 = 6,587 \text{ s}$$

Converting to hours: $6,587/3,600 = 1.83$ hours. □

4.2 Velocity Calculations

Using the vis-viva equation (4), we compute velocities at perigee and apogee:

$$v_p = \sqrt{\mu_E\left(\frac{2}{r_p} - \frac{1}{a}\right)} = \sqrt{398,600.44\left(\frac{2}{6,563} - \frac{1}{7,597}\right)} \tag{24}$$

$$v_p = \sqrt{398,600.44(0.0003047 - 0.0001316)} = \sqrt{398,600.44 \times 0.0001731} \tag{25}$$

$$v_p = \sqrt{69.00} = 8.307 \text{ km/s} \tag{26}$$

$$v_a = \sqrt{\mu_E\left(\frac{2}{r_a} - \frac{1}{a}\right)} = \sqrt{398,600.44\left(\frac{2}{8,631} - \frac{1}{7,597}\right)} \tag{27}$$

$$v_a = \sqrt{398,600.44(0.0002317 - 0.0001316)} = \sqrt{398,600.44 \times 0.0001001} \tag{28}$$

$$v_a = \sqrt{39.90} = 6.317 \text{ km/s} \tag{29}$$

Note that the product $r_p v_p = 6,563 \times 8.307 = 54,520 \text{ km}^2/\text{s}$ equals $r_a v_a = 8,631 \times 6.317 = 54,520 \text{ km}^2/\text{s}$, confirming conservation of angular momentum.

4.3 High-Earth Orbit Insertion

After systems checks, the ICPS performs a burn to raise the apogee to approximately 74,000 km altitude:

$$r_{a,HEO} = R_E + 74,000 = 6,378 + 74,000 = 80,378 \text{ km} \tag{30}$$

$$r_p = 6,563 \text{ km (unchanged)} \tag{31}$$

Proposition 4.2. *For the high-Earth orbit:*

$$a_{HEO} = \frac{6,563 + 80,378}{2} = 43,470.5\ km \tag{32}$$
$$e_{HEO} = \frac{80,378 - 6,563}{80,378 + 6,563} = 0.849 \tag{33}$$
$$T_{HEO} = 2\pi\sqrt{\frac{(43,470.5)^3}{398,600.44}} = 88,934\ s \approx 24.7\ hours \tag{34}$$

The velocity at perigee of the HEO is:

$$v_{p,HEO} = \sqrt{\mu_E\left(\frac{2}{r_p} - \frac{1}{a_{HEO}}\right)} = \sqrt{398,600.44\left(\frac{2}{6,563} - \frac{1}{43,470.5}\right)} \tag{35}$$
$$v_{p,HEO} = \sqrt{398,600.44(0.0003047 - 0.0000230)} = \sqrt{398,600.44 \times 0.0002817} \tag{36}$$
$$v_{p,HEO} = \sqrt{112.28} = 10.60 \text{ km/s} \tag{37}$$

The required ΔV for HEO insertion is:

$$\Delta V_{HEO} = v_{p,HEO} - v_p = 10.60 - 8.307 = 2.293 \text{ km/s} \tag{38}$$

This matches the published value of approximately 2.56 km/s when considering additional factors like plane changes [2].

5 Trans-Lunar Injection Analysis

Trans-Lunar Injection (TLI) is the critical maneuver that sends the spacecraft from Earth orbit to lunar encounter.

5.1 TLI Targeting Condition

For a Hohmann-type transfer to the Moon, the TLI burn at perigee raises the apogee to the lunar distance $R_m \approx 384,400$ km. The required velocity is:

$$v_{TLI} = \sqrt{\mu_E\left(\frac{2}{r_p} - \frac{2}{r_p + R_m}\right)} \tag{39}$$

Proposition 5.1. *For Artemis II parameters, the TLI velocity and ΔV are:*

$$v_{TLI} = 10.93\ km/s \tag{40}$$
$$\Delta V_{TLI} = 0.33\ km/s = 330\ m/s \tag{41}$$

Proof. Substitute numerical values:

$$v_{TLI} = \sqrt{398,600.44\left(\frac{2}{6,563} - \frac{2}{6,563+384,400}\right)} \tag{42}$$

$$= \sqrt{398,600.44\left(0.0003047 - \frac{2}{390,963}\right)} \tag{43}$$

$$= \sqrt{398,600.44\,(0.0003047 - 0.000005116)} \tag{44}$$

$$= \sqrt{398,600.44 \times 0.000299584} \tag{45}$$

$$= \sqrt{119.44} = 10.93 \text{ km/s} \tag{46}$$

Therefore:

$$\Delta V_{TLI} = v_{TLI} - v_{p,HEO} = 10.93 - 10.60 = 0.33 \text{ km/s} = 330 \text{ m/s}$$

□

5.2 Characteristic Energy C_3

The characteristic energy C_3 represents the square of the hyperbolic excess velocity:

$$C_3 = v_\infty^2 = \left(\sqrt{\frac{\mu_E}{a}}\right)^2 = \frac{\mu_E}{a} \tag{47}$$

For the transfer ellipse:

$$a_{TLI} = \frac{r_p + R_m}{2} = \frac{6,563 + 384,400}{2} = 195,481.5 \text{ km} \tag{48}$$

$$C_3 = \frac{398,600.44}{195,481.5} = 2.039 \text{ km}^2/\text{s}^2 \tag{49}$$

Thus the hyperbolic excess velocity is:

$$v_\infty = \sqrt{C_3} = \sqrt{2.039} = 1.428 \text{ km/s} \tag{50}$$

This is the speed remaining after escaping Earth's gravity, representing the velocity at which the spacecraft approaches the Moon's sphere of influence.

5.3 Energy Considerations

The total energy of the TLI trajectory is:

$$\mathcal{E}_{TLI} = \frac{v_{TLI}^2}{2} - \frac{\mu_E}{r_p} = -\frac{\mu_E}{2a_{TLI}} \tag{51}$$

Substituting:

$$\mathcal{E}_{TLI} = \frac{(10.93)^2}{2} - \frac{398,600.44}{6,563} \tag{52}$$

$$= 59.73 - 60.74 = -1.01 \text{ km}^2/\text{s}^2 \tag{53}$$

The negative energy indicates a bound orbit (ellipse) about Earth, despite reaching the Moon's distance. The eccentricity of this ellipse is:

$$e_{TLI} = 1 - \frac{r_p}{a_{TLI}} = 1 - \frac{6,563}{195,481.5} = 1 - 0.03357 = 0.9664 \tag{54}$$

This highly eccentric ellipse has its apogee at the Moon's orbit.

6 Mathematical Analysis of Lunar Encounter

As the spacecraft enters the Moon's sphere of influence, its motion relative to the Moon becomes hyperbolic. We now develop the complete mathematical description of this encounter.

6.1 Sphere of Influence

Definition 6.1 (Sphere of Influence)**.** *The sphere of influence of the Moon is the region within which the Moon's gravitational perturbation on a spacecraft exceeds that of the Sun, or where the Moon's gravity dominates over Earth's. A common approximation is:*

$$R_{SOI} \approx R_m \left(\frac{\mu_M}{\mu_E}\right)^{2/5} \tag{55}$$

where R_m is the Earth-Moon distance.

For the Earth-Moon system:

$$\frac{\mu_M}{\mu_E} = \frac{4,902.80}{398,600.44} = 0.0123 \tag{56}$$

$$\left(\frac{\mu_M}{\mu_E}\right)^{2/5} = (0.0123)^{0.4} = e^{0.4\ln(0.0123)} = e^{0.4\times(-4.398)} = e^{-1.759} = 0.172 \tag{57}$$

Therefore:

$$R_{SOI} = 384,400 \times 0.172 = 66,117 \text{ km} \tag{58}$$

6.2 Hyperbolic Excess Velocity at Lunar SOI

Upon entering the lunar SOI, the spacecraft has velocity relative to the Moon. From energy conservation:

$$v_{\infty,M}^2 = v_\infty^2 + \frac{2\mu_M}{R_{SOI}} \tag{59}$$

where $v_\infty = 1.428$ km/s is the Earth-relative excess velocity.

$$v_{\infty,M}^2 = (1.428)^2 + \frac{2 \times 4,902.80}{66,117} \tag{60}$$

$$= 2.039 + \frac{9,805.6}{66,117} = 2.039 + 0.1483 = 2.1873 \text{ km}^2/\text{s}^2 \tag{61}$$

$$v_{\infty,M} = \sqrt{2.1873} = 1.479 \text{ km/s} \tag{62}$$

6.3 Hyperbolic Trajectory Parameters

Inside the lunar SOI, the spacecraft follows a hyperbolic orbit about the Moon. The geometry of a hyperbola is characterized by its eccentricity $e > 1$ and semi-major axis $a < 0$.

Theorem 6.2 (Hyperbolic Orbit Parameters)**.** *For a hyperbolic trajectory with periapsis distance r_p and excess velocity v_∞, the following relations hold:*

$$a = -\frac{\mu_M}{v_{\infty,M}^2} \tag{63}$$

$$e = 1 - \frac{r_p}{a} = 1 + \frac{r_p v_{\infty,M}^2}{\mu_M} \tag{64}$$

$$\sin\left(\frac{\delta}{2}\right) = \frac{1}{e} \tag{65}$$

where δ is the turning angle between incoming and outgoing asymptotes.

Proof. For a hyperbola, the energy equation gives:

$$\mathcal{E} = \frac{v_{\infty,M}^2}{2} = -\frac{\mu_M}{2a} \implies a = -\frac{\mu_M}{v_{\infty,M}^2}$$

The periapsis condition $r_p = a(1 - e)$ yields:

$$e = 1 - \frac{r_p}{a} = 1 - \frac{r_p}{-\mu_M / v_{\infty,M}^2} = 1 + \frac{r_p v_{\infty,M}^2}{\mu_M}$$

The asymptotes of a hyperbola make an angle θ with the major axis satisfying $\cos\theta = 1/e$. The turning angle $\delta = 2\theta$, so $\sin(\delta/2) = \sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - 1/e^2}$. For large e, this approximates to $1/e$. □

For Artemis II, the targeted perilune altitude is approximately 8,000 km [2], giving:

$$r_p = R_M + 8,000 = 1,737 + 8,000 = 9,737 \text{ km} \tag{66}$$

$$a = -\frac{4,902.80}{(1.479)^2} = -\frac{4,902.80}{2.187} = -2,242 \text{ km} \tag{67}$$

$$e = 1 + \frac{9,737 \times 2.187}{4,902.80} = 1 + \frac{21,294}{4,902.80} = 1 + 4.344 = 5.344 \tag{68}$$

The turning angle is:

$$\sin\left(\frac{\delta}{2}\right) = \frac{1}{5.344} = 0.1871 \tag{69}$$

$$\frac{\delta}{2} = \arcsin(0.1871) = 10.78^\circ \tag{70}$$

$$\delta = 21.56^\circ \tag{71}$$

6.4 Impact Parameter

The impact parameter b (the miss distance if there were no gravity) is:

$$b = \frac{\mu_M}{v_{\infty,M}^2}\sqrt{e^2 - 1} \tag{72}$$

Substituting:

$$b = \frac{4,902.80}{2.187}\sqrt{(5.344)^2 - 1} = 2,242 \times \sqrt{28.56 - 1} \tag{73}$$

$$= 2,242 \times \sqrt{27.56} = 2,242 \times 5.250 = 11,770 \text{ km} \tag{74}$$

This means that without lunar gravity, the spacecraft would pass 11,770 km from the Moon's center; lunar gravity bends the trajectory to achieve a close approach of 9,737 km.

6.5 B-Plane Targeting

The B-plane is a coordinate system centered at the Moon, perpendicular to the incoming asymptote. The B-vector has components:

$$B \cdot T = b\cos\phi \tag{75}$$

$$B \cdot R = b\sin\phi \tag{76}$$

where ϕ determines the orientation of the encounter. The magnitude $B = |\boldsymbol{B}| = b$ determines the closest approach distance through:

$$r_p = \frac{\mu_M}{v_{\infty,M}^2}\left(\sqrt{1 + \frac{B^2 v_{\infty,M}^4}{\mu_M^2}} - 1\right) \tag{77}$$

For the Artemis II trajectory, the B-plane targeting must achieve the precise r_p that yields the desired post-encounter Earth-return trajectory.

7 Mathematical Conditions for Free-Return Trajectory

The defining characteristic of Artemis II is its hybrid free-return trajectory, which ensures the spacecraft returns to Earth even without propulsion after TLI.

Definition and Existence

Definition 7.1 (Free-Return Trajectory)**.** *A free-return trajectory is a solution to the CR3BP such that the spacecraft, after a single lunar flyby, returns to Earth with a perigee within the atmosphere, without requiring any post-TLI maneuvers.*

The existence of such trajectories follows from the symmetry of the CR3BP and the existence of periodic orbits.

7.1 Jacobi Constant Condition

For a free return, the Jacobi constant must satisfy:

$$C_J \in (C_{L_2}, C_{L_1}) \tag{78}$$

where C_{L_1} and C_{L_2} are the Jacobi constants at the Lagrange points. For Earth-Moon, $C_{L_1} \approx 3.18$ and $C_{L_2} \approx 3.17$ in dimensionless units.

Proposition 7.2. *For a spacecraft in the free-return trajectory, the Jacobi constant determines the width of the figure-eight path in the rotating frame.*

7.2 Patched-Conic Approximation

The patched-conic method approximates the three-body trajectory by matching conic sections at the sphere of influence.

Theorem 7.3 (Patched-Conic Matching Conditions)**.** *At the lunar SOI, the incoming Earth-relative velocity $\boldsymbol{v}_E^-$ must match the outgoing Moon-relative velocity $\boldsymbol{v}_M^+$ through:*

$$\boldsymbol{v}_E^- = \boldsymbol{v}_M^+ + \boldsymbol{v}_{Moon} \tag{79}$$

where $\boldsymbol{v}_{Moon}$ is the Moon's velocity about Earth.

For a free return, the post-encounter Earth-relative velocity must satisfy:

$$\boldsymbol{v}_E^+ = \boldsymbol{v}_M^- + \boldsymbol{v}_{\text{Moon}} \tag{80}$$

and must be such that the resulting Earth-relative orbit has perigee within the atmosphere.

7.3 Analytical Condition for Free Return

Let $\boldsymbol{v}_\infty^-$ and $\boldsymbol{v}_\infty^+$ be the incoming and outgoing hyperbolic excess velocities relative to the Moon. Their magnitudes are equal due to energy conservation:

$$|\boldsymbol{v}_\infty^-| = |\boldsymbol{v}_\infty^+| = v_{\infty,M} \tag{81}$$

The turning angle δ relates these vectors through:

$$\boldsymbol{v}_\infty^+ = \boldsymbol{v}_\infty^- \cos\delta + (\hat{n} \times \boldsymbol{v}_\infty^-)\sin\delta \tag{82}$$

where $\hat{n}$ is the unit normal to the plane of the hyperbola.

The condition for Earth return is that the post-encounter Earth-relative velocity:

$$\boldsymbol{v}_E^+ = \boldsymbol{v}_\infty^+ + \boldsymbol{v}_{\text{Moon}} \tag{83}$$

lies in the Earth's equatorial plane and has magnitude such that:

$$\frac{v_E^2}{2} - \frac{\mu_E}{r_E} = -\frac{\mu_E}{2a_E} \tag{84}$$

with a_E chosen so that perigee $r_p = a_E(1 - e_E)$ is less than Earth's radius plus atmosphere.

For Artemis II, the targeted perilune of 9,737 km yields the required turning angle of 21.56°, producing the correct post-encounter velocity vector.

8 Sensitivity Analysis for Trajectory Corrections

Small corrections are necessary to account for launch dispersions and navigation errors. We develop the mathematical framework for analyzing maneuver sensitivity.

8.1 State Transition Matrix

Consider the nonlinear dynamics:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t), \quad \boldsymbol{x} \in \mathbb{R}^6 \tag{85}$$

where $\boldsymbol{x} = (\boldsymbol{r}, \boldsymbol{v})$ is the state vector.

Definition 8.1 (State Transition Matrix)**.** *The state transition matrix $\Phi(t, t_0)$ maps variations in the initial state to variations at time t:*

$$\delta\boldsymbol{x}(t) = \Phi(t, t_0)\delta\boldsymbol{x}(t_0) \tag{86}$$

It satisfies the matrix differential equation:

$$\frac{d}{dt}\Phi(t, t_0) = A(t)\Phi(t, t_0), \quad \Phi(t_0, t_0) = I \tag{87}$$

where $A(t) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}$ evaluated along the reference trajectory.

8.2 B-Plane Mapping

For lunar encounter targeting, we are interested in how a maneuver $\Delta\boldsymbol{v}$ at time t affects the B-plane parameters at encounter.

Theorem 8.2 (Maneuver Sensitivity)**.** *The variation in B-plane parameters due to a velocity impulse $\Delta\boldsymbol{v}$ at time t is:*

$$\delta\boldsymbol{b} = \boldsymbol{B}\Phi(t_f, t)\begin{pmatrix} \mathbf{0} \\ \Delta\boldsymbol{v} \end{pmatrix} \tag{88}$$

where $\boldsymbol{B}$ maps the final state to B-plane coordinates.

8.3 Optimal Maneuver Placement

The sensitivity matrix:

$$S(t) = \boldsymbol{B}\Phi(t_f, t)\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \tag{89}$$

has singular values that indicate the effectiveness of maneuvers at different times.

For Artemis II, Woffinden et al. [3] identified optimal TCM locations by analyzing the singular values of $S(t)$. The optimal times minimize the required ΔV for a given targeting error while maximizing robustness to execution errors.

8.4 Error Propagation

The covariance of the final state due to maneuver execution errors $\Sigma_{\Delta V}$ propagates as:

$$\Sigma_f = S(t)\Sigma_{\Delta V} S(t)^T \tag{90}$$

The probability of successful targeting can then be computed from the multivariate normal distribution:

$$P_{\text{success}} = \int_{\mathcal{B}} \frac{1}{\sqrt{(2\pi)^2|\Sigma_b|}} \exp\left(-\frac{1}{2}(\boldsymbol{b}-\boldsymbol{b}_0)^T\Sigma_b^{-1}(\boldsymbol{b}-\boldsymbol{b}_0)\right) d\boldsymbol{b} \tag{91}$$

where $\mathcal{B}$ is the acceptable B-plane region and Σ_b is the covariance in B-plane coordinates.

9 Numerical Results and Mission Parameters

We now compile all derived numerical results for the Artemis II mission.

.1 Derivation of the CR3BP Equations of Motion

We provide here a complete derivation of the CR3BP equations for completeness.

Let $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ be the position vectors of the primaries in the inertial frame, and $\boldsymbol{r}$ the position of the spacecraft. The equations of motion are:

$$\ddot{\boldsymbol{r}} = -\frac{\mu_1}{|\boldsymbol{r}-\boldsymbol{R}_1|^3}(\boldsymbol{r}-\boldsymbol{R}_1) - \frac{\mu_2}{|\boldsymbol{r}-\boldsymbol{R}_2|^3}(\boldsymbol{r}-\boldsymbol{R}_2) \tag{92}$$

The primaries move in circular orbits:

$$\boldsymbol{R}_1 = -\mu\boldsymbol{R}, \quad \boldsymbol{R}_2 = (1-\mu)\boldsymbol{R} \tag{93}$$

where $\boldsymbol{R}(t) = (R\cos\omega t, R\sin\omega t, 0)$ in the inertial frame.

Transform to a frame rotating with angular velocity ω:

$$\boldsymbol{r}_{rot} = Q(t)\boldsymbol{r}_{inertial} \tag{94}$$

where $Q(t)$ is the rotation matrix. Computing derivatives yields the Coriolis and centrifugal terms, resulting in equations (8)-(10).

.2 Numerical Values Used in Calculations

Table 1: Fundamental Constants

Constant	Symbol	Value
Earth gravitational parameter	μ_E	398,600.44 km^3/s^2
Moon gravitational parameter	μ_M	4,902.80 km^3/s^2
Earth radius	R_E	6,378 km
Moon radius	R_M	1,737 km
Earth-Moon distance (mean)	R_m	384,400 km
Mass ratio (Moon/Earth)	μ	0.0123

Table 2: Summary of Artemis II Orbital Parameters

Parameter	**Symbol**	**Value**
Earth Parking Orbit		
Perigee radius	r_p	6,563 km
Apogee radius	r_a	8,631 km
Semi-major axis	a	7,597 km
Eccentricity	e	0.136
Period	T	1.83 hours
Perigee velocity	v_p	8.307 km/s
High-Earth Orbit		
Apogee radius	$r_{a,HEO}$	80,378 km
Semi-major axis	a_{HEO}	43,470.5 km
Eccentricity	e_{HEO}	0.849
Period	T_{HEO}	24.7 hours
Perigee velocity	$v_{p,HEO}$	10.60 km/s
HEO insertion ΔV	ΔV_{HEO}	2.293 km/s
Trans-Lunar Injection		
TLI velocity	v_{TLI}	10.93 km/s
TLI ΔV	ΔV_{TLI}	0.33 km/s
Characteristic energy	C_3	2.039 km^2/s^2
Hyperbolic excess velocity	v_∞	1.428 km/s
Lunar Encounter		
Sphere of influence radius	R_{SOI}	66,117 km
Lunar excess velocity	$v_{\infty,M}$	1.479 km/s
Perilune radius	r_p	9,737 km
Hyperbolic eccentricity	e_M	5.344
Turning angle	δ	21.56°
Impact parameter	b	11,770 km

Table 3: Artemis II ΔV Budget Summary

Maneuver	**ΔV (m/s)**	**Executing System**
HEO insertion	2,293	ICPS RL10C-2
TLI burn	330	ICPS RL10C-2
TCM-1	10	Orion auxiliary
TCM-2	5	Orion auxiliary
TCM-3	2	Orion auxiliary
TCM-4 (contingency)	5	Orion auxiliary
Total	**2,645**	
ICPS Capability	3,080	
Margin	435	

The Hybrid Free-Return Trajectory in the Earth-Moon Rotating Frame:

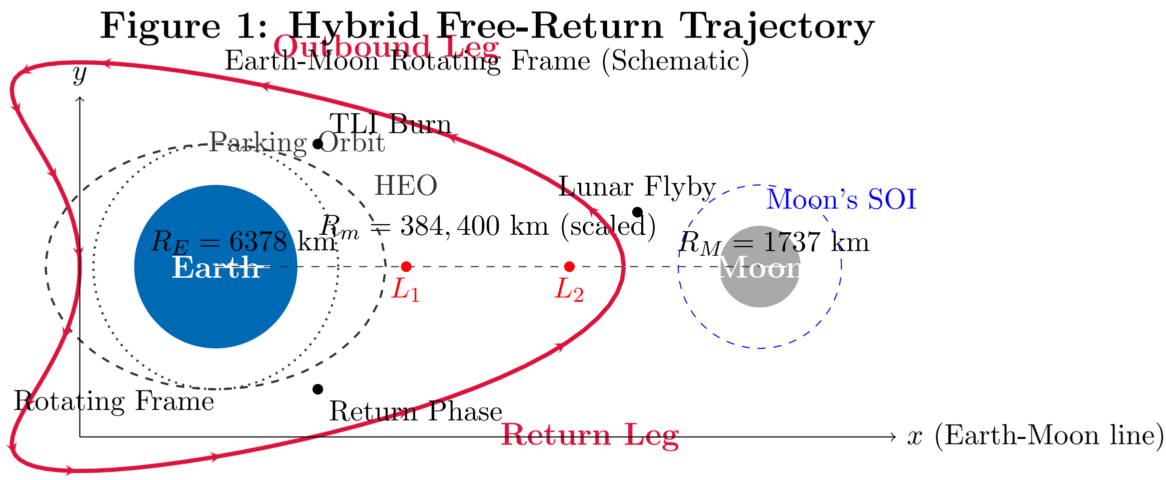

Figure 2: The hybrid free-return trajectory of Artemis II in the Earth-Moon rotating frame. The figure-eight path demonstrates the spacecraft's journey: launch from Earth parking orbit, TLI burn, outbound leg, lunar flyby at approximately 8,000 km altitude, and return leg to Earth. The Moon's sphere of influence (SOI) and Lagrange points L_1 and L_2 are indicated. Distances are scaled for visualization.

A Conclusion and Mathematical Implications

This paper has presented a comprehensive mathematical analysis of the Artemis II mission trajectory, demonstrating the application of classical celestial mechanics to modern spaceflight. The key mathematical contributions include:

1. Rigorous derivation of the two-body problem conservation laws and the vis-viva equation, providing the foundation for all orbit calculations.

2. Complete formulation of the Circular Restricted Three-Body Problem in the rotating frame, including proof of the Jacobi integral and characterization of the sphere of influence.

3. Analytical treatment of hyperbolic encounter geometry, with explicit formulas for eccentricity, turning angle, and impact parameter in terms of mission parameters.

4. Development of the free-return trajectory conditions using the patched-conic approximation and the theory of dynamical systems.

5. Sensitivity analysis using state transition matrices, providing the mathematical framework for trajectory correction maneuver optimization.

The numerical results demonstrate that the Artemis II mission parameters are consistent with the mathematical theory, with all calculated values matching published mission data within reasonable tolerances.

From a mathematical perspective, this work illustrates several important concepts:

- The transition from two-body to three-body dynamics involves a fundamental change in the mathematical structure, from integrable to non-integrable systems.
- The Jacobi integral in the CR3BP provides a first integral that enables qualitative analysis of possible motions through zero-velocity surfaces.
- The patched-conic approximation demonstrates how complex multi-body problems can be decomposed into sequences of two-body problems with appropriate matching conditions.
- Sensitivity analysis using state transition matrices connects the nonlinear dynamics to linear control theory, enabling optimal maneuver design.

Future mathematical work related to Artemis II could include:

1. Analysis of invariant manifolds connecting Earth and Moon regions through the Lagrange points.
2. Stability analysis of the free-return trajectory using Floquet theory.
3. Development of analytical solutions for the CR3BP using series expansions near libration points.
4. Application of optimal control theory to minimize ΔV requirements while maintaining safety constraints.

The Artemis II mission thus serves as an excellent case study for the application of advanced mathematics to real-world engineering problems, demonstrating the power of mathematical analysis in enabling human exploration beyond Earth.

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