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C H A P T E R SEVEN

Geometry without Space: Ancient Greek Mathematical Thought and Contemporary Consequences[1]

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In the 4th century BC Euclid composed his well-known text the Elements, introducing to the world the first known example of an axiomatic treatise and system that was also considered to be the first ‗textbook' designed for students. These thirteen books on mathematics, and primarily geometry, however, do not at any point define or even utilize the term ‗space'. The most obvious reason for this omission is that the ancient Greeks as a whole did not have a term exactly equivalent to that of space. Instead, the Greeks used either *topos*, meaning ‗place', or *diastēma*, meaning, for the most part, ‗distance', *chōra*, meaning ‗locus', *schēma*, meaning ‗figure', and *atopos* meaing ‗void' or more literally ‗no-place'. Of course, it is not necessarily the case that one must utilize an actual definition of space in order to present a viable account of geometric objects, yet it seems we would still be inclined, at least in modern thinking, to consider or else just assume geometrical objects to be spatial objects of some sort. Euclid's response is in no way mysterious; all we have to do is revisit his definitions of the different kinds of objects that he identifies in order to see the manner in which he characterizes them without utilizing the concept of space, at least not explicitly. Instead, perhaps the broadest or most general summation of Euclid's definitions of geometrical objects is that he relies on a principle of relation, whereby each object involves a particular kind of relation between its constituent parts, particularly, in this case, those of magnitudes and angles. Euclid, however, also does not define or really utilize a term equivalent to ‗relation' either, but rather more specifically utilizes and defines the concept of *logos*, i.e., ‗ratio', yet this is not found in his definitions of such objects themselves. The terms Euclid does mention are directional, e.g., lines that approach one another or that tend away from one another according to angle, as well as, of course, those that are parallel. One point worth noting here is that in Modern Greek *chōra* can indeed be translated as ‗space', but again in Ancient Greek the term is usually translated as a ‗locus' or spot in which an object is located. Euclid first employs another term, namely *schēma* or ‗figure' in Book I Def. 14: ―A figure is that which is contained by any boundary or boundaries.‖ We can see how none of these terms can be the

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equivalent of a general notion of space as such, for in the case of *schēma* (which is perhaps the closest term found in Euclid) it is used here to refer to an object, so that the closest conception we could stretch our translation to utilize would still be that of ‗a space', rather than space itself, Thus, ‗containment' and ‗boundaries' pertain here more to the concept of a figure rather than an explicit concept of space. In fact, we can now see how the very notion of ‗Euclidean space' is itself called into question, since there can only be an inferred statement about what Euclidean space might be since Euclid himself never employs the concept of space in an explicit manner. Thus, while it is not strictly the case that Euclidean geometry is a geometry without space in a strong sense of that phrase, it is nevertheless true that it is a geometry without a definite concept of space, and furthermore one that employs other notions, namely ‗figure', ‗distance', and ‗place' that only imply a general spatial concept or perception that is never explicitly defined or theorized about. It is therefore possible to conceive of a successful geometric system without any explicit notion of space as such.

When we consider this issue more generally, it is not all that surprising to have a geometric system which lacks a definition or theoretization of the nature of space itself. The abstract space of early modern science, particularly that of Newton and any such notion stemming from the Cartesian concept of extension, is far less necessary than we might think. From a philosophical point of view we can think of Russell's early work *An Essay on the Foundations of Geometry* as an example of what it means to derive epistemological conclusions regarding the status of space itself from modern geometrical systems. The closest ancient Greek parallel to such an investigation is probably that of Aristotle in the *Physics* and the *Metaphysics*, yet even there we do not find an inquiry into space as such, but rather a discussion of *topos*, locomotion, potentiality and actuality. The major difference between an inquiry such as Russell's and seemingly similar inquiries on the part of the Greeks, is that Russell theorizes about the ontological status of space as such, whereas the Greeks focus on the ontological status of geometric objects, including lines, points, angles, planes and the various figures that such objects constitute.

Even ancient discussions of the infinite do not hypostatize infinite space per se, but rather infinite objects such as lines (including the infinite division of lines), or an infinite cosmos, which is admittedly similar, yet still not identical with a reflection on space qua space. An example of this latter type of discussion is found in Cleomedes' *On the Heavens,* a late ancient introductory source on astronomy, which highlights the question of whether the cosmos has a limit, and then employs the thought experiment of reaching one's hand out beyond that limit and asking where is my hand at that point? Simplicius in his commentaries on Aristotle raises the same question.

Another important conception in this regard is that attributed to Democritus, which is also revisited by Aristotle, namely the concept of a ‗void' filled with atoms. This is probably the closest conception to abstract space that we can find in ancient thought, yet it is already a negative conception of ‗emptiness', or ‗*a-topos*', ‗no-place' or ‗placelessness'. It is significant also that Aristotle rejects the possibility of the ‗void', upholding the notion that an empty space would still need some ‗place' as a reference point and constituent, similar to the ‗locus' concept of *chōra* (which is also often translated as ‗receptacle' in Plato's *Timaeus*). In all of this, then, we can see how the questions raised are focused on objects or the constituent objects of space, and not on space itself, and thus space itself is simply not in the equation for the ancients. What is dealt with by ancient thinkers, particularly Plato, Aristotle, and Proclus, is the status of geometrical objects. In his *Commentary on Book I of Euclid's Elements*, Proclus emphasizes the epistemological aspect of such objects by associating them with *dianoia*, thereby placing them in an intermediate category between perception and thought, or *aesthesis* and *nous*.

This latter type of inquiry brings us back to our main interest, which is the question of how to understand and conceive of a geometrical system that does not employ or rest upon an explicit concept of space as such. In a modern context, we seem compelled to wonder, if geometry does not involve a theoretization of the nature of space itself, then what is being theorized about? The ancients seem to engage this question by investigating the status of geometric objects, and in some sense this is similar to the way in which contemporary theories of the cosmos move back into the realm physics, or specifically astrophysics, when dealing with the ‗structure' of the universe. In other words, modern geometrical questions about the nature of space have eventually come back to the question of what constitutes space, rather than questioning the ontological or epistemological status of space in itself. Thus, we find that the Newtonian or even Cartesian conception of space that we might attach to geometry is a phase of scientific thought that begins long after the ancients and that comes to an end with twentieth century physics. This is significant in that it frees us from having to conceive of geometry as a science that is ultimately based on the question of ‗what is space', since that notion is neither necessary for the ancients nor for contemporary science. Moreover, it frees us to gain a better understanding of what the ancients were really focused on in their mathematical and geometrical thinking insofar as we cannot falsely attribute or attach the concept of an abstract space to their inquiries. In this regard, one way we might better approach ancient geometrical thought is to consider it a systematic investigation of types of relation, or basically of *logos*, along with the resultant concepts of ratio, proportion, equality, similarity, and commensurability, which are the basic themes found in the successive books of Euclid's *Elements*. In conceiving of the Euclidean system in this way, we can also better situate Euclid's work within the larger context of Greek philosophy, science and logic, which in turn allows us to approach Euclid's geometry in closer connection with thinkers such as Plato and Aristotle, and thereby find greater consistency between ancient philosophy and ancient mathematics as a whole.

Just to be clear, I am not suggesting that the ancient Greeks were somehow different from us in terms of their physical perception or cognitive make-up. I am not putting forth a Whorffian thesis such as the lack of a linguistic term for space means that there is no perception of space as such. Rather, I am making a much less groundbreaking point, which is concerned with the nature of ancient mathematics in juxtaposition to modern mathematics. Thus, I am not in any way rejecting a Kantian a-priori notion of space in the minds of the ancient Greeks, because it simply does not concern the matter at hand, which instead is the difference between modern geometry and ancient geometry. The point is perhaps best stated when we consider that the concept of ‗Euclidean space' is in fact a modern concept. Thus the point in question is nothing extraordinary, it is simply a detail that historians of mathematics, philosophy and science (as well as philologists) should consider so as not to take for granted the idea that geometry is always concerned with defining the nature of space, for, as I have attempted to show, this simply was not the case for the ancients. Therefore, the history of mathematics involves a transition from relation-based theories regarding geometrical objects to early modern and modern theoretizations of space itself through geometry. Space as such can, generally speaking, be attributed to thinkers like Descartes, Galileo, Newton and Leibniz, thereby giving rise to the modern geometrical tradition and epistemological considerations such as Russell's which rely on Kant and his early modern predecessors.

It might be argued that even though the ancients themselves did not speak of ‗Euclidean space' per se, they nevertheless were working on this notion, and simply didn't realize it, i.e., that to do geometry at all is necessarily to deal with space qua space, regardless of what one thinks one is really doing. Such a position, however, makes an assertion about geometry itself, and even if it were true, it would not change the fact that the history of geometry involves an ancient approach that was not trying to define the nature of space in the manner of modern geometry. In this brief analysis, what is important, then, is neither the cognitive or epistemological underpinnings of an a-priori conception of space, nor the possible necessity of space in geometry, but rather the particular approach of the ancients toward geometrical objects in terms of relations that did not utilize and did not involve a term or theory of the nature of space itself. In this sense, the ancients did indeed work on a ‗geometry without space'. The contemporary consequences of this suggestion are not concerned with a meta conception of geometry as such, but rather with our characterization of the historical development of geometry and a greater understanding of the ancient Greek approach to geometrical objects which can be juxtaposed to a modern theorizing of space itself through the geometrical tradition. In the end, it is a small point that is modest in scope, yet still important for our continued discussions and sustained scholarship on an accurate account of the history of geometry and science.